THE CLASSIFICATION PROBLEM FOR S-LOCAL TORSION-FREE ABELIAN GROUPS OF FINITE RANK

SIMON THOMAS

ABSTRACT. Suppose that $n \geq 2$ and that S, T are sets of primes. Then the classification problem for the S-local torsion-free abelian groups of rank n is Borel reducible to the classification problem for the T-local torsion-free abelian groups of rank n if and only if $S \subseteq T$.

1. INTRODUCTION

This paper is a contribution to the project of determining the complexity of the classification problem for the torsion-free abelian groups of finite rank. Recall that, up to isomorphism, the torsion-free abelian groups of rank n are exactly the additive subgroups of the n-dimensional vector space $\mathbb{Q}^n$ which contain n linearly independent elements. Thus the classification problem for the torsion-free abelian groups of rank n can be naturally identified with the corresponding problem for

$$R(\mathbb{Q}^n) = \{A \leqslant \mathbb{Q}^n \mid A \text{ contains } n \text{ linearly independent elements}\}.$$

In 1937, Baer [4] solved the classification problem for the torsion-free abelian groups of rank 1. However, despite the efforts of such mathematicians as Kurosh [19] and Malcev [21], a satisfactory classification has not been found for the torsion-free abelian groups of finite rank $n \geq 2$. Thus it was natural to ask whether the classification problem was genuinely more difficult for the groups of rank $n \geq 2$. A major breakthough occurred in 1998, when Hjorth [13] proved that the classification problem for the rank 2 groups is strictly harder than that for the rank 1 groups. A few years later, making essential use of the work of Adams-Kechris [1], Thomas [29] proved that the complexity of the classification problem increases strictly with the rank n. More recently, a number of papers have been written on the classification

2000 *Mathematics Subject Classification.* Primary 03E15, 20K15, 37A20.

Key words and phrases. Borel equivalence relation, superrigidity, torsion-free abelian group.

Research partially supported by NSF Grants 0100794 and 0600940.

problem for the p-local torsion-free abelian groups of finite rank, including Thomas [31], Hjorth-Thomas [15] and Coskey [7, 8]. (Recall that an abelian group A is said to be p-local if A is q-divisible for every prime $q \neq p$.)

In this paper, we will consider the complexity of the classification problems for the S-local torsion-free abelian groups of a fixed rank $n \geq 2$, where S is an arbitrary set of primes. Here an abelian group A is said to be *S-local* if A is p-divisible for all primes $p \notin S$. For example, a torsion-free abelian group A is $\emptyset$-local if and only if A is divisible; while, on the other hand, every abelian group is $\mathbb{P}$-local, where $\mathbb{P}$ is the set of all primes. If $S \subseteq T$, then the class of S-local torsion-free abelian groups of rank n is included in the class of T-local torsion-free groups of rank n and so the classification problem for the S-local groups is trivially reducible to that for the T-local groups. The main result of this paper states that this is the only case in which a Borel reduction exists.

Theorem 1.1. *Let $n \geq 2$. If S, $T \subseteq \mathbb{P}$ are sets of primes, then the classification problem for the S-local torsion-free abelian groups of rank n is Borel reducible to that for the T-local groups of rank n if and only if $S \subseteq T$.*

Since the proof for the case when $n = 2$ is technically much more involved and does not introduce any new ideas, we will present a complete proof for the case when $n \geq 3$ in the main body of this paper and we will sketch the proof for the case when $n = 2$ in an appendix. The currently known proof for the case when $n = 2$ is essentially an amalgam of Thomas [32] and Hjorth-Thomas [15], together with the techniques in the main body of this paper. In particular, since the proof relies on Zimmer's supperigidity theorem [33, Chapter 10] for products of real and p-adic Lie groups, it requires a familiarity with the associated machinery of induced spaces, finite ergodic extensions, etc. On the other hand, the proof for the case when $n \geq 3$ proceeds via a much more direct argument which makes use of the recent Ioana superrigidity theorem [16] for profinite actions of Kazhdan groups; and this has enabled us to write the main body of the paper so as to be intelligible to readers who are unfamiliar with the notions and techniques of superrigidity theory.[1]

[1]It is currently not known whether Ioana's theorem also holds for the more general class of groups with Property (τ). If so, then working with $PSL_2(\mathbb{Z}[1/p])$ instead of $PSL_2(\mathbb{Z})$, the case when $n = 2$ can also be handled using the methods in the main body of this paper.

This paper is organized as follows. In Section 2, we will recall some basic notions and results concerning the theory of Borel equivalence relations and ergodic theory. In Section 3, following Kurosh [19] and Malcev [21], we will relate the classification problem for the torsion-free abelian groups of rank n to the natural actions of $GL_n(\mathbb{Q})$ on the standard Borel spaces $\mathrm{Sp}(\mathbb{Q}_p^n)$ of vector subspaces of $\mathbb{Q}_p^n$; and we will state a result which captures an important aspect of the fundamental incompatibility between the actions of $GL_n(\mathbb{Q})$ on $\mathbb{Q}_p^n$ and $\mathbb{Q}_q^n$ for distinct primes $p \neq q$. In Section 4, we will discuss the superrigidity results which will be used in the proof of Theorem 1.1 for the case when $n \geq 3$. In Section 5, we will prove an ergodicity result which will allow us to focus our attention on classes of torsion-free abelian groups A with a *fixed* automorphism group $\mathrm{Aut}(A) = D \leqslant Z(GL_n(\mathbb{Q}))$. As an application of this result, we will present an alternative proof of the result that the complexity of the classification problem for the torsion-free abelian groups of rank $n \geq 2$ increases strictly with n. In Section 6, we will present the proof of Theorem 1.1 for the case when $n \geq 3$. Finally, in Appendix A, we will sketch the proofs of two superrigidity theorems from Section 4; and in Appendix B, we will sketch the proof of Theorem 1.1 for the case when $n = 2$.

Acknowledgements. I would like to thank David Arnold and Alexander Kechris for helpful discussions concerning the material in this paper.

2. Preliminaries

In this section, we will recall some basic notions and results concerning the theory of Borel equivalence relations and ergodic theory.

2.1. **Borel equivalence relations.** Suppose that $(X, \mathcal{T})$ is a Polish space; i.e., a separable completely metrizable topological space. Then the associated *standard Borel space* is $(X, \mathcal{B})$, where $\mathcal{B}$ is the σ-algebra of Borel subsets of $(X, \mathcal{T})$. As usual, we will write X instead of $(X, \mathcal{B})$. If X, Y are standard Borel spaces, then the map $f : X \to Y$ is said to be *Borel* if $\mathrm{graph}(f)$ is a Borel subset of $X \times Y$. Equivalently, $f : X \to Y$ is Borel if $f^{-1}(Z)$ is a Borel subset of X for each Borel subset $Z \subseteq Y$. The notion of a Borel map is intended to capture the intuitive idea of an *explicit* map.

An equivalence relation E on the standard Borel space X is said to be *Borel* if $E \subseteq X^2$ is a Borel subset of X^2. The Borel equivalence relation E is said to be *countable* if every E-equivalence class is countable. If E, F are Borel equivalence relations on the standard Borel spaces X, Y, then a Borel map $f : X \to Y$ is said to be a *homomorphism* from E to F if for all $x, y \in X$,

$$x \, E \, y \quad \Longrightarrow \quad f(x) \, F \, f(y).$$

If f satisfies the stronger property that for all $x, y \in X$,

$$x \, E \, y \quad \Longleftrightarrow \quad f(x) \, F \, f(y),$$

then f is said to be a *Borel reduction* and we write $E \leq_B F$. If both $E \leq_B F$ and $F \leq_B E$, then we write $E \sim_B F$ and we say that E, F are *Borel bireducible*. Finally, if both $E \leq_B F$ and $F \nleq_B E$, then we write $E <_B F$.

Most of the Borel equivalence relations that we will consider in this paper arise from group actions as follows. Let G be a countable group. Then a *standard Borel G-space* is a standard Borel space X equipped with a Borel action $(g, x) \mapsto g \cdot x$ of G on X. The corresponding G-orbit equivalence relation on X, which we will denote by E_G^X, is a countable Borel equivalence relation. Conversely, by a classical result of Feldman-Moore [10], if E is an arbitrary countable Borel equivalence relation on the standard Borel space X, then there exists a countable group G and a Borel action of G on X such that $E = E_G^X$.

