MATHEMATICS 300 — SPRING 2019

Introduction to Mathematical Reasoning

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HOMEWORK ASSIGNMENT NO. 4, DUE ON MONDAY, MARCH 11, FOR SECTION 9, AND ON THURSDAY, MARCH 12 FOR SECTION 2

1. Problem 2 from the lectures notes, Part II, pages 25-26.

2. ***Prove*** by induction each of the following propositions:

   1. $(\forall n \in \mathbb{N})\Big(\sum_{k=1}^{n} k = \frac{n(n+1)}{2}\Big)$,
   2. $(\forall n \in \mathbb{N})\Big(\sum_{k=1}^{n} k^2 = \frac{(2n+1)n(n+1)}{6}\Big)$,
   3. $(\forall n \in \mathbb{N})\left(\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2\right)$,
   4. $(\forall n \in \mathbb{N}) n < 2^n$,
   5. $(\forall n \in \mathbb{N}) n^2 < 2^n + 2$,
   6. $(\forall n \in \mathbb{N}) n^3 < 2^n + 257$

   You do not need to prove again the results of the previous problem (i.e., of Problem 2 of Part II of the lecture notes). You are allowed to use those results.

3. ***Prove*** that the propositions

   1. $(\forall n \in \mathbb{N}) n^2 < 2^n + 1$,
   2. $(\forall n \in \mathbb{N}) n^3 < 2^n + 256$

   are not true.

4. ***Prove*** by induction that if n is a natural number and x, y, z are real numbers such that $|x| \leq z$ and $|y| \leq z$ then $|x^n - y^n| \leq nz^{n-1}|x - y|$. (Recall that, by definition, $a^0 = 1$ for any $a \in \mathbb{R}$, so " z^{n-1} " makes sense even when $n = 1$.) HINT: Use the ***triangle inequality***: $|a + b| \leq |a| + |b|$ for every $a, b \in \mathbb{R}$.

   REMARK: If you remember your Freshman Calculus ccurses, the result of this problem is a simple special case of the Mean Value Theorem (MVT). However, here I am asking you to prove it directly, without using derivatives and the MVT.