THE COMPLEXITY OF THE QUASI-ISOMETRY RELATION

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Abstract. We prove that the quasi-isometry relation on the Polish space $\mathcal{G}$ of finitely generated groups does not Borel reduce to an essentially free countable Borel equivalence relation.

1. Introduction

Let $\mathcal{G}$ be the Polish space of (marked) finitely generated groups. (For example, see Grigorchuk [6, 7] or Champetier [2].) Suppose that G, $H \in \mathcal{G}$ have finite generating sets $S \subseteq G \smallsetminus 1$, $T \subseteq H \smallsetminus 1$ respectively. Let $\operatorname{Cay}(G,S)$, $\operatorname{Cay}(H,T)$ be the corresponding Cayley graphs and let d_S, d_T be the corresponding path metrics. Then G, H are *quasi-isometric*, written $G \approx_{QI} H$, if there exists a map $\varphi : G \to H$, together with constants $\lambda \geq 1$ and $C \geq 0$, such that the following conditions are satisfied:

- For all x, $y \in G$, $\frac{1}{\lambda} d_S(x,y) - C \leq d_T(\varphi(x), \varphi(y)) \leq \lambda d_S(x,y) + C$.
- For all $z \in H$, $d_T(z, \varphi[G]) \leq C$.

(This definition does not depend upon the choices of the finite generating sets $S \subseteq G \smallsetminus 1$, $T \subseteq H \smallsetminus 1$.) By Whyte [28], if G, $H \in \mathcal{G}$ are nonamenable, then $G \approx_{QI} H$ if and only G, H are *Lipschitz equivalent*; i.e. there exists a bijection $\varphi : G \to H$, together with a constant $\lambda \geq 1$, such that for all x, $y \in G$,

$$\frac{1}{\lambda} d_S(x,y) \leq d_T(\varphi(x), \varphi(y)) \leq \lambda d_S(x,y).$$

It is well known that if G, $H \in \mathcal{G}$ are virtually isomorphic, then $G \approx_{QI} H$. (For example, see de la Harpe [9, Section IV.B].) Here, two groups G_1, $G_2 \in \mathcal{G}$ are said to be *virtually isomorphic*, written $G_1 \approx_{VI} G_2$, if there exist subgroups $N_i \leqslant H_i \leqslant G_i$ such that:

- $[G_1 : H_1]$, $[G_2 : H_2] < \infty$;
- N_1, N_2 are finite normal subgroups of H_1, H_2 respectively; and
- $H_1/N_1 \cong H_2/N_2$.

Thus, letting $\cong_{\mathcal{G}}$ denote the isomorphism relation on $\mathcal{G}$, we have that

$$\cong_{\mathcal{G}} \subseteq \approx_{VI} \subseteq \approx_{QI} .$$

It is natural to conjecture that these equivalence relations form a strictly increasing chain with respect to Borel reducibility. In Thomas [19], it was shown that $\approx_{VI}$ is strictly more complex than $\cong_{\mathcal{G}}$ with respect to Borel reducibility; but, until recently, it was only known that $\approx_{QI}$ was nonsmooth. (See Thomas [20].) In this paper, utilizing an approach developed by Tucker-Drob [27], we will prove the following significantly stronger result. However, it should be emphasized that it remains an open question whether $\cong_{\mathcal{G}}$ Borel reduces to $\approx_{QI}$.

Theorem 1.1. *The quasi-isometry relation $\approx_{QI}$ on the Polish space $\mathcal{G}$ of finitely generated groups does not Borel reduce to an essentially free countable Borel equivalence relation.*

This paper is organized as follows. In Section 2, we will recall some basic notions from the theory of countable Borel equivalence relations and ergodic theory; and we will discuss some of the remaining open questions. In Section 3, we will discuss a recursion/group theoretic embedding theorem of Harvey Friedman, which will play a key role in the proof of Theorem 1.1. In Section 4, following Tucker-Drob [27], we will discuss some obstructions to quasi-isometries between finitely generated groups. Finally, in Section 5, we will present the proof of Theorem 1.1.

