MATHEMATICS 300 — SPRING 2019

Introduction to Mathematical Reasoning

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HOMEWORK ASSIGNMENT NO. 2, DUE ON WEDNESDAY, FEBRUARY 20, FOR SECTION 9, AND ON THURSDAY, FEBRUARY 21, FOR SECTION 2

For each of the sentences in the list below, ***except for the first six, whose answers I am giving to you after the end of the list***:

i. Translate the sentence into reasonable English. (Please give a translation that is in correct English and sounds reasonable. Do not translate "$(\forall n \in \mathbb{Z})$" as "for all members n of the set of integers", because that sounds horrible. Write instead "for every integer", which sounds a lot better and says the same thing. Do not translate "$(\forall n \in \mathbb{Z})(2|n \implies 2|n^2)$" as "for all members n of the set of integers if 2 divides n then 2 divides n^2". Write instead "The square of every even integer is even", which says the same thing and is much better.)

ii. ***Whenever possible, try to give a translation in which the bound variables do not appear at all.***
This will reflect the fact that if, say, n is a bound variable in a sentence S, then the sentence isn't really about a number called n. For example: if I see the sentence "$2|n$", and I want to ask you if the sentence is true, I have to say something like "is n even?"; I cannot fail to mention n, because the sentence is about n. But if I see the sentence "$(\forall n \in \mathbb{Z})2|n$", and I want to ask you if this sentence is true[1], then I can say something like "is every integer even?". There is no need for me to mention n, because the sentence is not about n. Similarly, If I see the sentence "$k^2 = 9$" and I want to ask you if the sentence is true, I have to say something like "is k^2 equal to 9?"; I cannot fail to mention k, because the sentence is about k. But if I see the sentence "$\sum_{k=1}^{3} k^2 = 14$", and I want to ask you if this sentence is true, I can say something like "is the sum of the first three squares equal to 14?". There is no need for me to mention k, because the sentence is not about k.)
Sometimes this may be very hard to do, so it's better to use the names of the variables. For example, you could translate the sentence

$$(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})\Big((a+b)^2 = a^2 + b^2 + 2ab\Big)$$

[1] Of course the sentence is false, but that's another story.

as "the square of the sum of two real numbers is equal to the sum of the squares of the numbers plus twice their product", but if you try to do the same thing with the sentence

$$(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})\Big((a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\Big) \quad (\&)$$

then it gets too complicated, and it's better to use the letters[2].

iii. ***List*** all the variables that occur in the sentence, and indicate which ones are free (i.e. open) and which ones are bound (i.e., dummy, or closed). If a variable occcurs in the sentence more than once, it may happen that some of the occurrences are free and others are bound. If this happens, say it.

iv. ***Indicate*** whether the sentence is a proposition (i.e., has no open variables) or not.

v. If the sentence is a proposition, then

   a. ***indicate*** whether it is true or false,
   b. ***prove*** the assertion that you made to answer part a. (That is, prove that the sentence is true if you have asserted that ii is, and prove that it is false if you have asserted that ii is false.)

vi. If the sentence is a not a proposition, then give a detailed characterization of the set of values of the open variables for whichthe sentence is true. Your characterization should have the form "the sentence is true for x (or for x amd y, or whatever the open variables may be) if and only if C, where C is some condition involving the open variables. For example: you should not say "the sentence '$(\exists n \in \mathbb{Z})m > n^2 + 3$' is true if $m > 4$", because this does not tell me whether or not it is true for $m \leq 4$. You should write "the sentence is true for m if and only if $m > 4$".

And here is the list (***remember that the answers to the first six are given right after the list, so you should only write the solutions for sentences (7) to (19)***:

$$(\exists x \in \mathbb{R})ax = b\,, \tag{1}$$

$$(\forall b \in \mathbb{R})(\exists x \in \mathbb{R})ax = b\,, \tag{2}$$

$$(\forall a \in \mathbb{R})(\exists x \in \mathbb{R})ax = b\,, \tag{3}$$

$$(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})(\exists x \in \mathbb{R})ax = b\,, \tag{4}$$

$$(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})(a \neq 0 \Longrightarrow (\exists x \in \mathbb{R})ax = b)\,, \tag{5}$$

[2]If you can find a nice way to translate (&) into English without using the letters for the variables, please show it to me. I just cannot think of one.