Notice that $R(\mathbb{Q}^n)$ is a Borel subset of the Polish space $\mathcal{P}(\mathbb{Q}^n)$ of all subsets of $\mathbb{Q}^n$ and hence $R(\mathbb{Q}^n)$ can be regarded as a standard Borel space. (Here we are identifying $\mathcal{P}(\mathbb{Q}^n)$ with the Polish space $2^{\mathbb{Q}^n}$ of all functions $h : \mathbb{Q}^n \to \{0, 1\}$ equipped with the product topology.) Furthermore, the natural action of $GL_n(\mathbb{Q})$ on the vector space $\mathbb{Q}^n$ induces a corresponding Borel action on $R(\mathbb{Q}^n)$; and it is easily checked that if A, $B \in R(\mathbb{Q}^n)$, then $A \cong B$ if and only if there exists an element $\varphi \in GL_n(\mathbb{Q})$ such that $\varphi(A) = B$.

Definition 2.1. For each $S \subseteq \mathbb{P}$, the standard Borel space of S-local torsion-free abelian groups of rank n is defined to be

$$R^S(\mathbb{Q}^n) = \{ \, A \in R(\mathbb{Q}^n) \mid A \text{ is } S\text{-local } \}.$$

Throughout this paper, the isomorphism relations on $R(\mathbb{Q}^n)$ and $R^S(\mathbb{Q}^n)$ will be denoted by $\cong_n$ and $\cong_n^S$. If $S = \{ p \}$, then we will write $R^p(\mathbb{Q}^n)$ and $\cong_n^p$ instead of $R^{\{p\}}(\mathbb{Q}^n)$ and $\cong_n^{\{p\}}$.

A detailed development of the general theory of countable Borel equivalence relations can be found in Jackson-Kechris-Louveau [17]. Here we will just recall some of the basic theory of hyperfinite Borel equivalence relations. The countable Borel equivalence relation E on the standard Borel space X is said to be *hyperfinite* if there exists an increasing sequence

$$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots$$

of finite Borel equivalence relations on X such that $E = \bigcup_{n \in \mathbb{N}} F_n$. (Here an equivalence relation F is said to be *finite* if every F-equivalence classes is finite.) For example, the *Vitali equivalence relation*, defined on $2^{\mathbb{N}}$ by

$$x \, E_0 \, y \iff x(n) = y(n) \text{ for all but finitely many } n,$$

is hyperfinite. By Dougherty-Jackson-Kechris [9], a countable Borel equivalence relation E is hyperfinite if and only if $E \leq_B E_0$. It is interesting to note that Baer's classification of the torsion-free abelian groups of rank 1 shows that $\cong_1$ is Borel bireducible with E_0.

2.2. **Ergodic theory.** Suppose that Γ is a countable group and that X is a standard Borel Γ-space. If μ is a Γ-invariant probability measure on X, then the action of Γ on (X, μ) is said to be *ergodic* if for every Γ-invariant Borel subset $A \subseteq X$, either $\mu(A) = 0$ or $\mu(A) = 1$. The following characterization of ergodicity is well-known.

Theorem 2.2. *If μ is a Γ-invariant probability measure on the standard Borel Γ-space X, then the following statements are equivalent.*

(i) *The action of Γ on (X, μ) is ergodic.*
(ii) *If Y is a standard Borel space and $f : X \to Y$ is a Γ-invariant Borel function, then there is a Γ-invariant Borel subset $M \subseteq X$ with $\mu(M) = 1$ such that $f \restriction M$ is a constant function.*

Suppose that Γ is a countable group and that X is a standard Borel Γ-space with an invariant ergodic probability measure μ. Let $\Lambda \leqslant \Gamma$ be a subgroup such that $[\Gamma : \Lambda] < \infty$. Then a Λ-invariant Borel subset $Z \subseteq X$ is said to be an *ergodic component* for the action of Λ on X if

- $\mu(Z) > 0$; and
- Λ acts ergodically on (Z, μ_Z), where μ_Z is the probability measure defined on Z by $\mu_Z(A) = \mu(A)/\mu(Z)$.

It is easily checked that there exists a partition $Z_1 \sqcup \cdots \sqcup Z_d$ of X into finitely many ergodic components and that the collection of ergodic components is uniquely determined up to μ-null sets.

The following strengthening of the notion of ergodicity will play an important role in this paper. Suppose that Γ is a countable group and that X is a standard Borel Γ-space with an invariant probability measure μ. Let F be a Borel equivalence relation on the standard Borel space Y. Then (E_Γ^X, μ) is said to be F-*ergodic* if for every Borel homomorphism $f : X \to Y$ from E_Γ^X to F, there exists an Γ-invariant Borel subset $M \subseteq X$ with $\mu(M) = 1$ such that f maps M into a single F-class. In this case, for simplicity, we will usually say that E_Γ^X is F-ergodic. (For example, the action of Γ on (X, μ) is ergodic if and only if E_Γ^X is $\Delta(Y)$-ergodic for every standard Borel space Y, where $\Delta(Y)$ is the identity relation on Y.) The proof of Theorem 1.1 for the case when $n \geq 3$ makes use of the following strong ergodicity theorem, which is an immediate consequence of the results of Schmidt [27] and Jones-Schmidt [18]. (For more details, see Appendix A of Hjorth-Kechris [14].)

Theorem 2.3. *If Γ is a countable Kazhdan group and X is a standard Borel Γ-space with invariant ergodic probability measure μ, then E_Γ^X is E_0-ergodic.*

It is not not necessary to be familiar with the definition of a Kazhdan group in order to understand this paper. Instead, it will be enough to know that if $n \geq 3$ and Γ is either $PSL_n(\mathbb{Z})$ or $PSL_n(\mathbb{Z}[1/q])$ for some prime q, then every subgroup $\Delta \leqslant \Gamma$ of finite index is a Kazhdan group. Here $\mathbb{Z}[1/q]$ is the subring of $\mathbb{Q}$ consisting of the rational numbers of the form z/q^ℓ for some $z \in \mathbb{Z}$ and $\ell \geq 0$. (A clear account of the basic theory of Kazhdan groups can be found in Lubotsky [20].)

3. Groups acting on p-adic spaces and the Kurosh-Malcev p-adic completion technique

The proof of Theorem 1.1 is based upon a fundamental incompatibility between the actions of $GL_n(\mathbb{Q})$ on $\mathbb{Q}_p^n$ and $\mathbb{Q}_q^n$ for distinct primes $p \neq q$. In this section, we will first state a result which captures an important aspect of this incompatibility; and then, following Kurosh [19] and Malcev [21], we will relate the classification problem for the torsion-free abelian groups of rank n to the natural actions of $GL_n(\mathbb{Q})$ on the standard Borel spaces $\mathrm{Sp}(\mathbb{Q}_p^n)$ of vector subspaces of $\mathbb{Q}_p^n$. (The results in this section first appeared in the unpublished preprint [31] and have recently played a role in the papers of Coskey [7, 8].)

Fix some prime p and let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis of the n-dimensional vector space $\mathbb{Q}_p^n$ over the field of p-adic numbers. Then, by identifying $\mathbb{Q}^n$ with $\bigoplus_{i=1}^n \mathbb{Q}\,\mathbf{e}_i$, we can regard $\mathbb{Q}^n$ as an additive subgroup of $\mathbb{Q}_p^n$. With this identification, the natural action of $GL_n(\mathbb{Q})$ on $\mathbb{Q}^n$ extends to an action on $\mathbb{Q}_p^n$ and so we can also regard $GL_n(\mathbb{Q})$ as a subgroup of $GL_n(\mathbb{Q}_p)$.

