MATHEMATICS 300 — FALL 2017

Introduction to Real Analysis

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ANSWERS TO THE PROBLEMS

Problem 1. ***Prove or disprove*** each of the following statements, using the $\varepsilon - \delta$ definitions of continuity and uniform continuity:

1. If $f : \mathbb{R} \mapsto \mathbb{R}$ and $g : \mathbb{R} \mapsto \mathbb{R}$ are continuous functions, then the product fg is continuous.

ANSWER. *This statement is true.* Here is the proof.

Let $x_0 \in \mathbb{R}$. We want to prove that fg is continuous at x_0.

Let ε be a positive real number. We want to prove that

(*) *There exists a positive δ such that*

$$|f(x)g(x) - f(x_0)g(x_0)| < \varepsilon \qquad \text{whenever} \quad |x - x_0| < \delta\,. \tag{0.1}$$

Let

$$\varepsilon' = \min\left(1, \frac{\varepsilon}{1 + |f(x_0)| + |g(x_0)|}\right).$$

Since f is continuous at x_0, we may pick a positive δ_1 such that

$$|f(x) - f(x_0)| < \varepsilon' \qquad \text{whenever} \quad |x - x_0| < \delta_1\,. \tag{0.2}$$

Since g is continuous at x_0, we may pick a positive δ_2 such that

$$|g(x) - g(x_0)| < \varepsilon' \qquad \text{whenever} \quad |x - x_0| < \delta_2\,. \tag{0.3}$$

Let

$$\delta = \min(\delta_1, \delta_2)\,. \tag{0.4}$$

This is going to be the δ that we will use to prove (*). We have to prove that for this δ Equation (0.1) holds.

Let x be such that $|x - x_0| < \delta$. We want to prove that

$$|f(x)g(x) - f(x_0)g(x_0)| < \varepsilon\,. \tag{0.5}$$

Since $|x - x_0| < \delta$, it follows that

(a) $|x - x_0| < \delta_1$, so $|f(x) - f(x_0)| < \varepsilon'$,

(b) $|x - x_0| < \delta_2$, so $|g(x) - g(x_0)| < \varepsilon'$.

Therefore, if $|x - x_0| < \delta$, we have

$$\begin{aligned}
&|f(x)g(x) - f(x_0)g(x_0)| \\
&\qquad = |f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)| \\
&\qquad \leq |f(x)g(x) - f(x)g(x_0)| + |f(x)g(x_0) - f(x_0)g(x_0)| \\
&\qquad = |f(x)(g(x) - g(x_0))| + |(f(x) - f(x_0))g(x_0)| \\
&\qquad = |f(x)||g(x) - g(x_0)| + |f(x) - f(x_0)||g(x_0)| \\
&\qquad \leq |f(x)|\varepsilon' + |g(x_0)|\varepsilon' \\
&\qquad \leq (|f(x)| + |g(x_0)|)\varepsilon' \\
&\qquad = (|f(x) - f(x_0) + f(x_0)| + |g(x_0)|)\varepsilon' \\
&\qquad \leq (|f(x) - f(x_0)| + |f(x_0)| + |g(x_0)|)\varepsilon' \\
&\qquad < (\varepsilon' + |f(x_0)| + |g(x_0)|)\varepsilon' \\
&\qquad \leq (1 + |f(x_0)| + |g(x_0)|)\varepsilon' \\
&\qquad \leq \varepsilon\,.
\end{aligned}$$

So we have shown that $|f(x)g(x) - f(x_0)g(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$. That is, we have proved (0.1).

We have therefore proved (*), and this proves that fg is continuous at x_0. **Q.E.D.**

2. If $f : \mathbb{R} \mapsto \mathbb{R}$ and $g : \mathbb{R} \mapsto \mathbb{R}$ are uniformly continuous functions, then the product fg is uniformly continuous.

ANSWER. *The statement is false.* To prove this, we give a counterexample.

Let $f(x) = x$ and $g(x) = x$ for $x \in \mathbb{R}$. Then f and g are uniformly continuous, but the product fg is not, because $(fg)(x) = x^2$, and the function $\mathbb{R} \ni x \mapsto x^2$ (that is, the function h such that $h(x) = x^2$ for $x \in \mathbb{R}$) is not uniformly continuous.

Problem 2.

