MATHEMATICS 300 — SPRING 2019

Introduction to Mathematical Reasoning

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HOMEWORK ASSIGNMENT NO. 8, DUE ON WEDNESDAY, APRIL 24, FOR SECTION 9, AND ON THURSDAY, APRIL 25, FOR SECTION 2

This assignment consists of just one problem.

Introduction

In the lecture notes we defined "finite set" as follows:

Definition F1. *A set A is finite if there exist a nonnegative integer n and a function $f : \mathbb{N}_n \mapsto A$ which is onto A.* □

And we defined the "number of members", or "cardinality", of a set, as follows:

Definition F2. *If n is a nonnegative integer and A is a set, we say that A has n members, or that A hac cardinality n, if there exists a bijection from $\mathbb{N}_n$ to A, that is, if there exists a function $f : \mathbb{N}_n \mapsto A$ which is onto A and one-to-one.* □

And we proved:

Theorem FC. *For every finite set A therr exists a unique nonnegative integer n such that A has cardinality n.*

And, thanks to Theorem FC, we can talk about ***the*** cardinality (or ***the*** number of members) of a finite set.

We are now going to add a new cardinality to the ones we have so far, and expand the class of finite sets to the larger class of "countable sets". It will turn out that all countable sets that are not finite have the same cardinality, and this cardinality will be called[1] "$\aleph_0$" (i.e., "aleph-zero").

[1] "$\aleph$" ("aleph") is the first letter of the Hebrew alphabet. The name "$\aleph_0$" was introduced by Georg Cantor, who also uses $\aleph_1$, $\aleph_2$, and so on, for higher infinite cardinals.

Precisely, here are our definitions:

Definition C1. *A set A is countable if either (a) A is empty, or (b) there exists a function $f : \mathbb{N} \mapsto A$ which is onto A.* □

A function with domain $\mathbb{N}$ is called a sequencr.

A fuinction $f : \mathbb{N} \mapsto A$ which is onto A is called an enumeration of A.

So Definition C1 says: a set A is countable if there exists an enumeration of A.

Definition C2. *A set A is countably infinite if there exists a bijection from $\mathbb{N}$ to A. i.e., a function $f : \mathbb{N} \mapsto A$ which is one-to-one and onto A.* □

Definition C3. *We say that a set A has cardinality $\aleph_0$ if it is oountably infinite.* □

So $\aleph_0$ is the cardinality of $\mathbb{N}$ and of all the sets that have the same cardinality as $\mathbb{N}$.

The problem

Problem 1. ***Prove***, ***using the definitions given here***, that

1. *Every finite set is countable*[2]
2. *Every countably infinite set is countable*[3].
3. *Every subset of a countable set is countable*[4].
4. *Every countable set is either finite or countably infinite*[5].
5. *The set of all twin primes is countable.* (A twin prime is a prime number p such that $p + 2$ is also prime. For example, 3, 5, 11, 17, 29, 41, 59, 71, 101 are twin primes.)

[2]Do not worry if you think this is too easy and too simple, that there must be a catch. It ***is*** very easy and very simple. I just want to make sure that you know how to do it and how to write it up properly.

[3]Same comment as in the previous footnote.

[4]Same comment as in the two previous footnotes.

[5] ***This*** part requires some hard work. You want to start with a function $f : \mathbb{N} \mapsto A$ which is onto A, and you want to end up with a function $g : \mathbb{N} \mapsto A$,—or maybe $g : \mathbb{N}_n \mapsto A$ for some $n \in \mathbb{N} \cup \{0\}$—that is also onto A, but ***in addition is one-to-one***. The question is: how do we do that? That is, fow do we eliminate the repetitions? Take a look at the discussion of "how to eliminate repetitions" below.

