# HOMEWORK 9

**Question 1.** Let $F_2$ be the free group on two generators. Prove that if $P \subseteq F_2$ is an arbitrary subset, then there exists a finite subset $Q \subseteq P$ such that for all homomorphisms $\varphi$, $\psi : F_2 \to F_2$, if $\varphi \restriction Q = \psi \restriction Q$, then $\varphi \restriction P = \psi \restriction P$.

**Some hints:**

(1) Regard $F_2$ as the subgroup of $SL_2(\mathbb{Z})$ freely generated by

$$a = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

(2) Let $R$ be the polynomial ring $\mathbb{Z}[x_1, \cdots, x_8, y_1, \cdots, y_8]$ and define

$$M_a = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \quad M_b = \begin{pmatrix} x_5 & x_6 \\ x_7 & x_8 \end{pmatrix}$$

and

$$N_a = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \quad N_b = \begin{pmatrix} y_5 & y_6 \\ y_7 & y_8 \end{pmatrix}.$$

Then for each $w = w(a, b) \in F_2$, there are corresponding matrices $M_w$, $N_w$ obtained as suitable products of the matrices $M_a$, $M_b$, $N_a$, $N_b$.

(3) Note that if $\varphi$, $\psi : F_2 \to F_2$ are homomorphisms, then $\varphi(a)$, $\varphi(b)$ can be obtained by substituting appropriate values $\varphi(x_i) \in \mathbb{Z}$ into $M_a$, $M_b$. Then if $w = w(a, b) \in F_2$, we see that $\varphi(w)$ is the result of substituting $\varphi(x_i)$ into $M_w$. Similarly, $\psi(a)$, $\psi(b)$ can be obtained by substituting appropriate values $\psi(y_i) \in \mathbb{Z}$ into $N_a$, $N_b$, etc.

(4) Given a subset $P \subset F_2$, consider the ideal of $R$ corresponding to the set of matrix equations

$$\{M_w = N_w \mid w \in P\}.$$

**Definition 1.** If $\varphi \in \mathrm{Aut}(F_2)$, then the *fixed subgroup* is

$$\mathrm{Fix}(\varphi) = \{w \in F_2 \mid \varphi(w) = w\}$$

and the *recurrent subgroup* is

$$\mathrm{Rec}(\varphi) = \{w \in F_2 \mid \varphi^k(w) = w \text{ for some } k \geq 1\}.$$

**Question 2.** Prove that if $\varphi \in \mathrm{Aut}(F_2)$, then there exists $k \geq 1$ such that

$$\mathrm{Rec}(\varphi) = \mathrm{Fix}(\varphi^k).$$

**Hint:** Apply Question 1 to the subset

$$\mathrm{Rec}(\varphi) = \bigcup_{n \geq 1} \{w \in F_2 \mid \varphi^{n!}(w) = id(w)\}.$$