Cayley graphs of finitely generated groups

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Cayley graphs of finitely generated groups

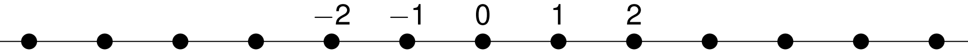
Definition

Let G be a f.g. group and let $S \subseteq G \smallsetminus \{1\}$ be a finite generating set. Then the **Cayley graph** $\mathrm{Cay}(G, S)$ is the graph with vertex set G and edge set

$$E = \{\{x, y\} \mid y = xs \text{ for some } s \in S \cup S^{-1}\}.$$

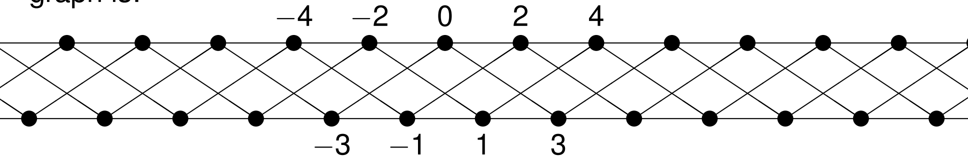
The corresponding **word metric** is denoted by d_S.

For example, when $G = \mathbb{Z}$ and $S = \{1\}$, then the corresponding Cayley graph is:



But which Cayley graph?

However, when $G = \mathbb{Z}$ and $S = \{2, 3\}$, then the corresponding Cayley graph is:



Question

Does there exist an explicit choice of generators for each f.g. group such that isomorphic groups are assigned isomorphic Cayley graphs?

The Main Result

Definition

$\mathcal{G}$ is the space of f.g. groups with underlying set $\mathbb{N}$.

Theorem (S.T.)

There does not exist a Borel map $\varphi : \mathcal{G} \to \mathrm{Fin}(\mathbb{N})$ *such that for each* $G, H \in \mathcal{G}$*:*

- $\varphi(G)$ *generates* G.
- *If* $G \cong H$, *then* $\mathrm{Cay}(G, f(G)) \cong \mathrm{Cay}(H, f(H))$.

The space of f.g. groups

- Consider $2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ equipped with the product topology.
- Identify each group

  $$A \in \mathcal{G} \longleftrightarrow m_A \in 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$$

  with the graph of its multiplication operation.
- Then $\mathcal{G}$ is a Borel subset of $2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$.
- Hence $\mathcal{G}$ is a standard Borel space; i.e. a complete separable metric space equipped with its σ-algebra of Borel subsets.

Theorem (Kuratowski)

There exists a unique uncountable standard Borel space up to isomorphism.

Borel maps

Definition

- Let X, Y be standard Borel spaces.
- Then the map $\varphi : X \to Y$ is Borel iff graph(φ) is a Borel subset of $X \times Y$.
- Equivalently, $\varphi : X \to Y$ is Borel iff $\varphi^{-1}(B)$ is a Borel set for each Borel set $B \subseteq Y$.

Church's Thesis for Real Mathematics

EXPLICIT = BOREL

A reason to believe

Question

Why is the Theorem “obviously true”?

Answer

Because the isomorphism problem for f.g. groups is much harder than that for Cayley graphs.

Borel reductions

Definition

Let E, F be equivalence relations on the standard Borel spaces X, Y.

- $E \leq_B F$ iff there exists a Borel map $\varphi : X \to Y$ such that

  $$x \, E \, y \iff \varphi(x) \, F \, \varphi(y).$$

  In this case, φ is called a Borel reduction from E to F.
- $E \sim_B F$ iff both $E \leq_B F$ and $F \leq_B E$.
- $E <_B F$ iff both $E \leq_B F$ and $E \nsim_B F$.

Smooth vs Nonsmooth

Definition

The equivalence relation E on the standard Borel space X is smooth iff there exists a Borel map $\varphi : X \to \mathbb{N}^{\mathbb{N}}$ such that

$$x \, E \, y \iff \varphi(x) = \varphi(y).$$

Example

The classification problem for countable divisible abelian groups is smooth.

Theorem (S.T.-Velickovic)

The classification problem for f.g. groups is not smooth.

Vertex transitive graphs of finite valency

Definition

Let $\mathcal{C}$ be the standard Borel space of graphs Γ with underlying set $\mathbb{N}$ which satisfy the following conditions:

- Each vertex $v \in \Gamma$ has finite degree.
- $\mathrm{Aut}(\Gamma)$ acts transitively on Γ.

Observation

$\mathcal{C}$ includes the Cayley graphs of f.g. groups.

Theorem

The isomorphism relation on $\mathcal{C}$ is smooth.

Borel homomorphisms

Definition

Let E, F be equivalence relations on the standard Borel spaces X, Y. Then the Borel map $\varphi : X \to Y$ is a Borel homomorphism iff

$$x \, E \, y \implies \varphi(x) \, F \, \varphi(y).$$

Theorem (S.T.)