What is remarkable about this example is that nobody knows whether the set of twin primes is finite or infinite[6]***.*** So ***you will not be able to produce a bijection from*** $\mathbb{N}_n$ ***or*** $\mathbb{N}$ ***to the set, because you don't even know if the rigth domain to use is*** $\mathbb{N}_n$ ***or*** $\mathbb{N}$***.***

6. *The union of two countable sets is countable.* (That is, if A and B are countable then $A \cup B$ is countable.) HINT: Sequences can be combined by ***intertwining*** them. For example, if f, g are two sequences, i.e, two functions with domain $\mathbb{N}$, then you can form a sequence $f\#g$, as follows: $f\#g$ is the sequence $(f(1), g(1), f(2), g(2), f(3), g(3), \ldots)$. That is, more precisely,

$$(f\#g)(n) = \begin{cases} f\left(\frac{n+1}{2}\right) & \text{if} \quad n \text{ is odd} \\ g\left(\frac{n}{2}\right) & \text{if} \quad n \text{ is even} \end{cases} .$$

Using this, you must prove that $\text{Ran}(f\#g) = \text{Ran}(f) \cup \text{Ran}(g)$, and with that the rest is smooth sailing.

7. *The union of a sequence of countable sets is countable.* That is, if $\mathbf{s}$ is a sequence of countable sets (or, in other words, $\mathbf{s}(1)$, $\mathbf{s}(2)$, $\mathbf{s}(3)$, ..., are countable sets) then the union $\bigcup_{j=1}^{\infty} \mathbf{s}(j)$ is a countable set. (The union $\bigcup_{j=1}^{\infty} \mathbf{s}(j)$ is the set $\{x : (\exists j \in \mathbb{N})x \in \mathbf{s}(j)$.) HINT: You have to figure out how to intertwine infinitely many sequences. Idea:

$$(f_1(1), f_1(2), f_2(1), f_1(3), f_2(2), f_3(1), f_1(4), f_2(3), f_3(2), f_4(1), f_1(5), f_2(4),$$

$$f_3(3), f_4(2), f_5(1), f_1(6), f_2(5), f_3(4), f_4(3), f_5(2), f_6(1), f_1(7), f_2(6), f_3(5), \cdots)$$

Figure out the pattern and define the infinite concatenation rigorously.)

8. *The set* $\mathbb{Q}$ *of all rational numbers ountably infinire.* (BINT: To prove that $\mathbb{Q}$ is countable use the result of the previous part. For each $n \in \mathbb{N}$, let A_n be the set of all rational numbers x such that $(\exists m \in \mathbb{Z})x = \frac{m}{n}$. Prove that A_n is countable and that $\mathbb{Q} = \bigcup_{n=1}^{\infty} A_n$.)

How do we eliminate repetitions?

[6]This is a big open problems in Mathematics. If you can solve it you will get an A in the course and become famous.

Suppose $f : \mathbb{N} \mapsto A$, so f is an infinite list, or sequence. We would like to produce a function $g : \mathbb{N} \mapsto A$ (or perhaos $g : \mathbb{N}_n \mapsto A$ for some $n \in \mathbb{N} \cup \{0\}$) such that $\mathrm{Ran}(g) = A$ and g is one-to-one. How do we do that? I'll sketch one way to do it, and ask you to write it up cleanly with all the details.

The ***first step*** is to call an index $j \in \mathbb{N}$ "bad" if the value $f(j)$ has already appeared before as one of the entries. That is: j is bad if

$$(\exists j \in \mathbb{N})(j < k \wedge f(j) = f(k)\,.$$

And then we call a pair $(j, f(j))$ "bad" if the index j is bad. (Remember that the function f is a set of ordered pairs, More precisely, it is the set of all ordered pairs $(j, f(j))$, for all $j \in \mathbb{N}$.)

The ***second step*** is to remove from f all the bad pairs.

For example, if f was a set of ordered pairs

$$\Big\{ (1,a), (2,g), (3,a), (4,a), (5,b), (6,c), (7,b), (8,c), (9,d), (10,b), (11,h), \ldots \Big\},$$

then the bad pairs are $(3,a)$, $(4,a)$, $(7,b)$, $(8,c)$, $(10,b)$, ...

After we remove the bad pairs we get

$$\Big\{ (1,a), (2,g), (5,b), (6,c), (9,d), (11,h) \ldots \Big\},$$

The ***third and last step*** is to renumber the inputs, so that they will be $1, 2, 3, 4, 5, \ldots$ in order. In our example, this yields

$$\Big\{ (1,a), (2,g), (3,b), (4,c), (5,d), (6,h) \ldots \Big\},$$

which means that our function is the list

$$(a, g, b, c, d, h, \ldots)\,.$$

(In geberal: going from left to right, if $(j_k, f(j_k)$ is the k-th pair, we change the index to k, so the pair becomes $(k, f(j_k))$.)