2. Borel equivalence relations

Suppose that E, F are Borel equivalence relations on the Polish spaces X, Y respectively. Then a Borel map $f : X \to Y$ is said to be a *homomorphism* from E to F if for all $x, y \in X$,

$$x \, E \, y \quad \Longrightarrow \quad f(x) \, F \, f(y).$$

If f satisfies the stronger property that for all $x, y \in X$,

$$x \, E \, y \quad \Longleftrightarrow \quad f(x) \, F \, f(y),$$

then f is said to be a *Borel reduction* and we write $E \leq_B F$. The Borel equivalence relations E, F are said to be *Borel bireducible*, written $E \sim_B F$, if $E \leq_B F$ and $F \leq_B E$. Finally we write $E <_B F$ if both $E \leq_B F$ and $F \nleq_B E$. Clearly a Borel homomorphism is a reduction if and only if the induced map $\bar{f} : X/E \to Y/F$ is an injection. A Borel homomorphism $f : X \to Y$ is a *weak Borel reduction* if the induced map $\bar{f} : X/E \to Y/F$ is countable-to-one, in which case we write $E \leq_B^w F$.

A Borel equivalence relation is *countable* if every E-class is countable. A countable Borel equivalence relation E is *universal* if $F \leq_B E$ for every countable Borel equivalence relation F and is *weakly universal* if $F \leq_B^w E$ for every countable Borel equivalence relation F. For example, by Thomas-Velickovic [26], the isomorphism relation $\cong_{\mathcal{G}}$ on the Polish space $\mathcal{G}$ of finitely generated groups is countable universal; and, by Thomas [22, Corollary 4.9], the Turing equivalence relation $\equiv_T$ is countable weakly universal. It is currently not known whether every countable weakly universal Borel equivalence relation is countable universal. However, Dougherty-Kechris [3] have shown that if Martin's Conjecture [11] on $\equiv_T$-invariant Borel maps holds, then $\equiv_T$ is not universal.

Let E be a countable Borel equivalence relation on the Polish space X. Then, by Feldman-Moore [5], there exists a countable group G and a Borel action $G \curvearrowright X$ such that $E = E_G^X$ is the corresponding orbit equivalence. We say that E is *free* if there exists a countable group G with a free Borel action $G \curvearrowright X$ such that $E_G^X = E$; and we say E is *essentially free* if there exists a free countable Borel equivalence relation F such that $E \leq_B F$. By Thomas [22, Corollary 4.8], if E is essentially free, then E is not weakly universal.

Suppose that E is a countable Borel equivalence relation on a Polish space and that μ is a probability measure on X. Then μ is said to be E*-invariant* if μ is G-invariant for some (equivalently every) Borel action $G \curvearrowright X$ of a countable group such that $E_G^X = E$. For example, let G be a countably infinite group, let $G \curvearrowright 2^G$ be the shift action, and let E_G be the corresponding orbit equivalence relation. Then the uniform product probability measure μ is E_G-invariant.

Definition 2.1. Suppose that E is a countable Borel equivalence relation on the Polish space X and that μ is an E-invariant probability. If F is a Borel equivalence relation on the Polish space Y, then E is said to be F*-ergodic* if for every Borel homomorphism $f : X \to Y$ from E to F, there exists a Borel subset $M \subseteq X$ with $\mu(M) = 1$ such that f maps M into a single F-class.

The following result, which is a straightforward consequence of the Popa Superrigidity Theorem [17], will play an important role in the proof of Theorem 1.1.