There does not exist a Borel homomorphism

$$\varphi : \langle \mathcal{G}, \cong \rangle \to \langle \mathcal{C}, \cong \rangle$$

such that for all $G \in \mathcal{G}$,

$$\varphi(G) \cong \mathrm{Cay}(G, S)$$

for some finite generating set $S \subset G$.

Getting closer

Theorem (S.T.)

If $\varphi : \langle \mathcal{G}, \cong \rangle \to \langle \mathcal{C}, \cong \rangle$ is any Borel homomorphism, then there exist groups G, H $\in \mathcal{G}$ such that:

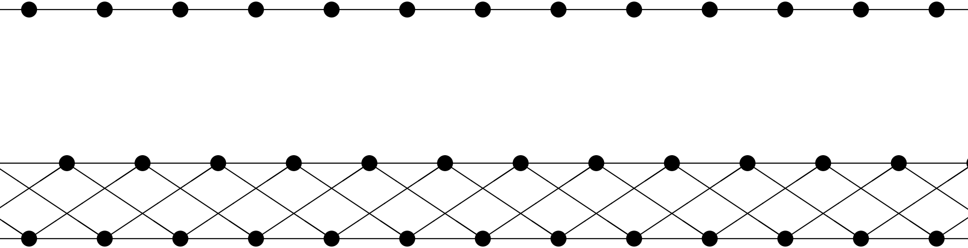
- *G and H don't have isomorphic Cayley graphs.*
- $\varphi(G) \cong \varphi(H)$.

Question

But how can we be sure that two f.g. groups do not have isomorphic Cayley graphs with respect to any finite generating sets?

The basic idea of geometric group theory

Although the Cayley graphs of a f.g. group G with respect to different generating sets S are usually nonisomorphic, they always have the same large scale geometry.



The quasi-isometry relation

Definition (Gromov)

Let G, H be f.g. groups with word metrics d_S, d_T respectively. Then G, H are said to be quasi-isometric, written $G \sim H$, iff there exist

- constants $\lambda \geq 1$ and $C \geq 0$, and
- a map $\varphi : G \to H$

such that for all $x, y \in G$,

$$\frac{1}{\lambda} d_S(x, y) - C \leq d_T(\varphi(x), \varphi(y)) \leq \lambda d_S(x, y) + C;$$

and for all $z \in H$,

$$d_T(z, \varphi[G]) \leq C.$$

When $C = 0$

Definition (Gromov)

Let G, H be f.g. groups with word metrics d_S, d_T respectively. Then G, H are said to be Lipschitz equivalent iff there exist

- a constant $\lambda \geq 1$, and
- a map $\varphi : G \to H$

such that for all $x, y \in G$,

$$\frac{1}{\lambda} d_S(x,y) \leq d_T(\varphi(x), \varphi(y)) \leq \lambda d_S(x,y);$$

and for all $z \in H$,

$$d_T(z, \varphi[G]) = 0.$$

The quasi-isometry relation

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and for all $z \in H$,

$$d_T(z, \varphi[G]) \leq C.$$

As expected

Observation

If S, T are finite generating sets for G, then

$$id : \langle G, d_S \rangle \to \langle G, d_T \rangle$$

is a quasi-isometry.

Corollary

If the f.g. groups G, H have isomorphic Cayley graphs with respect to some finite generating sets, then G, H are quasi-isometric.

Almost there

Theorem (S.T.)

If $\varphi : \langle \mathcal{G}, \cong \rangle \to \langle \mathcal{C}, \cong \rangle$ is any Borel homomorphism, then there exist groups G, $H \in \mathcal{G}$ such that:

- *G and H are not quasi-isometric.*
- *$\varphi(G) \cong \varphi(H)$.*

Remark

Of course, in order to prove this, we must actually show that there are "many such pairs".

A little ergodic theory

Definition

The measure preserving action of G on the probability space (X, ν) is ergodic iff whenever $Y \subseteq X$ is a G-invariant Borel subset, then $\nu(Y) = 0, 1$.

Remark

Equivalently, if $\psi : X \to \mathbb{N}^{\mathbb{N}}$ is a G-invariant Borel function, then there exists a Borel subset $M \subseteq X$ with $\nu(M) = 1$ such that $\psi \restriction M$ is a constant function.

Example

Let μ be the usual product probability measure on $2^{\mathbb{Z}}$ and consider the shift action of $\mathbb{Z}$ on $2^{\mathbb{Z}} = \mathcal{P}(\mathbb{Z})$. Then μ is $\mathbb{Z}$-invariant and $\mathbb{Z}$ acts ergodically on $(2^{\mathbb{Z}}, \mu)$.

Some easy consequences

Definition

Let $E_{\mathbb{Z}}$ be the orbit equivalence relation for the action of $\mathbb{Z}$ on $2^{\mathbb{Z}}$.