1. ***Define*** "interval".

ANSWER. An <u>interval</u> is a subset I of the real line $\mathbb{R}$ such that

$$(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})(\forall c \in \mathbb{R})\Big((a < b < c \wedge a \in I \wedge c \in I) \Longrightarrow b \in I\Big).$$

2. ***Prove*** that if I is an interval, $f : I \mapsto \mathbb{R}$ is continuous, and $a, b \in I$ are such that $f(a)$ and $f(b)$ have opposite signs (that is, $f(a)f(b) < 0$), then there exists $c \in I$ such that $f(c) = 0$.

ANSWER. Either $a < b$ or $b < a$. (It cannot be the cass that $a = b$, because if $a = b$ then it would follow that $f(a)f(b) = f(a)^2 \geq 0$, contradicting our assumption that $f(a)f(b) < 0$.)

The proof for the case when $b < a$ is identical to the proof for the case when $a < b$, so we just do the proof for the case when $a < b$.

So let us assume that $a < b$. We are going to carry a "divide and conquer" process, which will result in either

(i) finding a point c such that $f(c) = 0$,

or

(ii) producing infinite sequences $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$, $\mathbf{b} = \{b_n\}_{n=1}^{\infty}$, such that $a_1 = a$, $b_1 = b$, and, for every positive integer j,

- (ii.1) $a_j \leq a_{j+1} < b_{j+1} \leq b_j$,
- (ii.2) f changes sign in the interval $[a_j, b_j]$ (that is. $f(a_j)f(b_j) < 0$),
- (ii.3) $b_j - a_j = \frac{b_1 - a_1}{2^{j-1}}$.

That is, we will be producing a sequence $\{I_n\}_{n=1}^{\infty}$ of closed bounded intervals I_n, where $I_n = [a_n, b_n]$, such that

(iii.1) $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$,

(iii.2) f changes sign on each I_j,

(iii.3) $|I_j| = \frac{b_1 - a_1}{2^{j-1}}$ for every j, where, for an interval I, "$|I|$" denotes the length of I.

We start by defining the first interval, I_1, by letting

$$a_1 = a\,, \quad b_1 = b\,, \quad I_1 = [a_1, b_1]\,.$$

In order to construct I_2, we divide I_1 into two halves, by letting

$$d_1 = \frac{a+b}{2}\,,$$

so d_1 is the midpoint of the interval I_1.

Then one of the following two possibilities occurs: $f(d_1) = 0$ or $f(d_1) \neq 0$.

If $f(d_1) = 0$, then our process stops, because we have found a c such that $f(c) = 0$, namely, $c = d_1$.

If $f(d_1) \neq 0$, then we observe that f must change sign on one of the intervals $[a_1, d_1]$, $[d_1, b_1]$, because one of the numbers $f(a_1)f(d_1)$, $f(d_1)f(b_1)$ must be < 0. (Reason: Neither of these numbers can be zero, because we are assuming that $f(a_1) \neq 0$, $f(b_1) \neq 0$, and $f(d_1) \neq 0$. Suppose both numbers were positive. Then we would have $f(a_1)f(d_1) > 0$, $f(d_1)f(b_1) > 0$, so $f(a_1)f(d_1)^2 f(b_1) > 0$, and then, since $f(d_1)^2 > 0$, we would have $f(a_1)f(b_1) > 0$, contradicting our assumption that $f(a_1)f(b_1) < 0$.)

Now that we know that f changes sign on one of the intervals $[a_1, d_1]$, $[d_1, b_1]$, we define our second interval, I_2, to be the one of these two intervals where f changes sign. That is, we let

$$I_2 = \begin{cases} [a_1, d_1] & \text{if} \quad f(a_1)f(d_1) < 0 \\ [d_1, b_1] & \text{if} \quad f(d_1)f(b_1) < 0 \end{cases}\,.$$

In other words, we define

$$a_2 = \begin{cases} a_1 & \text{if} \quad f(a_1)f(d_1) < 0 \\ d_1 & \text{if} \quad f(d_1)f(b_1) < 0 \end{cases}\,,$$

and

$$b_2 = \begin{cases} d_1 & \text{if} \quad f(a_1)f(d_1) < 0 \\ b_2 & \text{if} \quad f(d_1)f(b_1) < 0 \end{cases}\,.$$

Clearly, I_2 has length $\frac{b_1 - a_1}{2}$.