Theorem 2.2 (Thomas [22]). *Suppose that H is a countable group and that $H \curvearrowright Y$ is a free Borel action of H on a Polish space Y. Then there exists a finitely generated group G such that E_G is E_H^Y-ergodic.*

By Thomas [19], we have that $\cong_{\mathcal{G}} <_B \approx_{VI}$, and it follows that $\approx_{VI}$ is not *essentially countable*; i.e. there does not exist a countable Borel equivalence relation E such that $\approx_{VI} \leq_B E$. Thus if we wish to understand the precise Borel complexity of the virtual isomorphism relation $\approx_{VI}$ and the quasi-isometry relation $\approx_{QI}$, then we must work within a strictly larger class of Borel equivalence relations than the relatively well-understood class of countable Borel equivalence relations.

Definition 2.3. An equivalence relation E on a Polish space X is $\mathbf{K}_\sigma$ if E is the union of countably many compact subsets of $X \times X$.

It is easily checked that $\cong_{\mathcal{G}}$, $\approx_{VI}$ and $\approx_{QI}$ are $\mathbf{K}_\sigma$ equivalence relations on the Polish space $\mathcal{G}$ of finitely generated groups. By Kechris [10] and Louveau-Rosendal [13], there exists a universal $\mathbf{K}_\sigma$ equivalence relation. In fact, Rosendal [18] has shown that the relation of Lipschitz equivalence between compact metric spaces is a universal $\mathbf{K}_\sigma$ equivalence relation. This suggests the following conjecture.

Conjecture 2.4. The quasi-isometry relation $\approx_{QI}$ on $\mathcal{G}$ is a universal $\mathbf{K}_\sigma$ equivalence relation.

By Thomas [21], $\approx_{VI}$ is not universal $\mathbf{K}_\sigma$; and thus Conjecture 2.4 implies that

$$\cong_{\mathcal{G}} <_B \approx_{VI} <_B \approx_{QI} .$$

We will close this section by mentioning two Borel non-selection conjectures. Note that since $\cong_{\mathcal{G}} <_B \approx_{VI}$, it follows that there does not exist a Borel map which selects an isomorphism class within each virtual isomorphism class. In other words, there does not exist a Borel map $\sigma : \mathcal{G} \to \mathcal{G}$ such that for all G, $H \in \mathcal{G}$,

(i) $\sigma(G) \approx_{VI} G$; and
(ii) if $G \approx_{VI} H$, then $\sigma(G) \cong_{\mathcal{G}} \sigma(H)$.

For clearly such a map σ would be a Borel reduction from $\approx_{VI}$ to $\cong_{\mathcal{G}}$. Similarly, the following would be a consequence of Conjecture 2.4.

Conjecture 2.5. There does not exist a Borel selection of a virtual isomorphism class within each quasi-isometry class of finitely generated groups.

Conjecture 2.5 seems to be very difficult; but perhaps the following consequence of Conjecture 2.5 might be more approachable.

Conjecture 2.6. There does not exist a Borel selection of an isomorphism class within each quasi-isometry class of finitely generated groups.

It should be pointed out that it is not necessary to show that $\cong_{\mathcal{G}} <_B \approx_{QI}$ in order to prove Conjecture 2.6. For example, let $\approx_C$ be the (abstract) commensurability relation on $\mathcal{G}$; i.e. if G_1, $G_2 \in \mathcal{G}$, then $G_1 \approx_C G_2$ if there exist subgroups $H_i \leqslant G_i$ of finite index such that $H_1 \cong H_2$. Then, by Thomas [24, Theorem 1.1], $\approx_C$ and $\cong_{\mathcal{G}}$ are Borel bireducible. However, by Thomas [24, Theorem 1.3], there does not exist a Borel selection of an isomorphism class within each commensurability class of finitely generated groups.

3. The Friedman embedding theorem

In this section, we will discuss a recursion/group theoretic embedding theorem of Harvey Friedman, which will play a key role in the proof of Theorem 1.1. For each $k \geq 1$, fix an effective enumeration $\{ w_n(x_1, \cdots, x_k) \mid n \in \mathbb{N} \}$ of the free group $\mathbb{F}_k$ on k generators.