Lemma

If $\psi : \langle 2^{\mathbb{Z}}, E_{\mathbb{Z}} \rangle \to \langle \mathbb{N}^{\mathbb{N}}, = \rangle$ *is a Borel homomorphism, then there exists a Borel subset* $M \subseteq 2^{\mathbb{Z}}$ *with* $\mu(M) = 1$ *such that* $\psi \restriction M$ *is a constant function.*

Corollary

If $\psi : \langle 2^{\mathbb{Z}}, E_{\mathbb{Z}} \rangle \to \langle \mathcal{C}, \cong \rangle$ *is a Borel homomorphism, then there exists a Borel subset* $M \subseteq 2^{\mathbb{Z}}$ *with* $\mu(M) = 1$ *such that* ψ *maps* M *into a single* $\cong$*-class.*

The heart of the matter

Main Lemma

There exists a Borel homomorphism

$$\begin{aligned} \theta : \langle 2^{\mathbb{Z}}, E_{\mathbb{Z}} \rangle &\to \langle \mathcal{G}, \cong \rangle \\ T &\mapsto G_T \end{aligned}$$

such that the set

$$\{ \langle S, T \rangle \in 2^{\mathbb{Z}} \times 2^{\mathbb{Z}} \mid G_S, G_T \text{ are not quasi-isometric} \}$$

has $\mu \times \mu$-measure 1.

Proof of Theorem

- Suppose that $\varphi : \langle \mathcal{G}, \cong \rangle \to \langle \mathcal{C}, \cong \rangle$ is a Borel homomorphism.
- Consider the composite Borel homomorphism

$$\psi : \langle 2^{\mathbb{Z}}, E_{\mathbb{Z}} \rangle \xrightarrow{\theta} \langle \mathcal{G}, \cong \rangle \xrightarrow{\varphi} \langle \mathcal{C}, \cong \rangle.$$

- Then there exists a Borel subset $M \subseteq 2^{\mathbb{Z}}$ with $\mu(M) = 1$ such that ψ maps M into a single $\cong$-class.
- Since the set

$$\{ \langle S, T \rangle \in 2^{\mathbb{Z}} \times 2^{\mathbb{Z}} \mid G_S, G_T \text{ are not quasi-isometric } \}$$

  has $\mu \times \mu$-measure 1, there exist S, $T \in M$ such that G_S, G_T are not quasi-isometric.

A slight variant of Champetier's construction

- Let $\mathbb{F}$ be the free group on $\{a, b\}$ and let $g \in \mathrm{Aut}(\mathbb{F})$ be the automorphism defined by

$$g(a) = ab \text{ and } g(b) = ab^2.$$

- Let $n \in \mathbb{N}^+$ be sufficiently large and let

$$w = ab^3 ab^4 \cdots ab^{n+2}.$$

- Let $\theta : 2^{\mathbb{Z}} \to \mathcal{G}$ be the Borel map defined by $\theta(S) = G_S$, where

$$G_S = \langle a, b \mid g^k(w) = 1 \text{ iff } k \in S \rangle.$$

- Clearly if S, $T \in 2^{\mathbb{Z}}$ lie in the same $\mathbb{Z}$-orbit, then $G_S \cong G_T$ and so θ is a Borel homomorphism from $\langle 2^{\mathbb{Z}}, E_{\mathbb{Z}} \rangle$ to $\langle \mathcal{G}, \cong \rangle$.

Combining ideas of Champetier and Bowditch

Definition

The subsets $C, D \subseteq \mathbb{N}^+$ are related, written $C \approx D$, iff there exists $k \geq 1$ such that:

- For every $c \in C$, there exists $d \in D$ such that $c/k \leq d \leq kc$.
- For every $d \in D$, there exists $c \in C$ such that $d/k \leq c \leq kd$.

Lemma

If G_S, G_T are quasi-isometric, then

$$\{\, \text{length}(g^s(w)) \mid s \in S\} \approx \{\, \text{length}(g^t(w)) \mid t \in T\}$$

Corollary

The set $\{\langle S, T\rangle \in 2^{\mathbb{Z}} \times 2^{\mathbb{Z}} \mid G_S, G_T$ are not quasi-isometric $\}$ has $\mu \times \mu$-measure 1.

Some open questions

Theorem (Higman, Neumann, Neumann 1949)

Any countable group G can be embedded into a 2-generator group.

Conjecture

There does not exist a Borel assignment $G \mapsto G^*$ such that

- *G^* is a 2-generator group into which G embeds.*
- *If $G \cong H$, then $G^* \cong H^*$.*

Some open questions

Theorem (S.T.)

The quasi-isometry relation on the space $\mathcal{G}$ of f.g. groups is not smooth.

Conjecture

- *The problem of classifying f.g. groups up to quasi-isometry is strictly harder than that of classifying them up to isomorphism.*
- *More precisely, the quasi-isometry problem for f.g. groups is a universal $\mathbf{K}_\sigma$ equivalence relation.*