$$(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})a \neq 0 \Longrightarrow (\exists x \in \mathbb{R})ax = b\,, \tag{6}$$

$$(\exists y \in \mathbb{R})y < x\,, \tag{7}$$

$$(\forall x \in \mathbb{R})y < x\,, \tag{8}$$

$$(\forall x \in \mathbb{R})(x > 0 \Longrightarrow y < x)\,, \tag{9}$$

$$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})y < x\,, \tag{10}$$

$$(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})y < x\,, \tag{11}$$

$$(\forall x \in \mathbb{R})\Big(x > 0 \Longrightarrow (\exists y \in \mathbb{R})(0 < y \wedge y < x)\Big)\,, \tag{12}$$

$$(\forall x \in \mathbb{R})x > 0 \Longrightarrow (\exists y \in \mathbb{R})(0 < y \wedge y < x)\,, \tag{13}$$

$$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\exists z \in \mathbb{R})(x < z \wedge z < y)\,, \tag{14}$$

$$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})\Big(x < y \Longrightarrow (\exists z \in \mathbb{R})(x < z \wedge z < y)\Big)\,, \tag{15}$$

$$(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})\Big(x < y \Longrightarrow (\exists z \in \mathbb{Z})(x < z \wedge z < y)\Big)\,, \tag{16}$$

$$(\forall x \in \mathbb{R})(\exists y \in \mathbb{N})x < y\,, \tag{17}$$

$$(\forall x \in \mathbb{R})\Big(\big(x > 0 \wedge x^2 \leq 2\big) \Longrightarrow (\exists y \in \mathbb{R})(x < y \wedge y^2 < 2)\Big)\,, \tag{18}$$

$$(\forall x \in \mathbb{Q})\Big(\big(x > 0 \wedge x^2 \leq 2\big) \Longrightarrow (\exists y \in \mathbb{R})(x < y \wedge y^2 < 2)\Big)\,. \tag{19}$$

(***Warning:*** sentences (18) and (19) may look "almost the same" to you, but they are *very* different!)

Solution of (1): The sentence says "there is a real number that multiplied by a yields b". The variables are a, b and x. The variables a and b are free, and x is bound. The sentence is *not* a proposition. Let us call our sentence $P(a, b)$, reflecting the fact that a and b are free variables. Then $P(a, b)$ *is true for two real numbers* a, b *if and only if* $a \neq 0 \vee (a = 0 \wedge b = 0)$. That is:

$$P(a, b) \Longleftrightarrow \big(a \neq 0 \vee (a = 0 \wedge b = 0)\big)\,. \tag{20}$$

Reason: Assume $P(a, b)$. Let us prove that $a \neq 0 \vee (a = 0 \wedge b = 0)$ using Rule $\vee_{prove}$. Assume $\sim a \neq 0$. Then $a = 0$. Since $(\exists x \in \mathbb{R})ax = b$, we can pick a witness w, so $aw = b$. Since $a = 0$, it follows that $b = 0$. So $a = 0 \wedge b = 0$. So we have proved that $a = 0 \wedge b = 0$ assuming $\sim a \neq 0$. Hence $a \neq 0 \vee (a = 0 \wedge b = 0)$ by Rule $\vee_{prove}$. So we have proved $a \neq 0 \vee (a = 0 \wedge b = 0)$ assuming $P(a, b)$, and then we get

$$P(a, b) \Longrightarrow \big(a \neq 0 \vee (a = 0 \wedge b = 0)\big)\,. \tag{21}$$

Now assume that $a \neq 0 \vee (a = 0 \wedge b = 0)\big)$. We use Rule $\vee_{use}$, i.e., we do a proof by cases. Suppose $a \neq 0$. Then we can choose $w = \frac{b}{a}$, and $aw = b$, so $(\exists x \in \mathbb{R})ax = b$ is true. Now consider the other case, $a = 0 \wedge b = 0$. In that case we can choose w to be any real number we want (for example, $w = 47$), and $aw = b$ will be true, so $(\exists x \in \mathbb{R})ax = b$ is true. We have estabilised the conclusion $(\exists x \in \mathbb{R})ax = b$ in both cases, so Rule $\vee_{use}$ enables us to conclude that $(\exists x \in \mathbb{R})ax = b$, that is, that $P(a, b)$, under the assumption that $a \neq 0 \vee (a = 0 \wedge b = 0)$. Therefore

$$\big(a \neq 0 \vee (a = 0 \wedge b = 0)\big) \Longrightarrow P(a, b)\,. \tag{22}$$

Since we have proved (21) and (22), (20) follows.