We now repeat the subdivision operation, by dividing I_2 into two equal halves. We do this by letting

$$d_2 = \frac{a_2 + b_2}{2}\,,$$

so d_2 is the midpoint of the interval I_2.

Then one of the following two possibilities occurs: $f(d_2) = 0$ or $f(d_2) \neq 0$.

If $f(d_2) = 0$, then our process stops, because we have found a c such that $f(c) = 0$, namely, $c = d_2$.

If $f(d_2) \neq 0$, then we observe that f must change sign on one of the intervals $[a_2, d_2]$, $[d_2, b_2]$, because one of the numbers $f(a_2)f(d_2)$, $f(d_2)f(b_2)$ must be < 0. (Reason: Neither of these numbers can be zero, because we are assuming that $f(a_2) \neq 0$, $f(b_2) \neq 0$, and $f(d_2) \neq 0$. Suppose both numbers were positive. Then we would have $f(a_2)f(d_2) > 0$, $f(d_2)f(b_2) > 0$, so $f(a_2)f(d_2)^2 f(b_2) > 0$, and then, since $f(d_2)^2 > 0$, we would have $f(a_2)f(b_2) > 0$, contradicting our assumption that $f(a_2)f(b_2) < 0$.)

Now that we know that f changes sign on one of the intervals $[a_2, d_2]$, $[d_2, b_2]$, we define our third interval, I_3, to be the one of these two intervals where f changes sign. That is, we let

$$I_3 = \begin{cases} [a_2, d_2] & \text{if} \quad f(a_2)f(d_2) < 0 \\ [d_2, b_2] & \text{if} \quad f(d_2)f(b_2) < 0 \end{cases} .$$

In other words, we define

$$a_3 = \begin{cases} a_2 & \text{if} \quad f(a_2)f(d_2) < 0 \\ d_2 & \text{if} \quad f(d_2)f(b_2) < 0 \end{cases} ,$$

and

$$b_3 = \begin{cases} d_2 & \text{if} \quad f(a_2)f(d_2) < 0 \\ b_2 & \text{if} \quad f(d_2)f(b_2) < 0 \end{cases} .$$

Clearly, I_3 has length $\frac{b_1 - a_1}{4}$.

Continuing in this way, we produce, for each natural number n, intervals $I_1, I_2, \ldots, I_n$, given by

$$I_j = [a_j, b_j] \qquad \text{for} \quad j = 1, \ldots, n \,,$$

such that

(iv.1) $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n$,

(iv.2) f changes sign on I_j for $j = 1, \ldots, n$,

(iv.3) $|I_j| = \frac{b_1 - a_1}{2^{j-1}}$ for $j = 1, \ldots, n$.

To go from I_n to I_{n+1}, we let $d_n = \frac{a_n+b_n}{2}$, and stop the process if $f(d_n) = 0$, in which case we have found the c we are looking for, namely, $c = d_n$.

If $f(d_n) \neq 0$, then it follows as before that f changes sign on one of the intervals $[a_n, d_n]$, $[d_n, b_n]$, and we define I_{n+1} to be the one of these two intervals on which f changes sign.

When this process is complete, we have either:

(I) found a c such that $f(c) = 0$,

or

(II) produced infinite sequences $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$, $\mathbf{b} = \{b_n\}_{n=1}^{\infty}$, $\mathbf{I} = \{I_n\}_{n=1}^{\infty}$ such that $a_1 = a$, $b_1 = b$, $I_n = [a_n, b_n]$ for each n, conditions (ii.1), (ii.2), (ii.3) hold for every positive integer j, and conditions (iii.1), (iii.2), (iii.3) hold.

If we are in case (II), we observe that the sequence $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ is increasing (because $a_{n+1} \geq a_n$ for every n), and bounded above (because, for example, $a_n \leq b_1$ for every n).

So $\mathbf{a}$ converges to a limit c. Since $b_n = a_n + \frac{b_1-a_1}{2^{n-1}}$. we see that $\{b_n\}_{n=1}^{\infty}$ also converges to c. Since f changes sign on each of the intervals I_n, we have $f(a_n)f(b_n) < 0$. Then $f(c)f(c) = \lim_{n\to\infty} f(a_n)f(b_n) \leq 0$. So $f(c)^2 \leq 0$. But the square of every real number is nonnegative, so $f(c)^2 \geq 0$. If follows that $f(c)^2 = 0$, and then $f(c) = 0$ as well.