Definition 3.1. If G is a group with finite generating set $s_1, \cdots, s_k$, then

$$\mathrm{Rel}_G = \{ n \in \mathbb{N} \mid w_n(s_1, \cdots, s_k) = 1 \}.$$

Thus Rel_G codes the word problem for the finitely generated group G. Of course, there is a slight abuse of notation here, since the set Rel_G depends on the choice of the generating set $s_1, \cdots, s_k$. However, this is harmless, since the Turing degree of Rel_G is independent of the choice of finite generating set.

Definition 3.2. If $A \in 2^{\mathbb{N}}$, then the group of *A-recursive permutations* is

$$\mathrm{Rec}^A(\mathbb{N}) = \{ \sigma \in \mathrm{Sym}(\mathbb{N}) \mid \sigma \leq_T A \}.$$

Here "$\sigma \leq_T A$" means that there is an algorithm which computes σ modulo an oracle which correctly answers queries of the form "is $n \in A$?" The following result is well known. (For example, see Morozov-Schupp [14] or Thomas [25].) Recall that a subgroup $G \leqslant \mathrm{Sym}(\Omega)$ is said to be *regular* if G acts freely and transitively on Ω.

Lemma 3.3. *Let G be an infinite finitely generated group with finite generating set $s_1, \cdots, s_k$. Then there exists an* Rel_G*-computable list*

$$1 = w_{i_0}(s_1, \cdots, s_k), w_{i_1}(s_1, \cdots, s_k), \cdots, w_{i_n}(s_1, \cdots, s_k), \cdots$$

of the distinct *elements of G; and for each $n \in \mathbb{N}$, the map $g_n : \mathbb{N} \to \mathbb{N}$, defined by*

$$w_{i_n}(s_1, \cdots, s_k) w_{i_m}(s_1, \cdots, s_k) = w_{i_{g_n(m)}}(s_1, \cdots, s_k)$$

is Rel_G*-computable. Consequently, G is isomorphic to regular subgroup of* $\mathrm{Rec}^{\mathrm{Rel}_G}(\mathbb{N})$.

Remark 3.4. For later use, note that the above lemma yields an isomorphic copy of G of the form $\langle \mathbb{N}, *, 0 \rangle$ such that for each $n \in \mathbb{N}$, the left multiplication map $m \mapsto n * m$ is Rel_G-computable.

Warning 3.5. It is not the case that a finitely generated group with an A-computable presentation necessarily embeds into $\mathrm{Rec}^A(\mathbb{N})$. In fact, by Morozov-Schupp [14, Corollary 1.2], a recursively presentable finitely generated group G is embeddable into the group $\mathrm{Rec}^{\emptyset}(\mathbb{N})$ of recursive permutations if and only if Rel_G is recursive.

The following recursion/group theoretic embedding result is due to Harvey Friedman. (A proof can be found in Thomas [25].)

Theorem 3.6 (The Friedman Embedding Theorem)**.** *There exists a Borel map $A \mapsto F_A$ from $2^{\mathbb{N}}$ to $\mathcal{G}$ with the following properties:*

(i) *If $A \equiv_T B$, then $F_A \cong F_B$.*
(ii) *F_A is a 2-generator group satisfying* $\mathrm{Rec}^A(\mathbb{N}) \leqslant F_A$.

Remark 3.7. Applying Lemma 3.3 and Clause 3.6(ii), we see that if $H \in \mathcal{G}$ satisfies $\mathrm{Rel}_H \leq_T A$, then $H \hookrightarrow F_A$. In particular, for all $A \in 2^{\mathbb{N}}$, the free group on two generators embeds into F_A and so F_A is nonamenable.

It is well known that for every $A \in 2^{\mathbb{N}}$, there exists a finitely generated group G such that $\mathrm{Rel}_G \equiv_T A$; and it follows that the map $A \mapsto F_A$ is a weak Borel reduction from $\equiv_T$ to $\cong_{\mathcal{G}}$.

Conjecture 3.8. The map $A \mapsto F_A$ is a weak Borel reduction from $\equiv_T$ to $\approx_{QI}$.