For each integer $1 \leq k < n$, let $V^{(k)}(n, \mathbb{Q}_p)$ be the standard Borel space consisting of the k-dimensional vector subspaces of $\mathbb{Q}_p^n$. Then it is easily shown that the compact group $SL_n(\mathbb{Z}_p)$ acts transitively on $V^{(k)}(n, \mathbb{Q}_p)$. (For example, see Thomas [30, Lemma 6.1].) Hence we can identify $V^{(k)}(n, \mathbb{Q}_p)$ with the coset space $SL_n(\mathbb{Z}_p)/L$, where L is a suitably chosen closed subgroup of $SL_n(\mathbb{Z}_p)$. For the remainder of this paper, μ_p will denote the corresponding Haar probability measure on $V^{(k)}(n, \mathbb{Q}_p)$. Since $SL_n(\mathbb{Z})$ is a dense subgroup of $SL_n(\mathbb{Z}_p)$, it follows that $SL_n(\mathbb{Z})$ acts ergodically on $(\, V^{(k)}(n, \mathbb{Q}_p), \mu_p \,)$. Since $Z(GL_n(\mathbb{Q}))$ acts trivially on $V^{(k)}(n, \mathbb{Q}_p)$, the above discussion also applies to the actions of $PGL_n(\mathbb{Q})$ and $PSL_n(\mathbb{Z})$ on $V^{(k)}(n, \mathbb{Q}_p)$. We will usually write $PG(n-1, \mathbb{Q}_p)$ instead of $V^{(1)}(n, \mathbb{Q}_p)$ for the standard Borel space of 1-dimensional vector subspaces of $\mathbb{Q}_p^n$.

Theorem 3.1 (Thomas [31], Coskey [8]). *Let $n \geq 3$. Suppose that $p \neq q$ are distinct primes and that $1 \leq k < n$. Let E_1 be the orbit equivalence relation arising from the action of $PSL_n(\mathbb{Z})$ on $PG(n-1, \mathbb{Q}_p)$ and let E_2 be the orbit equivalence relation arising from the action of $PGL_n(\mathbb{Q})$ on $V^{(k)}(n, \mathbb{Q}_q)$. Then E_1 is E_2-ergodic with respect to μ_p.*

Remark 3.2. Theorem 3.1 was originally proved in Thomas [31] via a technically complex argument which involved the Margulis [22] and Zimmer [33] superrigidity theorems, together with Ratner's measure classification theorem [24]. A much simpler proof was recently discovered by Coskey [8] using the Ioana superrigidity theorem [16], together with some ideas of Furman [12]. (Coskey [8] also generalized Theorem 3.1 to deal with the actions of $PSL_m(\mathbb{Z})$ on $PG(m-1, \mathbb{Q}_p)$ and $PGL_n(\mathbb{Q})$ on $V^{(k)}(n, \mathbb{Q}_q)$ for arbitrary m, $n \geq 3$.)

In the remainder of this section, following Kurosh [19] and Malcev [21], we will relate the classification problem for the torsion-free abelian groups of rank n to the natural actions of $GL_n(\mathbb{Q})$ on the standard Borel spaces $\mathrm{Sp}(\mathbb{Q}_p^n)$ of vector subspaces of $\mathbb{Q}_p^n$.

Definition 3.3. If $p \in \mathbb{P}$ and $A \in R(\mathbb{Q}^n)$, then the *p-adic completion* of A is defined to be $\widehat{A}_p = \mathbb{Z}_p \otimes A$.

We will regard each $\widehat{A}_p$ as an additive subgroup of $\mathbb{Q}_p^n$ in the usual way; i.e. $\widehat{A}_p$ is the subgroup consisting of all finite sums

$$\gamma_1 a_1 + \gamma_2 a_2 + \cdots + \gamma_t a_t,$$

where $\gamma_i \in \mathbb{Z}_p$ and $a_i \in A$ for $1 \leq i \leq t$. By Fuchs [11, Lemma 93.3], there exist integers $0 \leq k, \ell \leq n$ with $k + \ell = n$ and elements v_i, $w_j \in \widehat{A}_p$ such that

$$\widehat{A}_p = \bigoplus_{i=1}^{k} \mathbb{Q}_p v_i \oplus \bigoplus_{j=1}^{\ell} \mathbb{Z}_p w_j.$$

Definition 3.4. For each $p \in \mathbb{P}$ and $A \in R(\mathbb{Q}^n)$, let $V_p^A = \bigoplus_{i=1}^{k} \mathbb{Q}_p v_i$.

Suppose that $A \in R(\mathbb{Q}^n)$ and that $\dim V_p^A = k$. If $A \cong B$, then there exists $\pi \in GL_n(\mathbb{Q})$ such that $\pi(A) = B$. Regarding $GL_n(\mathbb{Q})$ as a subgroup of $GL_n(\mathbb{Q}_p)$, it follows that $\pi(\widehat{A}_p) = \widehat{B}_p$ and hence $\pi(V_p^A) = V_p^B$. Thus the $GL_n(\mathbb{Q})$-orbit of the subspace $V_p^A \in V^{(k)}(n, \mathbb{Q}_q)$ is an isomorphism invariant of A. (In fact, this is one of the much maligned Kurosh-Malcev invariants.) If $A \in R^p(\mathbb{Q}^n)$ is a p-local group, then this invariant comes close to determining A up to isomorphism. In order to explain this more precisely, it is necessary to introduce the quasi-equality and quasi-isomorphism relations on $R(\mathbb{Q}^n)$. (These notions will also play an important role in Section 5.)

Definition 3.5. Suppose that A, $B \in R(\mathbb{Q}^n)$.

(i) A and B are *quasi-equal*, written $A \approx B$, if $A \cap B$ has finite index in both A and B.

(ii) A and B are *quasi-isomorphic*, written $A \sim B$, if there exists $\pi \in GL_n(\mathbb{Q})$ such that $\pi(A) \approx B$.

Equivalently, A and B are quasi-isomorphic if and only if there exist subgroups $A_0 \leqslant A$, $B_0 \leqslant B$ with $[\,A : A_0\,]$, $[\,B : B_0\,] < \infty$ such that $A_0 \cong B_0$. The following result was proved in Thomas [29, Section 4].

Proposition 3.6. *If A, $B \in R^p(\mathbb{Q}^n)$, then*

(a) *$A \approx B$ if and only if $V_A = V_B$;*

(b) *$A \sim B$ if and only if there exists $\pi \in GL_n(\mathbb{Q})$ such that $\pi(V_A) = V_B$.*

The following result will play an important role in the proof of Theorem 1.1.

Theorem 3.7. *There exists a Borel map $\varphi : PG(n-1, \mathbb{Q}_p) \to R^p(\mathbb{Q}^n)$ such that for all x, $y \in PG(n-1, \mathbb{Q}_p)$,*

$$PGL_n(\mathbb{Q}) \cdot x = PGL_n(\mathbb{Q}) \cdot y \qquad \Longleftrightarrow \qquad \varphi(x) \cong \varphi(y).$$

The proof of Theorem 3.7 makes use of the following observation.

Lemma 3.8. *If A, $B \in R^p(\mathbb{Q}^n)$ and $\dim V_p^A = \dim V_p^B = n-1$, then the following are equivalent:*

(i) *$A \cong B$.*

(ii) *There exists $\pi \in GL_n(\mathbb{Q})$ such that $\pi(V_p^A) = V_p^B$.*

Proof. We have already noted that (i) implies (ii). On the other hand, if there exists $\pi \in GL_n(\mathbb{Q})$ such that $\pi(V_p^A) = V_p^B$, then A and B are quasi-isomorphic. By Exercises 32.5 and 93.1 of Fuchs [11], if $C \in R^p(\mathbb{Q}^n)$, then

$$\dim_{\mathbb{Q}_p} V_p^C = n - \dim_{\mathbb{F}_p} C/pC.$$

In particular, $\dim_{\mathbb{F}_p} A/pA = 1$ and it follows that $|A/qA| \leq q$ for every prime q. Hence, by Fuchs [11, Proposition 92.1], if C is any torsion-free abelian group which is quasi-isomorphic to A, then $A \cong C$. In particular, $A \cong B$. $\square$

Proof of Theorem 3.7. First we will define an analogous Borel map

$$\sigma : V^{(n-1)}(n, \mathbb{Q}_p) \to R^p(\mathbb{Q}^n).$$

Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis of $\mathbb{Q}_p^n$. For each $W \in V^{(n-1)}(n, \mathbb{Q}_p)$, let

$$\sigma(W) = (W \oplus \mathbb{Z}_p \mathbf{e}_{i_W}) \cap \mathbb{Q}^n,$$

where $1 \leq i_W \leq n$ is the least integer such that $\mathbf{e}_{i_W} \notin W$. Arguing as in the proof of Fuchs [11, Theorem 93.5], we easily obtain that $\sigma(W) \in R^p(\mathbb{Q}^n)$ and that

$$\mathbb{Z}_p \otimes \sigma(W) = W \oplus \mathbb{Z}_p \mathbf{e}_{i_W}$$

In other words, $V_p^{\sigma(W)} = W$. Hence, by Lemma 3.8, if W, $W' \in V^{(n-1)}(n, \mathbb{Q}_p)$, then $\sigma(W) \cong \sigma(W')$ if and only there exists $\pi \in GL_n(\mathbb{Q})$ such that $\pi(W) = W'$.

