MATHEMATICS 300 — SPRING 2019

Introduction to Mathematical Reasoning

H. J. Sussmann

HOMEWORK ASSIGNMENT NO. 6, DUE ON MONDAY, APRIL 8, FOR SECTION 9, AND ON TUESDAY, APRIL 9, FOR SECTION 2

This assignment has two parts. Part II will be posted on Thursday April 5.

PART I

Problem 1. Book, page 92:

1. Problem 4, parts (b), (d), (f), (g), (h), (i), (j),
2. Problem 5, parts (b), (d), (f), (h), (j), (l),
3. Problem 6, parts (b), (d), (e), (f).
4. Problem 7. (NOTE: When the authors say "if $x \notin B$ and $A \subseteq B$ then $x \notin A$", they mean "for every x, if $x \notin B$ and $A \subseteq B$ then $x \notin A$".)

Problem 2. Book, page 93:

1. Problem 10, all three parts.
2. Problem 11, all parts except (b).
3. Problem 12. (NOTE: The precise definition of "empty set" is this: *A set S is empty if it has no members.* It was proved in class that an empty set S exists, by taking S to be the set S given by $S = \{x : x \neq x\}$. But we did not prove that if T is any other empty set then $T = S$. Here you are asked to prove that.)

Problem 3. Book, page 93:

1. Problem 13. In addition to the question asked in the book, do the following:

   - One and only one of the following four sentences is true. ***Indicate*** which one is the true sentence and ***prove*** it, and ***disprove*** all the others.

$$(\forall a \in \mathbb{N})(\forall b \in \mathbb{N})(a\mathbb{Z} \subseteq b\mathbb{Z} \iff a \leq b) \tag{1}$$
$$(\forall a \in \mathbb{N})(\forall b \in \mathbb{N})(a\mathbb{Z} \subseteq b\mathbb{Z} \iff b \leq a) \tag{2}$$
$$(\forall a \in \mathbb{N})(\forall b \in \mathbb{N})(a\mathbb{Z} \subseteq b\mathbb{Z} \iff a|b) \tag{3}$$
$$(\forall a \in \mathbb{N})(\forall b \in \mathbb{N})(a\mathbb{Z} \subseteq b\mathbb{Z} \iff b|a)\,. \tag{4}$$

2. Problem 15, parts (b), (c), (d), (f), (h).

3. Problem 16, parts (b), (d).

4. Problem 17, parts (b), (d), (f),(h), (j), (l).

Problem 4. In this problem,

i. the Cartesian product of two sets A, B is the set $A \times B$ of all ordered pairs (x, y) such that $x \in A$ and $y \in B$. That is,

$$A \times B = \{u : (\exists x \in A)(\exists y \in B)u = (x, y)\}$$

or, equivalently,

$$A \times B = \{(x, y) : x \in A \wedge y \in B\}\,.$$

ii. The expression "$\mathbb{R}^2$" denotes the Cartesian product $\mathbb{R} \times \mathbb{R}$. The set $\mathbb{R}^2$ is called the Euclidean plane, and the members of $\mathbb{R}^2$ are referred to as "points of $\mathbb{R}^2$". So, for example, the pairs $(3, 7)$ and $(-\pi, 32.5)$ are points of $\mathbb{R}^2$.

iii. If a, b, c are real numbers and[1] $a^2 + b^2 > 0$, then the set

$$L_{a,b,c} = \{(x, y) : ax + by = c\}$$

is called the straight line with equation $ax + by = c$.

[1]Recall that "$a^2 + b^2 > 0$" is just another way to say "$a \neq 0 \vee b \neq 0$".

iv. If (p, q) is a point of $\mathbb{R}^2$ and R is a positive[2] real number, then the set

$$C_{p,q,R} = \{(x, y) : (x - p)^2 + (y - q)^2 = R^2\}$$

is called the circle with center (p, q) and radius R.

Of the following four statements, two are true and two are false. ***Indicate*** which ones are the true sentences and ***prove*** them, and ***disprove*** the other two:

$$(\forall a, b, c, \hat{a}, \hat{b}, \hat{c} \in \mathbb{R})\Big((a^2 + b^2 > 0 \wedge \hat{a}^2 + \hat{b}^2 > 0) \\ \implies \big(L_{a,b,c} = L_{\hat{a},\hat{b},\hat{c}} \iff (a = \hat{a} \wedge b = \hat{b} \wedge c = \hat{c})\big)\Big), \tag{5}$$

$$(\forall a, b, c, \hat{a}, \hat{b}, \hat{c} \in \mathbb{R})\Bigg((a^2 + b^2 > 0 \wedge \hat{a}^2 + \hat{b}^2 > 0) \\ \implies \Big(L_{a,b,c} = L_{\hat{a},\hat{b},\hat{c}} \iff (\exists r \in \mathbb{R})(a = r\hat{a} \wedge b = r\hat{b} \wedge c = r\hat{c})\Big)\Bigg), \tag{6}$$

$$(\forall p, q, R, \hat{p}, \hat{q}, \hat{R} \in \mathbb{R})\Bigg((R > 0 \wedge \hat{R} > 0) \\ \implies \Big(C_{p,q,R} = C_{\hat{p},\hat{q},\hat{R}} \iff (p = \hat{p} \wedge q = \hat{q} \wedge R = \hat{R})\Big)\Bigg), \tag{7}$$

$$(\forall p, q, R, \hat{p}, \hat{q}, \hat{R} \in \mathbb{R})\Bigg((R > 0 \wedge \hat{R} > 0) \\ \implies \Big(C_{p,q,R} = C_{\hat{p},\hat{q},\hat{R}} \iff (\exists r \in \mathbb{R})(p = r\hat{p} \wedge q = r\hat{q} \wedge R = r\hat{R})\Big)\Bigg). \tag{8}$$

[2]Remember that in this course "positive" means "> 0".