So, whether (I) or (II) holds, we have found a c such that $f(c) = 0$. **Q.E.D.**

Problem 3.

1. ***Define*** "compact subset of $\mathbb{R}$" , using the open covers definition.

ANSWER. A subset E of $\mathbb{R}$ is <u>compact</u> if for every cover $\{U_\alpha\}_{\alpha\in A}$ of E by open sets there exists a finite subset B of A such that $E \subseteq \bigcup_{\alpha\in B} U_\alpha$.

2. ***Prove*** that if E is a compact subset of $\mathbb{R}$ and $f : E \mapsto \mathbb{R}$ is continuous, then f has a maximum and a minimum.

ANSWER. Let us first prove that every continuous function $f : E \mapsto \mathbb{R}$ has a maximum.

So let $f : E \mapsto \mathbb{R}$ be continuous. We first prove that f is bounded above. For each point $p \in E$, pick a positive δ_p such that $|f(x) - f(p)| < 1$ whenever

$|x-p| < \delta_p$. Let $U_p = (p-\delta_p, p+\delta_p)$. Then $\{U_p\}_{p\in E}$ is an open cover of E. So there exists a finite subset B of E such that $E \subseteq \bigcup_{p\in B} U_p$. Let M be the largest of the the numbers $f(p)+1$, $p \in B$. (This is a finite set of numbers, so it has a largest member.) Then, if $x \in E$, we may pick $p \in B$ such that $x \in U_p$. But then

$$\begin{aligned} f(x) &= f(x) - f(p) + f(p) \\ &\leq |f(x) - f(p)| + f(p) \\ &\leq 1 + f(p) \\ &\leq M\,. \end{aligned}$$

So $f(x) \leq M$ for every $x \in E$. Therefore M is an upper bound for the set $\{f(x) : x \in E\}$, so f is bounded above.

Since f is bounded above, the set $\{f(x) : x \in E\}$ has a least upper bound u. We will prove that there exists $x \in E$ such that $f(x) = u$.

Suppose an x such that $f(x) = u$ does not exist. This means that for every $x \in E$, $f(x) < u$. Define a function $g : E \mapsto \mathbb{R}$ by letting

$$g(x) = \frac{1}{u - f(x)} \quad \text{for} \quad x \in E\,.$$

This function is continuous, because the denominator $u - f(x)$ is always different from zero, since we are assuming that $f(x)$ never equals u.

We have already proved that every continuous function on E is bounded above. So g is bounded above. This means that we may pick a real number A such that $g(x) \leq A$ for every $x \in E$. That is, the inequality

$$\frac{1}{u - f(x)} \leq A$$

holds for every $x \in E$. But then, if $x \in E$, we have

$$u - f(x) \geq \frac{1}{A}\,,$$

so

$$f(x) - u \leq -\frac{1}{A}\,,$$

and then

$$f(x) \leq u - \frac{1}{A}\,. \tag{0.6}$$

So we have shown that (0.6) holds for every $x \in E$. Therefore $u - \frac{1}{A}$ is an upper bound for the set $\{f(x) : x \in E\}$.

On the other hand, $u - \frac{1}{A} < u$. So u is not the least upper bound for the set $\{f(x) : x \in E\}$. This contradiction arose from assuming that there is no $x \in E$ sch that $f(x) = u$. Hence there exists an $x \in E$ such that $f(x) = u$. Clearly, then,

$$f(x) \geq f(y) \qquad \text{for every } \ y \in E \,.$$

So x is a maximum of f.

We have thus proved that every continuous function on E has a maximum. Now, given a continuous function $f : E \mapsto \mathbb{R}$, we can apply the result on the existence of a maximum to the function $-f$, and conclude that $-f$ has a maximum. Clearly, a maximum of $-f$ is a minimum of f. So f has a minimum.

So we have thus proved that every continuous function on E has a maximum and a minimum. **Q.E.D.**

Problem 4.

1. ***State*** the Mean Value Theorem.