Finally, in order to obtain (for purely aesthetic reasons) a corresponding map from $V^{(1)}(n, \mathbb{Q}_p)$, let V^* be the dual space of linear functionals $f : \mathbb{Q}_p^n \to \mathbb{Q}_p$. Then we can define a natural $GL_n(\mathbb{Q})$-equivariant bijection from $V^{(1)}(n, \mathbb{Q}_p)$ to the standard Borel space of $(n-1)$-dimensional subspaces of V^* by

$$U \mapsto U^{\perp} = \{ f \in V^* \mid f(u) = 0 \text{ for all } u \in U \}.$$

The result follows easily. □

4. Some superrigidity results

In this section, after a brief discussion of the notion of a Borel cocycle, we will state a cocycle superrigidity result that, in conjunction with various "non-embeddability results", will play an essential role in the proof of Theorem 1.1 for the case when $n \geq 3$. Then the remainder of the section will be devoted to an account of the relevant non-embeddability results.

We will begin by discussing the notion of a Borel cocycle. Until further notice, we will fix a countable group G, together with a standard Borel G-space X with an invariant probability measure μ.

Definition 4.1. If H is a countable group, then a Borel function $\alpha : G \times X \to H$ is called a *cocycle* if for all g, $h \in G$ and $x \in X$,

$$\alpha(hg, x) = \alpha(h, g \cdot x)\, \alpha(g, x).$$

Cocycles typically arise in the following manner. Suppose that Y is a standard Borel H-space and that H acts *freely* on Y; i.e., that $h \cdot y \neq y$ for all $y \in Y$ and $1 \neq h \in H$. If $f : X \to Y$ is a Borel homomorphism from E_G^X to E_H^Y, then we can define a corresponding Borel cocycle $\alpha : G \times X \to H$ by

$$\alpha(g,x) = \text{ the } \textit{unique} \text{ element } h \in H \text{ such that } h \cdot f(x) = f(g \cdot x).$$

Suppose now that $b : X \to H$ is a Borel map and that $f' : X \to Y$ is defined by $f'(x) = b(x) \cdot f(x)$. Then f' is also a Borel homomorphism from E_G^X to E_H^Y and the corresponding cocycle $\beta : G \times X \to H$ satisfies

$$\beta(g,x) = b(g \cdot x)\alpha(g,x)b(x)^{-1}$$

for all $g \in G$ and $x \in X$. This observation motivates the following definition.

Definition 4.2. If H is a countable group, then the cocycles α, $\beta : G \times X \to H$ are *equivalent* if there exist a Borel function $b : X \to H$ and a G-invariant Borel subset $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that

$$\beta(g,x) = b(g \cdot x)\alpha(g,x)b(x)^{-1}$$

for all $g \in G$ and $x \in X_0$.

Cocycle superrigidity theorems state that with suitable hypotheses on G, X and H, every Borel cocycle $\alpha : G \times X \to H$ is equivalent to a group homomorphism $\varphi : G \to H$. In this case, if α is the cocycle corresponding to a Borel homomorphism $f : X \to Y$ from E_G^X to E_H^Y and $f' : X \to Y$ is the "adjusted homomorphism" corresponding to φ, then

$$\varphi(g) \cdot f'(x) = f'(g \cdot x)$$

for all $g \in G$ and $x \in X_0$; i.e., the pair $(\,\varphi, f' \restriction X_0\,)$ is a *permutation group homomorphism* from $(\,G, X_0\,)$ to $(\,H, Y\,)$.

The following result, which is an immediate consequence of the Ioana cocycle superrigidity theorem [16], will be used repeatedly in the proof of Theorem 1.1 for the case when $n \geq 3$. (For the sake of completeness, we have sketched the proof of Theorem 4.3 in Appendix A.)

Theorem 4.3. *Let $n \geq 3$. Suppose that $X = PG(n-1, \mathbb{Q}_p)$ and that Γ is either $PSL_n(\mathbb{Z})$ or $PSL_n(\mathbb{Z}[1/q])$ for some prime $q \neq p$. If H is any countable group and*

$$\alpha : \Gamma \times X \to H$$

is a Borel cocycle, then there exists a subgroup $\Delta \leqslant \Gamma$ with $[\,\Gamma : \Delta\,] < \infty$ and an ergodic component $X_0 \subseteq X$ for the action of Δ on X such that $\alpha \restriction (\,\Delta \times X_0\,)$ is equivalent to either:

(a) *an embedding $\varphi : \Delta \to H$; or else*
(b) *the trivial homomorphism $\varphi : \Delta \to H$ which takes constant value 1.*

In each of our applications of Theorem 4.3, we will be proving the incompatibility of E_Γ^X with an equivalence relation induced by a free action of some countable group H. Consequently, in order to ensure that condition 4.3(b) holds, we will need a suitable non-embeddability result. The following result will enable us to rule out the possibility of an embedding $\varphi : \Delta \to H$ when H is "too small". Throughout this paper, $\bar{\mathbb{Q}}$ denotes the algebraic closure of the field $\mathbb{Q}$ of rational numbers.

Theorem 4.4. *Let $n \geq 2$ and let $\Gamma = PSL_n(\mathbb{Z})$. Suppose that $\Delta \leq \Gamma$ with $[\,\Gamma : \Delta\,] < \infty$ and that G is an algebraic $\bar{\mathbb{Q}}$-group such that $\dim G < n^2 - 1$. Then there does not exist an embedding $\varphi : \Delta \to G(\bar{\mathbb{Q}})$.*

When $n = 2$, Theorem 4.4 follows from the fact that if G is an algebraic group with $\dim G < 3$, then G is solvable-by-finite, together with the fact that $PSL_2(\mathbb{Z})$ contains a free nonabelian subgroup. When $n \geq 3$, Theorem 4.4 is an immediate consequence of the Margulis superrigidity theorems [22]. (Once again, we have sketched the proof of Theorem 4.4 in Appendix A.)

The next two results verify that the hypotheses of Theorem 4.4 hold in two settings that will occur in the proofs of Theorem 1.1 and Theorem 5.2.

Proposition 4.5. *Let S be a $\mathbb{Q}$-subalgebra of $\mathrm{Mat}_n(\mathbb{Q})$ and let S^* be the group of units of S. If the normalizer $N = N_{GL_n(\mathbb{Q})}(S)$ is a proper subgroup of $GL_n(\mathbb{Q})$, then there exists an algebraic $\mathbb{Q}$-group G with $\dim G < n^2 - 1$ such that $N/S^* \leqslant G(\mathbb{Q})$.*

Proof. Arguing as in the proof of Thomas [29, Lemma 3.7], we see that there exist algebraic $\mathbb{Q}$-groups L, $K \leqslant GL_n$ such that $L = N(\mathbb{Q})$ and $K(\mathbb{Q}) = S^*$. By Borel

[6, Theorem 6.8], $G = L/K$ is an algebraic $\mathbb{Q}$-group and

$$N/S^* = L(\mathbb{Q})/K(\mathbb{Q}) \leqslant G(\mathbb{Q}).$$

If N is a proper subgroup of $GL_n(\mathbb{Q})$, then L is a proper algebraic $\mathbb{Q}$-subgroup of GL_n and so $\dim L < n^2$. Since $Z(GL_n(\mathbb{Q})) \leqslant S^*$, it follows that $\dim K \geq 1$ and hence $\dim G < n^2 - 1$. □

Proposition 4.6. *Let W be a k-dimensional subspace of $\mathbb{Q}_q^n$, where $1 \leq k < n$, and let H be the setwise stabilizer of W in $GL_n(\mathbb{Q})$. Then there exists an algebraic $\bar{\mathbb{Q}}$-group G with $\dim G < n^2 - 1$ such that $H/Z(GL_n(\mathbb{Q})) \leqslant G(\bar{\mathbb{Q}})$.*