Suppose that Conjecture 3.8 holds. Then, since $\equiv_T$ is a weakly universal countable Borel equivalence relation, it follows that $\approx_{QI}$ does not Borel reduce to any countable Borel equivalence relation which is *not* weakly universal. Since there exist examples of countable Borel equivalence relations which are not weakly universal and also not essentially free, this would be a strict strengthening of Theorem 1.1. (Examples of such countable Borel equivalence relations can be found in Thomas [22, Section 3].)

4. High girth graphs and the graphical small cancellation condition

In this section, following Tucker-Drob [27], we will discuss some obstructions to quasi-isometries between finitely generated groups. Since the groups involved in the proof of Theorem 1.1 are nonamenable, it will be enough to find obstructions to Lipschitz equivalences between the Cayley graphs of finitely generated groups.

Definition 4.1. Let Δ, Γ be connected graphs with path metrics d_Δ, d_Γ and let $j \geq 1$. Then an injective map $\varphi : \Delta \to \Gamma$ is a *j-Lipschitz embedding* if for all x, $y \in \Delta$,

$$d_\Delta(\varphi(x), \varphi(y)) \leq j d_\Delta(x, y).$$

Example 4.2. Suppose that G, $H \in \mathcal{G}$ have finite generating sets $S \subseteq G \smallsetminus 1$, $T \subseteq H \smallsetminus 1$ and that $\varphi : H \to G$ is an embedding. Let $j = \max\{\, d_S(1, \varphi(t)) \mid t \in T \,\}$. Then φ is a j-Lipschitz embedding of $\mathrm{Cay}(H, T)$ into $\mathrm{Cay}(G, S)$.

The following result is a special case of Tucker-Drob [27, Lemma 3.2].

Lemma 4.3. *Let Γ be a 4-regular graph with the property that $\mathrm{Aut}(\Gamma)$ acts transitively on Γ. Then for each $j \geq 1$, there exists a constant $a(j)$ such that for all $s \geq 1$, there are at most $a(j)^s$ isomorphism classes of connected graphs on s vertices which admit an j-Lipschitz embedding into Γ.*

The following result is an easy consequence of Linial-Simkin [12, Theorem 1.2], as formulated by Tucker-Drob [27, Section 3].

Lemma 4.4. *Fix some real number $c \in (0,1)$. Then there exist constants $b > 0$ and $d \in \mathbb{N}$ such that for every even $s \geq d$, there are at least $(bs^{1/2})^s$ isomorphism classes of connected 3-regular graphs on s vertices with girth at least $c \log_2 s$.*

Applying Lemma 4.3 and Lemma 4.4, we see that whenever Γ is a Cayley graph of a 2-generator group and $j \geq 1$ is fixed, then if s is sufficiently large, most connected 3-regular graphs on s vertices with girth at least $c \log_2 s$ do not j-Lipschitz embed into Γ. We will next discuss when a sequence of disjoint copies of such graphs, with $s \to \infty$, embeds into a Cayley graph of a finitely generated group.

Let $(\Theta_n)_{n=1}^{\infty}$ be a sequence of 3-regular connected finite graphs. For each $n \geq 1$, let $E(\Theta_n)$ be the set of edges of Θ_n, let $g_n = \mathrm{girth}(\Theta_n)$ and let $\gamma_n = \lfloor \frac{1}{6} g_n \rfloor$. Suppose that the following conditions are satisfied:

(4.5)(i) $1 < \gamma_n < \gamma_{n+1}$ for every $n \geq 1$; and hence $g_n \to \infty$ as $n \to \infty$.

(4.5)(ii) There exists a constant $k \geq 2$ such that $|E(\Theta_n)| \leq k^{g_n}$ for all $n \geq 1$.