ANSWER. *Let a, b be real numbers such that $a < b$. Let $f : [a, b] \mapsto \mathbb{R}$ be a function such that f is continuous on $[a, b]$ and differentiable at every point of the open interval (a, b). Then there exists a point $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c) \,.$$

2' ***Prove*** the Mean Value Theorem.

ANSWER. Let a, b, f be as in the statement of the theorem.

Define a function $g : [a, b] \mapsto \mathbb{R}$ by letting

$$g(x) = \alpha + \beta x \ \text{ for } \ x \in [a.b] \,,$$

where the constants α, β will be chosen so that

$$f(a) = g(a) \qquad \text{and} \qquad f(b) = g(b) \,.$$

Then

$$f(a) = \alpha + \beta a \qquad \text{and} \qquad f(b) = \alpha + \beta b \,,$$

so

$$f(b)-f(a)=\beta(b-a)\,,$$

and then

$$\beta=\frac{f(b)-f(a)}{b-a}\,.$$

It follows that

$$\alpha=f(a)-\beta a=f(a)-\frac{f(b)-f(a)}{b-a}a\,,$$

so

$$\begin{aligned} g(x) &= f(a)-\frac{f(b)-f(a)}{b-a}a+\frac{f(b)-f(a)}{b-a}x \\ &= f(a)+\frac{f(b)-f(a)}{b-a}(x-a)\,, \end{aligned}$$

and it is easy to verify that $g(a)=f(a)$ and $g(b)=f(b)$. (Verification: $g(a)=f(a)+\frac{f(b)-f(a)}{b-a}(a-a)=f(a)$, and $g(b)=f(a)+\frac{f(b)-f(a)}{b-a}(b-a)=f(a)+f(b)-f(a)=f(b)$.)

Let $h(x)=f(x)-g(x)$. Then $h(a)=0$ and $h(b)=0$. By Rolle's Theorem, we may pick $c\in(a,b)$ such that $h'(c)=0$.

Then

$$0=h'(c)=f'(c)-g'(c)=f'(c)-\beta=f'(c)-\frac{f(b)-f(a)}{b-a}\,.$$

So $f'(c)=\frac{f(b)-f(a)}{b-a}$. **Q.E.D.**

3. ***Prove*** that if $x,y\in\mathbb{R}$ then $|\sin x-\sin y|\le|x-y|$.

ANSWER. Let $x,y\in\mathbb{R}$. If $x=y$ then $\sin x=\sin y$, so $\sin x-\sin y=0$, and then $|\sin x-\sin y|\le|x-y|$.

Next, assume that $x\neq y$. By the Mean Value Theorem, there exists $c\in\mathbb{R}$ such that

$$\frac{\sin x-\sin y}{x-y}=\cos c\,.$$

Then

$$\left|\frac{\sin x-\sin y}{x-y}\right|\le 1\,.$$

Therefore,

$$|\sin x-\sin y|\le|x-y|\,.$$

Q.E.D.

4. ***Prove*** that if $x, y \in \mathbb{R}$ and $0 < x \le y$ then $\sqrt[3]{y} - \sqrt[3]{x} \le \frac{y-x}{3\sqrt[3]{x^2}}$.

ANSWER. Let $x, y \in \mathbb{R}$ be such that $0 < x \le y$. If $y = x$ then $\sqrt[3]{y} - \sqrt[3]{x} = 0$, so the inequality $\sqrt[3]{y} - \sqrt[3]{x} \le \frac{y-x}{3\sqrt[3]{x^2}}$ holds.

Now assume that $x < y$. Let $f(u) = \sqrt[3]{u}$, for $u > 0$. Then $f(u) = u^{1/3}$. Therefore $f'(u) = \frac{1}{3}u^{-2/3}$. So the Mean Value Theorem implies that there exists $c \in \mathbb{R}$ such that $x < c < y$ and

$$\frac{\sqrt[3]{y} - \sqrt[3]{x}}{y - x} = \frac{1}{3}c^{-2/3}\,.$$

Let $g(u) = u^{-2/3}$ for $u > 0$. Then g is a decreasing function. Since $c > x$, it follows that $c^{-2/3} < x^{-2/3}$. Hence

$$\frac{\sqrt[3]{y} - \sqrt[3]{x}}{y - x} < \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} = \frac{1}{3\sqrt[3]{x^2}}\,.$$

It then follows that

$$\sqrt[3]{y} - \sqrt[3]{x} < \frac{y - x}{3\sqrt[3]{x^2}}\,.$$

Q.E.D.