Proof. Clearly the result holds if $H = Z(GL_n(\mathbb{Q}))$ and hence we can suppose that $H \neq Z(GL_n(\mathbb{Q}))$. We will consider the corresponding action of $GL_n(\mathbb{Q})$ on the exterior power $V = \bigwedge^k(\mathbb{Q}_q^n)$. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis of $\mathbb{Q}_q^n$. Let $d = \binom{n}{k}$ and let $\mathcal{B} = \{\mathbf{b}_j \mid 1 \leq j \leq d\}$ be the corresponding "standard basis" of V; i.e. $\mathcal{B}$ consists of the vectors $\mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_k}$, where $i_1 < \dots < i_k$. Finally, let $\overline{\mathbb{Q}}^d \cap V$ be the collection of vectors $\mathbf{v} \in V$ of the form

$$\mathbf{v} = a_1\mathbf{b}_1 + \dots + a_d\mathbf{b}_d,$$

where each $a_j \in \overline{\mathbb{Q}} \cap \mathbb{Q}_q$. A subspace $U \leqslant V$ is said to be a $\overline{\mathbb{Q}}$-subspace if there exists a (possibly empty) collection of vectors $\mathbf{u}_1, \dots, \mathbf{u}_t \in \overline{\mathbb{Q}}^d \cap V$ such that $U = \langle\, \mathbf{u}_1, \dots, \mathbf{u}_t \,\rangle$. Clearly if U, $U' \leqslant V$ are $\overline{\mathbb{Q}}$-subspaces, then $U \cap U'$ is also a $\overline{\mathbb{Q}}$-subspace. In particular, for each 1-dimensional subspace $\langle \mathbf{v} \rangle$ of V, there exists a unique minimal $\overline{\mathbb{Q}}$-subspace U such that $\langle \mathbf{v} \rangle \leqslant U$.

For each k-dimensional subspace $S = \langle \mathbf{s}_1, \dots, \mathbf{s}_k \rangle$ of $\mathbb{Q}_q^n$, let $\widetilde{S} = \langle \mathbf{s}_1 \wedge \dots \wedge \mathbf{s}_k \rangle$ be the corresponding 1-dimensional subspace of V. Let E be the unique minimal $\overline{\mathbb{Q}}$-subspace such that $\widetilde{W} \leqslant E$.

Claim 4.7. *E is a proper H-invariant $\overline{\mathbb{Q}}$-subspace of V.*

Proof of Claim 4.7. First note that if $h \in H$, then $\widetilde{W} = h(\widetilde{W}) \leqslant h(E)$ and so $\widetilde{W} \leqslant E \cap h(E)$. Hence, by the minimality of E, we must have that $h(E) = E$. To see that E is a proper subspace of V, let $h \in H \setminus Z(GL_n(\mathbb{Q}))$. Since $h(\widetilde{W}) = \widetilde{W}$, it follows that $\widetilde{W}$ is included in an eigenspace U for the induced action of h on V. Clearly U is a proper $\overline{\mathbb{Q}}$-subspace of V and hence the same is true of E. □

Clearly it is enough to show that the $\overline{\mathbb{Q}}$-subspace $E \leqslant V$ is not $GL_n(\mathbb{Q})$-invariant. To see this, recall that $SL_n(\mathbb{Z}_p)$ acts transitively on $V^{(k)}(n, \mathbb{Q}_q)$ and hence $SL_n(\mathbb{Z}_p)$ acts transitively on the subset $\{\widetilde{S} \mid S \in V^{(k)}(n, \mathbb{Q}_q)\} \subseteq V$. In particular, for each basis vector $\mathbf{b}_j \in \mathcal{B}$, there exists $g \in SL_n(\mathbb{Z}_p)$ such that $g(\widetilde{W}) = \langle \mathbf{b}_j \rangle$. Since $\widetilde{W} \leqslant E$ and E is a proper subspace of V, it follows that E is not $SL_n(\mathbb{Z}_p)$-invariant. Because $SL_n(\mathbb{Z})$ is a dense subgroup of $SL_n(\mathbb{Z}_p)$, it follows that E is also not $SL_n(\mathbb{Z})$-invariant. This completes the proof of Proposition 4.6. □

Our final non-embeddability result reflects the incompatibility of the actions of $GL_n(\mathbb{Q})$ on $\mathbb{Q}_p^n$ and $\mathbb{Q}_q^n$ for distinct primes $p \neq q$. For each prime q, let

$$\mathbb{Z}_{(q)} = \mathbb{Z}_q \cap \mathbb{Q} = \{a/b \in \mathbb{Q} \mid b \text{ is relatively prime to } q\}.$$

Of course, if $p \neq q$ are distinct primes, then $\mathbb{Z}[1/q] \subseteq \mathbb{Z}_{(p)}$ and $\mathbb{Z}[1/q] \nsubseteq \mathbb{Z}_{(q)}$.

Theorem 4.8. *Suppose that $n \geq 2$. Let q be a prime and let $\Gamma = PSL_n(\mathbb{Z}[1/q])$. Suppose that $\Delta \leq \Gamma$ with $[\,\Gamma : \Delta\,] < \infty$ and that $D \leqslant Z(GL_n(\mathbb{Z}_{(q)}))$. Then there does not exist an embedding $\varphi : \Delta \to GL_n(\mathbb{Z}_{(q)})/D$.*

Proof. Suppose that $\varphi : \Delta \to GL_n(\mathbb{Z}_{(q)})/D$ is an embedding. Let

$$\pi : GL_n(\mathbb{Z}_{(q)})/D \to PGL_n(\mathbb{Z}_{(q)}) = GL_n(\mathbb{Z}_{(q)})/Z(GL_n(\mathbb{Z}_{(q)}))$$

be the canonical surjective homomorphism and let $\psi = \pi \circ \varphi$. Then we claim that ψ is also an embedding. To see this, recall that by Margulis [22, Chapter VIII], if N is a nontrivial normal subgroup of Δ, then $[\,\Delta : N\,] < \infty$. In particular, if $\ker \psi \neq 1$, then $[\,\Delta : \ker \psi\,] < \infty$. But since φ embeds $\ker \psi$ into the abelian group $Z(GL_n(\mathbb{Z}_{(q)}))/D$, this means that Δ is abelian-by-finite and hence Γ is also abelian-by-finite, which is a contradiction. Similarly, after passing to a subgroup of finite index if necessary, we can suppose that $\psi(\Delta) \leqslant PSL_n(\mathbb{Z}_{(q)})$.

For each $t \geq 1$, let $K_t = \ker \theta_t$ denote the congruence subgroup of $PSL_n(\mathbb{Z}_{(q)})$ arising from the canonical surjective homomorphism

$$\theta_t : PSL_n(\mathbb{Z}_{(q)}) \to PSL_n(\mathbb{Z}_{(q)}/q^t\mathbb{Z}_{(q)}) \cong PSL_n(\mathbb{Z}/q^t\mathbb{Z}).$$

Then $PSL_n(\mathbb{Z}_{(q)})/K_1 \cong PSL_n(\mathbb{F}_q)$, where $\mathbb{F}_q$ is the finite field with q elements, and K_1/K_t is a finite q-group for each $t \geq 1$. After passing to a subgroup of finite index if necessary, we can suppose that $\psi(\Delta) \leqslant K_1$. For each $t \geq 1$, let $N_t = \ker(\theta_t \circ \psi)$.

Then Δ/N_t embeds into K_1/K_t and so Δ/N_t is a finite q-group. For later use, note that since ψ is injective and $\bigcap K_t = 1$, it follows that $\bigcap N_t = 1$.