Let $S = S_0 \sqcup S_0^{-1}$ be a finite alphabet with a fixed-point-free involution $s \mapsto s^{-1}$. A *labelling* ℓ of the disjoint union $\Theta = \sqcup_{n=1}^{\infty} \Theta_n$ by S is an assignment of a letter $\ell(e) \in S$ to each directed edge e of Θ so that the reverse of each directed edge e is assigned the corresponding inverse label $\ell(e)^{-1}$. If $P = e_1 \cdots e_k$ is a directed path in Θ, then the corresponding *word read along* P is $\ell(P) = \ell(e_1) \cdots \ell(e_k)$. The labelling ℓ is *reduced* if no two distinct directed edges with the same initial vertex are assigned the same letter. A reduced labelling satisfies the $C'(1/6)$ *graphical small cancellation condition* if for every n, no word of length at least $\frac{1}{6}\,\mathrm{girth}(\Theta_n)$ read along a directed non-backtracking path in Θ_n occurs as the word read along any other directed non-backtracking path in Θ.

The following result was extracted by Tucker-Drob [27, Section 4] from the results of Osajda [16] and Esperet-Giocanti [4]. (I have added the observation that the labelling can be computed from Θ.)

Theorem 4.6. *With the above hypotheses, the graph $\Theta = \sqcup_{n=1}^{\infty} \Theta_n$ admits a Θ-recursive reduced finite alphabet labelling ℓ satisfying the $C'(1/6)$ graphical small cancellation condition.*

Let ℓ be a reduced labelling by the finite alphabet $S = S_0 \sqcup S_0^{-1}$ of the graph $\Theta = \sqcup_{n=1}^{\infty} \Theta_n$ which satisfies the $C'(1/6)$ graphical small cancellation condition. A *rooted directed simple cycle* in Θ is a simple directed cycle with a choice of the initial vertex. Define the associated presentation by

$$G(\Theta, \ell) = \langle\, S_0 \mid \ell(C), C \text{ is a rooted directed simple cycle in } \Theta \,\rangle.$$

The following result was first proved by Ollivier [15]. We will use a formulation that was given by Gruber [8, Theorem 5.10].

Theorem 4.7. *With the above hypotheses, for each $n \geq 1$, the graph Θ_n embeds isometrically into* $\mathrm{Cay}(G(\Theta, \ell), S)$.

Finally, by Brown-Leary [1, Theorem 2.7], the presentation

$$G(\Theta, \ell) = \langle\, S_0 \mid \ell(C), C \text{ is a rooted directed simple cycle in } \Theta \,\rangle$$

has the *Dehn property*; i.e. if w is a reduced word in the alphabet S which represents the identity element in $G(\Theta, \ell)$, then there exists a rooted directed simple cycle C in Θ such that strictly more than half of the word $\ell(C)$ occurs as a subsequence of w. Of course, this implies that the word problem for $G(\Theta, \ell)$ is computable from the sequence $(\, \Theta_n, \ell \restriction \Theta_n \,)_{n=1}^{\infty}$ of finite labelled graphs.

5. The proof of Theorem 1.1

Suppose H is a countable group and that $H \curvearrowright Y$ is a free Borel action of H on a Polish space Y. Suppose that $\psi : \mathcal{G} \to Y$ is a Borel reduction from $\approx_{QI}$ to E_H^Y. By Theorem 2.2, there exists a finitely generated group G such that the orbit equivalence relation E_G of the shift action $G \curvearrowright (\, 2^G, \mu \,)$ is E_H^Y-ergodic. Furthermore, by Remark 3.4, we can suppose that $G = \langle\, \mathbb{N}, *, 0 \,\rangle$; and that for each $g \in G$, the map $n \mapsto g * n$ is Rel_G-recursive. In particular, the shift action of G on 2^G is given by a corresponding measure-preserving action $G \curvearrowright (\, 2^{\mathbb{N}}, \mu \,)$. From now on, if $g \in G$ and $m \in \mathbb{N}$, we will usually write $g(n)$ instead of $g * n$. Let $\varphi : 2^{\mathbb{N}} \to \mathcal{G}$ be the Borel map defined by

$$B \mapsto F_{B \oplus \mathrm{Rel}_G},$$

where $B \oplus \mathrm{Rel}_G$ is the recursive join of B and Rel_G; and $F_{B \oplus \mathrm{Rel}_G}$ is the corresponding 2-generator group, given by the Friedman Embedding Theorem.