Problem 5.

1. ***Prove*** that the equation $\cos x = x - 1$ has a solution.

ANSWER. Let $h(x) = \cos x - (x - 1)$, i.e., $h(x) = \cos x - x + 1$. Then $h(0) = 2$, and $h\left(\frac{\pi}{2}\right) = 1 - \frac{\pi}{2}$. So $h(0) > 0$ and $h\left(\frac{\pi}{2}\right) < 0$. By the Intermediate Value Theorem, there exists c such that $0 < c < \frac{\pi}{2}$ and $h(c) = 0$. **Q.E.D.**

2. ***Prove*** that the equation $e^x = x$ does not have a solution.

ANSWER. Let $h(x) = e^x - x$. Then $h'(x) = e^x - 1$. Clearly, $e^x < 1$ if $x < 0$, and $e^x > 1$ if $x > 0$. It follows that $h'(x) < 0$ if $x < 0$, and $h'(x) > 0$ if $x > 0$. Therefore the function h is decreasing on the half-line $(-\infty, 0]$, and increasing on the half-line $[0, \infty)$. So $h(0) \le h(x)$ if $x < 0$, and $h(0) \le h(x)$ also if $x > 0$. Hence h has a minimum at $x = 0$.

On the other hand, $h(0) = 1$. So $h(x) \ge 1 > 0$ for every x. It follows that there is no x such that $h(x) = 0$. **Q.E.D.**

3. ***Prove*** that the equation $e^x = 3x$ has a solution.

ANSWER. Let $h(x) = e^x - 3x$. Then $h(0) = 1$ and $h(1) = e - 3$. So $h(0) > 0$ and $h(1) < 0$. By the Intermediate Value Theorem, there exists c such that $0 < c < 1$ and $h(c) = 0$. **Q.E.D.**

Problem 6.

1. ***State*** the formula for the derivaticve of a product.

ANSWER. *Let $f, g : E \mapsto \mathbb{R}$ be functions such that both f and g are differentiable at a point x_0 of E. Then the product fg is differentiable at x_0, and*

$$(fg)'(x_0) = f'(x_0)h(x_0) + f(x_0)g'(x_0)\,.$$

2. ***Prove*** the formula for the derivaticve of a product.

ANSWER. For $x \in E$ such that $x \neq x_0$ we have

$$\begin{aligned}
\frac{(fg)(x) - (fg)(x_0)}{x - x_0} &= \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \\
&= \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} \\
&= \frac{(f(x) - f(x_0))g(x) + f(x_0)(g(x) - g(x_0))}{x - x_0} \\
&= \frac{(f(x) - f(x_0))g(x)}{x - x_0} + \frac{f(x_0)(g(x) - g(x_0))}{x - x_0} \\
&= \frac{f(x) - f(x_0)}{x - x_0}g(x) + f(x_0)\frac{g(x) - g(x_0)}{x - x_0}\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\lim_{x\to x_0}\left(\frac{(fg)(x) - (fg)(x_0)}{x - x_0}\right) \\
&\quad= \lim_{x\to x_0}\left(\frac{f(x) - f(x_0)}{x - x_0}g(x) + f(x_0)\frac{g(x) - g(x_0)}{x - x_0}\right) \\
&\quad= \lim_{x\to x_0}\left(\frac{f(x) - f(x_0)}{x - x_0}g(x)\right) + \lim_{x\to x_0}\left(f(x_0)\frac{g(x) - g(x_0)}{x - x_0}\right) \\
&\quad= \lim_{x\to x_0}\left(\frac{f(x) - f(x_0)}{x - x_0}\right) \times \lim_{x\to x_0} g(x) + f(x_0)\lim_{x\to x_0}\left(\frac{g(x) - g(x_0)}{x - x_0}\right) \\
&\quad= f'(x_0)g(x_0) + f(x_0)g'(x_0)\,,
\end{aligned}$$

where, in the last step, we used the fact that if g is differentiable at x_0 then g is continuous at x_0, so $\lim_{x\to x_0} g(x) = g(x_0)$.

Hence $(fg)'(x_0)$ exists and is equal to $f'(x_0)g(x_0) + f(x_0)g'(x_0)$. **Q.E.D**.