In order to simplify the notation, for the remainder of this proof, we will assume that n is odd so that $\Gamma = SL_n(\mathbb{Z}[1/q])$. Since $[\,\Gamma : \Delta\,] < \infty$, it follows that each $[\,\Gamma : N_t\,] < \infty$ and hence there is a normal subgroup M_t of Γ such that $M_t \leqslant N_t$. By Bass-Lazard-Serre [5] and Mennicke [23], Γ has the congruence subgroup property. (By Serre [28], Γ also has the congruence subgroup property when $n = 2$.) Hence, for each $t \geq 1$, we can suppose that there exists an integer $m_t \geq 1$ with $(m_t, q) = 1$ such that $\Gamma/M_t \cong SL_n(\mathbb{Z}/m_t\mathbb{Z})$. Fix an integer $t \geq 1$ such that m_t is much larger than $[\,\Gamma : \Delta\,]$ and let $m_t = p_1^{\ell_1} \cdots p_r^{\ell_r}$ be the corresponding prime factorization. Then

$$\Gamma/M_t \cong SL_n(\mathbb{Z}/p_1^{\ell_1}\mathbb{Z}) \times \cdots \times SL_n(\mathbb{Z}/p_r^{\ell_r}\mathbb{Z}).$$

Let $\tau : \Gamma/M_t \to SL_n(\mathbb{F}_{p_1}) \times \cdots \times SL_n(\mathbb{F}_{p_r})$ be the surjective homomorphism corresponding to the canonical map

$$SL_n(\mathbb{Z}/p_1^{\ell_1}\mathbb{Z}) \times \cdots \times SL_n(\mathbb{Z}/p_r^{\ell_r}\mathbb{Z}) \to SL_n(\mathbb{F}_{p_1}) \times \cdots \times SL_n(\mathbb{F}_{p_r}).$$

Then $\ker \tau = P_1 \times \cdots \times P_r$, where each P_i is a finite p_i-group. Let $K \trianglelefteq \Gamma$ be the normal subgroup such that $K/M_t = \ker \tau$. Then

$$(K \cap \Delta)N_t/N_t \cong (K \cap \Delta)/(K \cap N_t)$$

is a homomorphic image of

$$(K \cap \Delta)/M_t \leqslant K/M_t = \ker \tau.$$

Since Δ/N_t is a q-group and $(q, p_i) = 1$ for $1 \leq i \leq r$, it follows that $K \cap \Delta \leqslant N_t$. Thus Δ/N_t is a homomorphic image of $\Delta/(K \cap \Delta)$. Notice that $\Delta/(K \cap \Delta)$ is isomorphic to a subgroup of

$$\Gamma/K \cong SL_n(\mathbb{F}_{p_1}) \times \cdots \times SL_n(\mathbb{F}_{p_r})$$

of index at most $[\,\Gamma : \Delta\,]$ and that there are only finitely many primes p such that $SL_n(\mathbb{F}_p)$ has a proper subgroup of index at most $[\,\Gamma : \Delta\,]$. Thus, after re-indexing the primes $\{p_1, \cdots, p_r\}$ if necessary, we can suppose that

$$\Delta/(K \cap \Delta) \cong SL_n(\mathbb{F}_{p_1}) \times \cdots \times SL_n(\mathbb{F}_{p_s}) \times B$$

for some $s \leq r$, where B is a group of bounded order. Since $n \geq 3$, it follows that for each $1 \leq i \leq s$, the nontrivial homomorphic images of $SL_n(\mathbb{F}_{p_i})$ have the form $SL_n(\mathbb{F}_{p_i})/L_i$ for some central subgroup L_i. (This is also true when $n = 2$ and $p_i > 3$.) Since Δ/N_t is a q-group, it follows that Δ/N_t is a homomorphic image of B. But this means that Δ/N_t has bounded order, which contradicts the fact that $\bigcap N_t = 1$. □

5. Fixing The Automorphism Group

Let G, H be countable groups with Borel actions on the standard Borel spaces X, Y and let μ be a G-invariant probability measure on X. Suppose that $f : X \to Y$ is a Borel homomorphism from E^X_G to E^H_Y. In order to be able to apply a cocycle superrigidity result in this setting, we must first be able to define a corresponding cocycle and this is not always possible. Of course, as we have noted in Section 4, it is easy to define a corresponding cocycle if H acts freely on Y. More generally, suppose that there is a fixed subgroup $K \leqslant H$ such that $K = \{ h \in H \mid h(y) = y \}$ for all $y \in Y$. Then the quotient group $N_H(K)/K$ acts freely on Y and so we can define a corresponding cocycle taking values in $N_H(K)/K$. In this section, we will prove a result which will allow to reduce our analysis to this situation in the proof of Theorem 1.1 when $n \geq 3$. (Of course, the action of $GL_n(\mathbb{Q})$ on $R(\mathbb{Q}^n)$ is not free. In fact, if $A \in R(\mathbb{Q}^n)$, then the stabilizer of A in $GL_n(\mathbb{Q})$ is precisely the automorphism group $\mathrm{Aut}(A)$ and this always contains the nontrivial automorphism $a \mapsto -a$.)

Definition 5.1. Suppose that G is a countable group and that X is a standard Borel G-space with an invariant probability measure μ. If E is a countable Borel equivalence relation on the standard Borel space Y and $f : X \to Y$ is a Borel homomorphism from E^X_G to E, then f is said to be *μ-trivial* if there exists a Borel subset $Z \subseteq X$ with $\mu(Z) = 1$ such that f maps Z into a single E-class.

Theorem 5.2. *Suppose that $n \geq 3$. Let $X = PG(n-1, \mathbb{Q}_p)$ and let $\Gamma = PSL_n(\mathbb{Z})$. If $f : X \to R(\mathbb{Q}^n)$ is a μ_p-nontrivial Borel homomorphism from E^X_Γ to $\cong_n$, then there exists a Borel subset $X_0 \subseteq X$ with $\mu_p(X_0) = 1$ and a fixed central subgroup $D \leqslant Z(GL_n(\mathbb{Q}))$ such that $\mathrm{Aut}(f(x)) = D$ for all $x \in X_0$.*

Before we begin the proof of Theorem 5.2, we will first present an alternative proof of the result that the complexity of the classification problem for the torsion-free abelian groups of rank $n \geq 2$ increases strictly with n.

Corollary 5.3 (Thomas [29]). *If $2 \leq m < n$, then $\cong_m <_B \cong_n$.*

Proof. Let $t = n - m$. Then we can define a Borel reduction $g : R(\mathbb{Q}^m) \to R(\mathbb{Q}^n)$ from $\cong_m$ to $\cong_n$ by $g(A) = A \oplus \mathbb{Q}^t$. Suppose that $\cong_n \leq_B \cong_m$. Then, applying Theorem 3.7, it follows that there exists a Borel map $h : PG(n-1, \mathbb{Q}_p) \to R(\mathbb{Q}^m)$ such that for all x, $y \in PG(n-1, \mathbb{Q}_p)$,

$$PGL_n(\mathbb{Q}) \cdot x = PGL_n(\mathbb{Q}) \cdot y \qquad \Longleftrightarrow \qquad h(x) \cong h(y).$$

Let $X = PG(n-1, \mathbb{Q}_p)$ and $\Gamma = PSL_n(\mathbb{Z})$. Then $f = g \circ h : X \to R(\mathbb{Q}^n)$ is a countable-to-one Borel homomorphism from E_Γ^X to $\cong_n$. In particular, it follows that f is μ_p-nontrivial. Hence, by Theorem 5.2, there exists a Borel subset $X_0 \subseteq X$ with $\mu_p(X_0) = 1$ and a fixed central subgroup $D \leqslant Z(GL_n(\mathbb{Q}))$ such that $\mathrm{Aut}(f(x)) = D$ for all $x \in X_0$. But this is impossible, since if $x \in X$, then $f(x) = A \oplus \mathbb{Q}^t$ for some $A \in R(\mathbb{Q}^m)$ and so $\mathrm{Aut}(f(x))$ is *not* a central subgroup of $GL_n(\mathbb{Q})$. □

In the proof of Theorem 5.2, we will consider the action of $GL_n(\mathbb{Q})$ on the set of quasi-equality classes of elements of $R(\mathbb{Q}^n)$. For each group $A \in R(\mathbb{Q}^n)$, let $[A]$ be the corresponding $\approx$-class which contains A. In order to describe the setwise stabiliser in $GL_n(\mathbb{Q})$ of a $\approx$-class $[A]$, it is necessary to introduce the notions of a quasi-endomorphism and a quasi-automorphism. If $A \in R(\mathbb{Q}^n)$, then a linear transformation $\varphi \in \mathrm{Mat}_n(\mathbb{Q})$ is said to be a *quasi-endomorphism* of A if there exists an integer $m > 0$ such that $m\varphi \in \mathrm{End}(A)$. It is easily checked that the collection $\mathrm{QE}(A)$ of quasi-endomorphisms of A is a $\mathbb{Q}$-subalgebra of $\mathrm{Mat}_n(\mathbb{Q})$ and that if $A \approx B$, then $\mathrm{QE}(A) = \mathrm{QE}(B)$. A linear transformation $\varphi \in \mathrm{Mat}_n(\mathbb{Q})$ is said to be a *quasi-automorphism* of A if φ is a unit of the $\mathbb{Q}$-algebra $\mathrm{QE}(A)$. The group of quasi-automorphisms of A is denoted by $\mathrm{QAut}(A)$.