Lemma 5.1. *If $B \in 2^{\mathbb{N}}$ and $g \in G$, then $F_{B\oplus \mathrm{Rel}_G} \cong F_{g(B)\oplus \mathrm{Rel}_G}$.*

Proof. Since $g \leq_T \mathrm{Rel}_G$, it follows that $g(B) \leq_T B \oplus \mathrm{Rel}_G$ and hence

$$g(B) \oplus \mathrm{Rel}_G \leq_T B \oplus \mathrm{Rel}_G\,.$$

Similarly, $B \oplus \mathrm{Rel}_G \leq_T g(B) \oplus \mathrm{Rel}_G$ and so $B \oplus \mathrm{Rel}_G \equiv_T g(B) \oplus \mathrm{Rel}_G$. Thus $F_{B\oplus \mathrm{Rel}_G} \cong F_{g(B)\oplus \mathrm{Rel}_G}$. □

It follows that $\theta = \psi \circ \varphi$ is a Borel homomorphism from E_G to E_H^Y. Hence there exists a Borel subset $M \subseteq 2^{\mathbb{N}}$ with $\mu(M) = 1$ such that φ maps M into a single quasi-isometry class. Furthermore, since all of the groups $F_{B\oplus \mathrm{Rel}_G}$ are nonamenable, it follows that if $B, C \in M$, then $F_{B\oplus \mathrm{Rel}_G}$ and $F_{C\oplus \mathrm{Rel}_G}$ are Lipschitz equivalent.

Fix some real number $c \in (\,0, 1\,)$; and let $b > 0$ and $d \in \mathbb{N}$ be the constants given by Lemma 4.4. For each even $s \geq d$, let $\mathcal{H}_s$ be the set of isomorphism classes of connected 3-regular graphs on s vertices with girth at least $c \log_2 s$. Thus $|\mathcal{H}_s| \geq (bs^{1/2})^s$. Define

$$\gamma_{s,0} = \min\{\, \frac{1}{6}\,\mathrm{girth}(\Gamma) \mid \Gamma \in \mathcal{H}_s \,\} \quad \text{and} \quad \gamma_{s,1} = \max\{\, \frac{1}{6}\,\mathrm{girth}(\Gamma) \mid \Gamma \in \mathcal{H}_s \,\}.$$

Also, for each $j \geq 1$, let $a(j)$ be the constant given by Lemma 4.3. If we fix j, then for all sufficiently large s,

$$\frac{a(j)^s}{|\mathcal{H}_s|} \leq \frac{a(j)^s}{(bs^{1/2})^s} < \frac{1}{2^{j+1}};$$

and so, letting m be the integer such that $2^m \leq |\mathcal{H}_s| < 2^{m+1}$, we have that

$$\frac{a(j)^s}{2^m} = \frac{2a(j)^s}{2^{m+1}} < \frac{2a(j)^s}{|\mathcal{H}_s|} < \frac{1}{2^j}.$$

Hence we can inductively define a strictly increasing sequence of even s_j for $j \geq 1$ such that the following conditions are satisfied:

(5.2)(i) $1 < \gamma_{s_j,0} \leq \gamma_{s_j,1} < \gamma_{s_{j+1},0}$ for all $j \geq 1$.

(5.2)(ii) If m_j is the integer such that $2^{m_j} \leq |\mathcal{H}_{s_j}| < 2^{m_j+1}$, then $a(j)^{s_j}/2^{m_j} < 1/2^j$.