Solution of (2): The sentence says "for every real number there is a real number that multiplied by a yields the number". The variables are a, b and x. The variable a is free, and b and x are bound. The sentence is *not* a proposition. Let us call our sentence $P(a)$, reflecting the fact that a is the free variable. Then $P(a)$ *is true for a real numbers* a *if and only if* $a \neq 0$. That is:

$$P(a) \Longleftrightarrow a \neq 0\,. \tag{23}$$

Reason: Assume $P(a)$. Then for every b there exists x auch that $ax = b$. Choose b to be any real number other than 0. (For example, $b = 47$ will do.) Then $(\exists x \in \mathbb{R})ax = b$. Pick a witness w. Then $aw = b$. And then $a \neq 0$, because if a was 0 then aw would be 0, but we have picked b such that $b \neq 0$. So we have proved that $a \neq 0$ assuming $P(a)$, and this proves that

$$P(a) \Longrightarrow a \neq 0\,. \tag{24}$$

Now assume that $a \neq 0$. Then we can pick $w = \frac{b}{a}$, so that $aw = b$. So w is a witness for $(\exists x \in \mathbb{R})ax = b$. Hence we have proved $(\exists x \in \mathbb{R})ax = b$, i.e., $P(a)$, assuming $a \neq 0$. This proves

$$a \neq 0 \Longrightarrow P(a)\,. \tag{25}$$

Since we have proved (24) and (25), (23) follows.

Solution of (3): The sentence says "every real number can be multiplied by some real number to get b". The variables are a, b and x. The variable b is free, and a and x are bound. The sentence is *not* a proposition. Let us call our sentence $P(b)$, reflecting the fact that b is the free variable. Then $P(b)$ *is true for a real number* b *if and only if* $b = 0$. That is:

$$P(b) \Longleftrightarrow b = 0\,. \tag{26}$$

Reason: Assume $P(b)$. Then for every a there exists x auch that $ax = b$. Choose $a = 0$. Then $(\exists x \in \mathbb{R})0 \times x = b$. Pick a witness w. Then $0 \times w = b$. And then $b = 0$, because $0 \times w = 0$. So we have proved that $b = 0$ assuming $P(b)$, and this proves that

$$P(b) \Longrightarrow b = 0\,. \tag{27}$$

Now assume that $b = 0$. Then for every a we can pick w to be 0, and $aw = b$ will be true, because $b = 0$. So for every a the sentence $(\exists x \in \mathbb{R})ax = b$ is true. Hence we have proved $(\forall a \in \mathbb{R})(\exists x \in \mathbb{R})ax = b$, i.e., $P(b)$, assuming $b = 0$. This proves

$$b = 0 \Longrightarrow P(b)\,. \tag{28}$$

Since we have proved (27) and (28), (26) follows.

Solution of (4): This sentence says "for any two given real numbers there exista a real number that multiplied by the first given number yields the second given number". The variables are a, b and x. They are all bound. The sentence *is* a proposition. And it is ***false***. *Reason:* Pick $a = 0$ and $b = 1$. Then obviously there is no $x \in \mathbb{R}$ such that $ax = b$.

Solution of (5): This sentence says "for any two given real numbers, if the first real number is nonzero, then there exista a real number that multiplied by the first given number yields the second given number". The variables are a, b and x. They are all bound. The sentence *is* a proposition. And it is ***true***. *Reason:* Let a, b be arbitrary real numbers. Assume $a \neq 0$. Then we can let $w = \frac{b}{a}$, and then $aw = b$, so $(\exists x \in \mathbb{R})ax = b$ is true.

Solution of (6): This sentence says "if for every pair of real numbers the first one is always nonzero, then there exists a real number that multiplied by a yields b". The variables are a, b and x. The variable x is bound. The variable a occurs three times; the first two occurrences are bound and the third one is free. The variable b occurs twice; the first occurrence is bound and the second one is free. The sentence is *not* a proposition. The sentence is true for all values of a and b, because for each a, b it is an implication with a false premise.