Lemma 5.4 (Thomas [29]). *If $A \in R(\mathbb{Q}^n)$, then $\mathrm{QAut}(A)$ is the setwise stabilizer of $[A]$ in $GL_n(\mathbb{Q})$.*

The following result, which is due to Reid [26, Theorem 5.5], will play a crucial role in the proof of Theorem 5.2. (In order to obtain the precise statement of

Lemma 5.5, it is necessary to combine Reid [26, Theorem 5.5] with the proof of Reid [26, Corollary 5.8].) For the sake of completeness, we have included a proof of Lemma 5.5.

Lemma 5.5. *Suppose that $A \in R(\mathbb{Q}^n)$ is a torsion-free abelian group of rank n. If* $\mathrm{QE}(A) = \mathrm{Mat}_n(\mathbb{Q})$*, then there exists a rank 1 torsion-free abelian group C such that*

$$A \cong \underbrace{C \oplus \cdots \oplus C}_{n \text{ copies}}.$$

Proof. Let $M = \mathrm{Mat}_n(\mathbb{Q})$ and let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis of $\mathbb{Q}^n$. For each $1 \le i \le n$, let $p_i \in M$ be the idempotent element corresponding to the projection $\mathbb{Q}^n \to \mathbb{Q}\mathbf{e}_i$. By Reid [25, Theorem 3.5], the decomposition

$$M = Mp_1 \oplus Mp_2 \oplus \cdots \oplus Mp_n$$

of $M = \mathrm{QE}(A)$ into a direct sum of left ideals induces a corresponding quasi-decomposition

$$A \approx p_1(A) \oplus p_2(A) \oplus \cdots \oplus p_n(A)$$

of A into a direct sum of rank 1 torsion-free abelian groups. Furthermore, by Reid [25, Theorem 3.4], since the left ideals Mp_i are pairwise isomorphic as M-modules, it follows that the abelian groups $p_i(A)$ are pairwise quasi-isomorphic. Hence, by Arnold [2, Lemma 3.1.1], since each $p_i(A)$ has rank 1, it follows that the abelian groups $p_i(A)$ are pairwise isomorphic. Thus there exists a rank 1 torsion-free abelian group C such that

$$A \sim \underbrace{C \oplus \cdots \oplus C}_{n \text{ copies}}.$$

Finally, by Arnold [2, Corollary 3.2.7], this implies that

$$A \cong \underbrace{C \oplus \cdots \oplus C}_{n \text{ copies}}.$$

□

We are now ready to present the proof of Theorem 5.2. Let $f : X \to R(\mathbb{Q}^n)$ be a μ_p-nontrivial Borel homomorphism from E_Γ^X to $\cong_n$. For each $x \in X$, let $A_x = f(x) \in R(\mathbb{Q}^n)$. First notice that there are only countably possibilities for the $\mathbb{Q}$-algebra $\mathrm{QE}(A_x)$. Hence there exists a Borel subset $X_1 \subseteq X$ with $\mu_p(X_1) > 0$ and a fixed $\mathbb{Q}$-subalgebra S of $\mathrm{Mat}_n(\mathbb{Q})$ such that $\mathrm{QE}(A_x) = S$ for all $x \in X_1$. Since Γ acts ergodically on (X, μ_p), it follows that $\mu_p(\Gamma \cdot X_1) = 1$. In order to simplify notation, we will assume that $\Gamma \cdot X_1 = X$. After slightly adjusting f if necessary, we can suppose that $\mathrm{QE}(A_x) = S$ for all $x \in X$. In particular, for each $x \in X$, we have that $\mathrm{QAut}(A_x) = S^*$, the group of units of S. Now suppose that x, $y \in X$ and that $\gamma \cdot x = y$ for some $\gamma \in \Gamma$. Then $A_x \cong A_y$ and so there exists $\varphi \in GL_n(\mathbb{Q})$ such that $\varphi(A_x) = A_y$. Since

$$\mathrm{QE}(A_x) = S = \mathrm{QE}(A_y),$$

it follows that $\varphi \in N = N_{GL_n(\mathbb{Q})}(S)$.

Lemma 5.6. *$N = GL_n(\mathbb{Q})$.*

Proof. Suppose that N is a proper subgroup of $GL_n(\mathbb{Q})$ and consider the induced action of N on the corresponding set $\{[A_x] \mid x \in X\}$ of $\approx$-classes. By Lemma 5.4, for each $x \in X$, the setwise stabilizer of $[A_x]$ in N is $\mathrm{QAut}(A_x) = S^*$. Let $H = N/S^*$ and for each $\varphi \in N$, let $\overline{\varphi} = \varphi S^*$. Then we can define a Borel cocycle $\alpha : \Gamma \times X \to H$ by

$$\alpha(\gamma, x) = \text{ the unique element } \overline{\varphi} \in H \text{ such that } \varphi([A_x]) = [A_{\gamma \cdot x}].$$

By Theorem 4.3, there exists a subgroup $\Delta \leqslant \Gamma$ with $[\,\Gamma : \Delta\,] < \infty$ and an ergodic component $X_0 \subseteq X$ for the action of Δ on X such that $\alpha \restriction (\,\Delta \times X_0\,)$ is equivalent to either:

(a) an embedding $\varphi : \Delta \to H$; or else

(b) the trivial homomorphism $\varphi : \Delta \to H$ which takes constant value 1.

Applying Proposition 4.5 and Theorem 4.4, it follows that there does not exist an embedding $\varphi : \Delta \to H$. Thus, after deleting a null subset of X_0 if necessary, we can suppose that there exists a Borel map $b : X_0 \to N$ such that

$$b(\gamma \cdot x)\alpha(\gamma, x)b(x)^{-1} \in S^*$$

for all $x \in X_0$ and $\gamma \in \Delta$. Hence if $f' : X_0 \to R(\mathbb{Q}^n)$ is the Borel map defined by $f'(x) = b(x)f(x)$, then $[\,f'(x)\,] = [\,f'(\gamma \cdot x)\,]$ for all $x \in X_0$ and $\gamma \in \Delta$. In other words, f' is a Borel homomorphism from $E_\Delta^{X_0}$ to the quasi-equality relation $\approx_n$ on $R(\mathbb{Q}^n)$. By Theorem 2.3, since Δ is a Kazhdan group, it follows that $E_\Delta^{X_0}$

is E_0-ergodic. By Thomas [29, Theorem 3.8], the quasi-equality relation $\approx_n$ is hyperfinite and hence we can suppose that f' maps X_0 into a single quasi-equality class. It follows that f maps X_0 into a single quasi-isomorphism class. By Thomas [29, Lemma 3.2], each quasi-isomorphism class consists of only countably many isomorphism classes. Hence there exists a Borel subset $X_1 \subseteq X_0$ with $\mu_p(X_1) > 0$ such that f maps X_1 to a fixed group $A \in R(\mathbb{Q}^n)$. Finally, using the ergodicity of the action of Γ on (X, μ_p), it follows that $\Gamma \cdot X_1$ is a μ_p-measure 1 subset of X which is mapped by f into a single $\cong_n$-class, which is a contradiction. $\square$

Since $GL_n(\mathbb{Q})$ normalizes S, it follows that $GL_n(\mathbb{Q})$ also normalizes S^*. Hence, since $Z(GL_n(\mathbb{Q})) \leqslant S^*$, one of the following two cases must occur:

(I) $SL_n(\mathbb{Q}) \leqslant S^*$.

(II) $S^* = Z(GL_n(\mathbb{Q}))$.

