HOMEWORK 4

Question 1. Compute the Galois group over $\mathbb{Q}$ of the polynomial

$$x^6 + 22x^5 - 9x^4 + 12X^3 - 37x^2 - 29x - 15$$

(*Hint:* Reduce mod 2,3,5.)

Question 2. Let $f(x) = x^4 + ax^2 + b \in \mathbb{Q}[x]$ be an irreducible polynomial with roots $\pm\alpha$, $\pm\beta$ and splitting field K.

(a) Prove that the Galois group $G = \mathrm{Aut}_{\mathbb{Q}} K$ is isomorphic to a transitive subgroup of the Sylow 2-subgroup D_4 of $\mathrm{Sym}(4)$; and hence G is isomorphic to one of the following groups:

$$(i)\ \mathbb{Z}/4\mathbb{Z} \quad (ii)\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \quad (iii)\ D_4.$$

(b) Prove that:

- Case (i) holds if and only if
$$\frac{\alpha}{\beta} - \frac{\beta}{\alpha} \in \mathbb{Q}.$$
- Case (ii) holds if and only if $\alpha\beta \in \mathbb{Q}$.
- Otherwise, case (iii) holds.

Question 3. Suppose that K is a field of characteristic 0 and that $\sigma \in \mathrm{Aut}_K K^{alg}$. Let

$$F = \{u \in K^{alg} \mid \sigma(u) = u\}$$

be the corresponding fixed field. Prove that every finite extension of F is cyclic.