MATHEMATICS 300 — SPRING 2019

Introduction to Mathematical Reasoning

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HOMEWORK ASSIGNMENT NO. 6, DUE ON MONDAY, APRIL 8, FOR SECTION 9, AND ON TUESDAY, APRIL 9, FOR SECTION 2

This is Part II of the assigment. Part I had 4 problems, problems 1, 2, 3 and 4. This part consists of two problems: problem 5 and problem 6.

PART II

Definition 1. A relation is a set of ordered pairs. That is: A set R is a relation if and only if

$$(\forall u)\Big(u \in R \Longrightarrow (\exists x)(\exists y)u = (x, y)\Big)\,.$$

(That is, every member of R is an ordered pair.) □

Notational convention 1. If R is a relation, then

- "xRy" means "$(x, y) \in R$",

For example, let us call "divides"s the relation

$$\{(m, n) \in \mathbb{Z} \times \mathbb{Z} : (\exists k \in \mathbb{Z})n = mk\}\,. \tag{1}$$

Then the following are true statements:

$$(3, 6) \in \text{divides}\,,\ (3, 7) \notin \text{divides}\,,\ (-4, 8) \in \text{divides}\,,\ (0, 5) \notin \text{divides}\,.$$

Using notational convention 1, we can write instead,

$$3\,\text{divides}\,6\,,\ \sim 3\,\text{divides}\,7\,,\ -4\,\text{divides}\,8\,,\ \sim 0\,\text{divides}\,5\,.$$

Even better, if instead of callig the set (1) "divides" we call it "|", then we can write

$$3|6\,,\ \sim 3|7\,,\ -4|8\,,\ \sim 0|5\,.$$

(For example, the expression "3|6" now means "$(3, 6) \in |$", where "|" is the set (1). So "3|6" is still equivalent, to "$(\exists k \in \mathbb{Z})6 = 3k$".)

Definition 2. If A, B are sets, a relation from A to B is a subset of $A \times B$. □

Definition 3. If R is a relation, then the inverse of R is the relation R^{-1} given by

$$R^{-1} = \{(x, y) : (y, x) \in R\}.$$

Definition 4. A relation R is symmetric if $(\forall x)(\forall y)(xRy \implies yRx)$. □

Problem 5.

1. ***Prove*** that if A, B are sets, and R is a relation from A to B, then R^{-1} is a relation from B to A.
2. ***Prove*** that if R is a relation then R is symmetric if and only if $R = R^{-1}$.
3. Book, page 161, problem 6.
4. For each the twelve relations of the previous question (problem 6 of page 161 of the book)
   - ***indicate*** whether the relation is symmetric,
   - ***prove*** your answer.

Definition 5. A function is a relation f such that

$$(\forall x)(\forall y)(\forall z)\Big(\big((x, y) \in f \wedge (x, z) \in f \big) \implies y = z \Big). \tag{2}$$

NOTE: Condition (2) is the ***unique output property***. Think of a relation R as an "input-output device": you give it an input x, and then the relation gives you as outputs values of y. The pairs (x, y) that belong to R are the pairs (x, y) such that y is one of the outputs for the input x. Then (2) says: "for every input x, the output produced by f is unique". □

Definition 6. The domain of a function f is the set $\mathrm{Dom}(f)$ given by

$$\mathrm{Dom}(f) = \{x : (\exists y)(x, y) \in f\}.$$

Definition 7. The range of a function f is the set $\mathrm{Ran}(f)$ given by

$$\mathrm{Ran}(f) = \{y : (\exists x)(x, y) \in f\}.$$

Definition 8. If f is a function, and x is a member of $\mathrm{Dom}(f)$, then the value of f at x is the unique y such that $(x, y) \in f$. □

NOTE: If $x \in \mathrm{Dom}(f)$, then

(a) a y such that $(x, y) \in f$ exists because $x \in \mathrm{Dom}(f)$,

(b) y is unique because of the unique output property. □

Notational convention 2. If f is a function, and x is a member of $\mathrm{Dom}(f)$, then we write "$f(x)$" to denote the value of f at x. □

Definition 9. If f is a function and C is a set, we say that f takes values in C if $f(x) \in C$ for every $x \in \mathrm{Dom}(f)$. In other words: f takes values in C if and only if $\mathrm{Ran}(f) \subseteq C$. □

Definition 10. A codomain of a function f is a set C such that f takes values in C. In other words: C is a codomain of f if and only if $\mathrm{Ran}(f) \subseteq C$. □

WARNING. The book makes a serious mistake when it talks, the "Definitions" box on page 204, about "the" codomain of f. ***There is no such thing as "the" codomain of a function*** f**.** A function f has lots of different codomains, so one can talk about ***a*** codomain of f, but not about ***the*** codomain of f. □

Definition 11. If A, B are sets, and f is a function, we say that f is a function from A to B (or that f maps A to B, or that f sends A to B or that f goes from A to B) if $\mathrm{Dom}(f) = A$ and f takes values in B. □

Notational convention 3. The expression

$$f : A \mapsto B$$

means "f is a function from A to B".

Examples.

- Let
$$f = \{(x, y) \in \mathbb{R}^2 : y = x^2\}.$$
Then $f : \mathbb{R} \mapsto \mathbb{R}$, and also $f : \mathbb{R} \mapsto \mathbb{R}_{\geq 0}$, where $\mathbb{R}_{\geq 0}$ is the set $\{x \in \mathbb{R} : x \geq 0\}$.

- Let
$$f = \left\{(x, y) \in \mathbb{R}^2 : y = \frac{1}{1 + x^2}\right\}.$$
Then $f : \mathbb{R} \mapsto \mathbb{R}$, and also $f : \mathbb{R} \mapsto \mathbb{R}_{\geq 0}$, $f : \mathbb{R} \mapsto [0, 1]$ (where $[0, 1]$ is the closed interval $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$), and $f : \mathbb{R} \mapsto]0, 1]$ (where $]0, 1]$ is the interval $\{x \in \mathbb{R} : 0 < x \leq 1\}$). □

Problem 6.

1. For each the twelve relations of problem 6 of page 161 of the book
   - ***indicate*** whether the relation is a function;
   - ***prove*** your answer.
2. ***Prove*** that if f and g are functions then

$$f = g \iff \Big(\mathrm{Dom}(f) = \mathrm{Dom}(g) \wedge (\forall x) \Big(x \in \mathrm{Dom}(f) \implies f(x) = g(x) \Big) \Big) .$$

   (In other words: *two functions f, g are equal if and only if they have the same domain and f and g have the same value at every x in that domain.*

3. Book, problem 1, page 209. (NOTE: a codomain of a function f is a set C such that $\mathrm{Ran}(f) \subseteq C$. In other words, C is a codomain of f if f takes values in C, that is, if $f(x) \in C$ for every $x \in \mathrm{Dom}(f)$.)