# MATHEMATICS 300 — SPRING 2019

*Introduction to Mathematical Reasoning*

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## HOMEWORK ASSIGNMENT NO. 5, DUE ON WEDNESDAY, MARCH 27, FOR SECTION 9, AND ON THURSDAY, MARCH 28 FOR SECTION 2

### 1. List of practice problems

*The following is a list of problems that you should do in order to practice doing problems by induction. The list of problems you have to hand in is given at the end.*

- Book, pages 123 and 124, problem 4, all parts; notice that Part (e) has already appeared in a previous homework assignment;
- Book, page 124, problem 5, parts (a) through (o); notice that part (n) is exactly Theorem 36 the lecture notes, Part II, section 4.3; this proof was discussed in class and is done in detail in the lecture notes, Part II, section 4.3.
- Book, page 125, problem 6, all parts.
- Problem 39 of the lecture notes, Part II, page 110.
- In this problem, $P(k)$ is the statement "the product of any $k$ consecutive integers is divisible by $k!$". (That is, on formal language, $P(k)$ is the statement

  $$(\forall n \in \mathbb{Z})\ k! \big| a(n,k)\,,$$

  where we write $a(n,k)$ for the product of $k$ consecutive integers starting at $n$, i.e., $a(n,k) = n \times (n+1) \times \cdots \times (n+k-1)$, or, equivalently, $a(n,k) = \prod_{j=1}^{k}(n+j-1)$.)

  1. ***Prove*** $P(1)$.
  2. ***Prove*** the implication $P(1) \Longrightarrow P(2)$. (Use induction going forward and backward.)

3. ***Prove*** the implication $P(2) \Longrightarrow P(3)$. (Use induction going forward and backward.)
4. ***Prove*** the implication $P(3) \Longrightarrow P(4)$. (Use induction going forward and backward.)
5. ***Prove*** the implication $P(4) \Longrightarrow P(5)$. (Use induction going forward and backward.)
6. ***Prove*** the implication $P(k) \Longrightarrow P(k+1)$ for general $k$. That is, prove that $(\forall k \in \mathbb{N})(P(k) \Longrightarrow P(k+1))$. (Use induction going forward and backward.)

NOTE: This was done in detail in class and is discussed in detail in the lecture notes, Part II, section 5.1. ***You should study carefully the explanation of this proof given in the notes***, because it is much more detailed and clearer than what was said in class.

- [**HI**] Problem 31 of the lecture notes, Part II, page 100. In this proof, you are only allowed to use

  (a) the fact that $\mathbb{Z}$ is a commutative ring with identity, so the axioms for a commutative ring with identity (i.e,, axioms RA1 throguhn RA11) hold for $\mathfrak{R} = \mathbb{Z}$,

  (b) the basic facts BFN1, BFN2, BFN3, BFN4 about the natural numbers listed in the lecture notes, Part II, section 4.3,

  (c) the PMI,

  (d) Theorems 27 and 28 of the lecture notes part II.

  (In Theorem 27 we prove that if $n$ is a natural number and $n \neq 1$ then $n-1$ is a natural number. In Theorem 28 we prove that the sum of two natural numbers is a natural number. You are asked to prove that the product of two natural numbers is a natural number ***using only*** the facts listed above. So, for example, you cannot say "if $m$ and $n$ are natural numbers, then $m$ and $n$ are integers, and they are positive; since the product of two integers is an integer the product of two positive integers is positive, then $mn$ is a positive integer, so $mn$ is a natural number, **Q.E.D.**". You cannot do this because at this point you are not supposed toknow anything about "positive numbers". All you are allowed to use is the facts (a), (b), (c), (d) listed above. Use as a model the proof of Theorem 28 given in the notes.)

- ***Prove*** that the proposition

$$(\forall n \in \mathbb{Z})\ k | n^k - n$$

is true for $k = 2, 3, 5, 7$ and 11, but is not true for $n = 4, 6, 8, 9$ and 10. HINT: the ***binomial formula***

$$(a+b)^k = \sum_{j=0}^{k} \binom{k}{j} a^j b^{n-j} \tag{1}$$

should be useful. Here the ***binomial coefficients*** $\binom{k}{j}$ are defined by

$$\binom{k}{j} = \frac{k!}{j!(k-j)!} \text{ for } k \in \mathbb{Z}\,,\ j \in \mathbb{Z}\,,\ 0 \le j \le k\,..$$

NOTE: The definition of "factorial" is:

$$\begin{aligned} 1! &= 1\,, \\ (n+1)! &= n! \times (n+1) \text{ for } n \in \mathbb{N}\,, \end{aligned}$$

and in addition we also define

$$0! = 1\,.$$

So, for example,

$$\begin{aligned} \binom{7}{3} &= \frac{7!}{3! \times 4!} \\ &= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1) \times (4 \times 3 \times 2 \times 1)} \\ &= \frac{7 \times 6 \times 5}{3 \times 2 \times 1} \\ &= \frac{210}{6} \\ &= 35\,. \end{aligned}$$

## 2. List of problems to hand in

1. Book, pages 123 and 124, problem 4, parts (c), (d), (f) and (j).
2. Book, pages 125, problem 5, parts (l), (m) and (o).
3. Book, page 126, problem 6, parts (h) and (j).
4. The problem marked "**[HI]**" in the list of practice problems above.