Note that $\Gamma \in \mathcal{H}_{s_j}$, then $g = \mathrm{girth}(\Gamma) \geq c \log_2 s_j$ and so $s_j \leq 2^{g/c}$. Since $g \geq 1$ and Γ is 3-regular, it follows that

$$|E(\Gamma)| = \frac{3}{2} s_j \leq 2^g 2^{g/c} = (2^{1+1/c})^g.$$

Thus letting $k = 2^{1+1/c}$, we also have that:

(5.2)(iii) $|E(\Gamma)| \leq k^{\mathrm{girth}(\Gamma)}$ for all $j \geq 1$ and $\Gamma \in \mathcal{H}_{s_j}$.

Next we can effectively form a sequence of lists

$$\Gamma_{0,j}, \dots, \Gamma_{2^{m_j}-1,j}, \qquad j \geq 1,$$

such that each $\Gamma_{i,j} \in \mathcal{H}_{s_j}$; and express $\mathbb{N}$ as a disjoint union of increasing finite intervals

$$\mathbb{N} = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_j \sqcup \cdots ,$$

where $|I_j| = m_j$. If $B \in 2^{\mathbb{N}}$, then then we can identify $B \restriction I_j$ with the binary representation of an integer $0 \leq b_j \leq 2^{m_j} - 1$; and thus $b_j = B \restriction I_j$ determines the graph $\Gamma_{b_j,j} \in \mathcal{H}_{s_j}$. It follows that the sequence of graphs $(\Gamma_{b_j,j})_{j\geq 1}$ is B-recursive. By Theorem 4.6, there exists an B-recursive reduced labelling ℓ_B by a finite alphabet S_B of the graph $\Theta_B = \bigsqcup_{j=1}^{\infty} \Gamma_{b_j,j}$ satisfying the $C'(1/6)$ graphical small cancellation condition; and it follows that the corresponding presentation of the group $G(\Theta_B, \ell_B)$ is also B-recursive. Since this presentation satisfies the Dehn property, it follows that the word problem for $G(\Theta_B, \ell_B)$ is B-recursive and hence $G(\Theta_B, \ell_B)$ embeds into $F_{B\oplus \mathrm{Rel}_G}$.

Fix some $C \in M$. Let $F = F_{C\oplus \mathrm{Rel}_G}$ and let X be a fixed 2-generator set. Then clearly $\mathrm{Cay}(F, X)$ satisfies the hypotheses of Lemma 4.3. For each $j \geq 1$, let A_j be the set of $0 \leq i \leq 2^{m_j} - 1$ such that there exists a j-Lipschitz embedding of $\Gamma_{i,j}$ into $\mathrm{Cay}(F, X)$. Then

$$\mu(\{ B \in 2^{\mathbb{N}} \mid B \restriction I_j \in A_j \}) \leq \frac{a(j)^{s_j}}{2^{m_j}} < \frac{1}{2^j}.$$

Hence, by Borel–Cantelli Lemma, for μ-a.e $B \in 2^{\mathbb{N}}$, we have that $B \restriction I_j \notin A_j$ for all but finitely many $j \geq 1$; and so this is true for μ-a.e $B \in M$. Fix such a $B \in M$. Then $G(\Theta_B, \ell_B)$ embeds into $F_{B\oplus \mathrm{Rel}_G}$ and $F_{B\oplus \mathrm{Rel}_G}$ is Lipschitz equivalent to F. It follows that there exists $k \geq 1$ such that $\mathrm{Cay}(G(\Theta_B, \ell_B), S_B)$ admits a k-Lipschitz embedding into $\mathrm{Cay}(F, X)$. Let $j \geq k$ be such that $b_j = B \restriction I_j \notin A_j$. Since $\Gamma_{b_j,j}$ embeds isometrically into $\mathrm{Cay}(G(\Theta_B, \ell_B), S_B)$, it follows that there is a k-Lipschitz embedding $\psi : \Gamma_{b_j,j} \to \mathrm{Cay}(F, X)$. But then ψ is a j-Lipschitz embedding of $\Gamma_{b_j,j}$ into $\mathrm{Cay}(F, X)$ and so $b_j = B \restriction I_j \in A_j$, which is a contradiction. This completes the proof of Theorem 1.1.

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