(For example, see Artin [3, Theorem 4.9].) First suppose that $SL_n(\mathbb{Q}) \leqslant S^*$. Since $SL_n(\mathbb{Q})$ is a generating set for the $\mathbb{Q}$-algebra $\mathrm{Mat}_n(\mathbb{Q})$, it follows that $S = \mathrm{Mat}_n(\mathbb{Q})$. Hence, by Lemma 5.5, for each $x \in X$, there exists a torsion-free abelian group C_x of rank 1 such that

$$A_x \cong \underbrace{C_x \oplus \cdots \oplus C_x}_{n \text{ copies}}$$

and so $\cong_n \restriction f(X)$ is a hyperfinite equivalence relation. But, using the E_0-ergodicity of the action of Γ on (X, μ_p), this implies that there exist a μ_p-measure 1 subset of X which is mapped by f into a single $\cong_n$-class, which is a contradiction.

Hence we must have that $S^* = Z(GL_n(\mathbb{Q}))$. Note that for each $x \in X$,

$$\mathrm{Aut}(A_x) \leqslant \mathrm{QAut}(A_x) = S^*$$

and so $D_x = \mathrm{Aut}(A_x) \leqslant Z(GL_n(\mathbb{Q}))$. Furthermore, since each automorphism group D_x is central in $GL_n(\mathbb{Q})$, it follows that the map $x \mapsto D_x$ is Γ-invariant. Hence, by the ergodicity of the action of Γ on (X, μ_p), there exists a Borel subset $X_0 \subseteq X$ with $\mu_p(X_0) = 1$ and a fixed subgroup $D \leqslant Z(GL_n(\mathbb{Q}))$ such that $D_x = D$ for all $x \in X_0$. This completes the proof of Theorem 5.2.

6. The proof of Theorem 1.1 for $n \geq 3$

In this section, we will present the proof of Theorem 1.1 for the case when $n \geq 3$. Suppose that S, $T \subseteq \mathbb{P}$. If $S \subseteq T$, then $R^S(\mathbb{Q}^n) \subseteq R^T(\mathbb{Q}^n)$ and it is clear that

$(\cong_n^S) \leq_B (\cong_n^T)$. From now on, suppose that $S \nsubseteq T$ and that $(\cong_n^S) \leq_B (\cong_n^T)$. Fix some prime $p \in S \smallsetminus T$ and let $X = PG(n-1, \mathbb{Q}_p)$. Applying Theorem 3.7, since $(\cong_n^p) \leq_B (\cong_n^S) \leq_B (\cong_n^T)$, there exists a Borel reduction

$$f : X \to R^T(\mathbb{Q}^n)$$

from the orbit equivalence relation corresponding to the action of $PGL_n(\mathbb{Q})$ on $PG(n-1, \mathbb{Q}_p)$ to $\cong_n^T$. For each $x \in X$, let $A_x = f(x) \in R^T(\mathbb{Q}^n)$. Until further notice, we will work with the action of the subgroup $PSL_n(\mathbb{Z}) \leqslant PGL_n(\mathbb{Q})$ on X. Since $PSL_n(\mathbb{Z})$ acts ergodically on (X, μ_p), there exists a $PSL_n(\mathbb{Z})$-invariant Borel subset $X_0 \subseteq X$ with $\mu_p(X_0) = 1$ such that:

(a) for each prime $q \in \mathbb{P}$, there exists a fixed integer $0 \leq d_q \leq n$ such that $\dim_{\mathbb{Q}_q} V_q^{A_x} = d_q$ for all $x \in X_0$.

Furthermore, by Theorem 5.2, after slightly shrinking X_0 if necessary, we can also suppose that:

(b) there exists a fixed subgroup $D \leqslant Z(GL_n(\mathbb{Q}))$ such that $\mathrm{Aut}(A_x) = D$ for all $x \in X_0$.

Note that if $q \notin T$, then A_x is q-divisible and hence $\dim_{\mathbb{Q}_q} V_q^{A_x} = n$. In particular, it follows that $d_p = n$.

Lemma 6.1. *For each $q \in \mathbb{P}$, either $d_q = 0$ or $d_q = n$.*

Proof. Suppose that there exists a prime $q \in \mathbb{P}$ such that $1 \leq d_q = k < n$. Then $q \neq p$ and we can define a Borel homomorphism

$$g : X_0 \to V^{(k)}(n, \mathbb{Q}_q)$$
$$x \mapsto V_q^{A_x}$$

from $E_{PSL_n(\mathbb{Z})}^{X_0}$ to the orbit equivalence relation arising from the action of $PGL_n(\mathbb{Q})$ on $V^{(k)}(n, \mathbb{Q}_q)$. Applying Theorem 3.1, it follows that there exists a $PSL_n(\mathbb{Z})$-invariant Borel subset $X_1 \subseteq X_0$ with $\mu_p(X_1) = 1$ such that g maps X_1 into a single $PGL_n(\mathbb{Q})$-orbit. After slightly adjusting f if necessary, we can suppose that there exists a fixed subspace $W \in V^{(k)}(n, \mathbb{Q}_q)$ such that $g(x) = W$ for all $x \in X_1$. Suppose that x, $y \in X_1$ and that $\gamma \cdot x = y$ for some $\gamma \in PSL_n(\mathbb{Z})$. Then $A_x \cong A_y$

and so there exists $\pi \in GL_n(\mathbb{Q})$ such that $\pi(A_x) = A_y$. Clearly

$$\pi(W) = \pi(V_q^{A_x}) = V_q^{A_y} = W$$

and so π is an element of the setwise stabilizer $G_{\{W\}}$ of W in $GL_n(\mathbb{Q})$. Furthermore, if $D \leqslant Z(GL_n(\mathbb{Q}))$ is the fixed group such that $\mathrm{Aut}(A_x) = D$ for all $x \in X_0$, then $D \leqslant G_{\{W\}}$. Hence, letting $H = G_{\{W\}}/D$, we can define a Borel cocycle $\alpha : PSL_n(\mathbb{Z}) \times X_1 \to H$ by

$$\alpha(\gamma, x) = \text{ the unique element } \pi \in H \text{ such that } \pi(A_x) = A_{\gamma \cdot x}.$$

By Theorem 4.3, there exists a subgroup $\Delta \leqslant PSL_n(\mathbb{Z})$ with $[\, PSL_n(\mathbb{Z}) : \Delta \,] < \infty$ and an ergodic component $Z \subseteq X_1$ for the action of Δ on X_1 such that $\alpha \restriction (\, \Delta \times Z \,)$ is equivalent to either:

(a) an embedding $\varphi : \Delta \to H$; or else
(b) the trivial homomorphism $\varphi : \Delta \to H$ which takes constant value 1.

We claim that there does not exist an embedding $\varphi : \Delta \to H$. To see this, suppose that $\varphi : \Delta \to H$ is an embedding and let $\pi : H \to G_W/Z(GL_n(\mathbb{Q}))$ be the canonical surjective homomorphism. Applying Proposition 4.6 and Theorem 4.4, it follows that $\psi = \pi \circ \varphi : \Delta \to G_W/Z(GL_n(\mathbb{Q}))$ is not an embedding. Hence, by Margulis [22, Chapter VIII], it follows that $[\, \Delta : \ker \psi \,] < \infty$. But since φ embeds $\ker \psi$ into the abelian group $Z(GL_n(\mathbb{Q}))/D$, this means that Δ is abelian-by-finite and hence $PSL_n(\mathbb{Z})$ is also abelian-by-finite, which is a contradiction. Thus, after deleting a null subset of Z if necessary, we can suppose that there exists a Borel map $b : Z \to G_{\{W\}}$ such that

$$b(\gamma \cdot x)\alpha(\gamma, x)b(x)^{-1} \in D$$

for all $x \in Z$ and $\gamma \in \Delta$. Hence if $f' : Z \to R(\mathbb{Q}^n)$ is the Borel map defined by $f'(x) = b(x)f(x)$, then $f'(x) = f'(\gamma \cdot x)$ for all $x \in Z$ and $\gamma \in \Delta$. Using the ergodicity of Δ on Z, we can suppose that f' maps Z to a fixed group $A \in R(\mathbb{Q}^n)$. But then f maps Z to the isomorphism class containing A, which contradicts the fact that f is countable-to-one. $\square$

Since $d_q = n$ if and only if the group A_x is q-divisible, it follows that there exists at least one prime q such that $d_q = 0$. We will fix such a prime q for the remainder of this section. Of course, since $d_p = n$, it follows that $p \neq q$. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis of $\mathbb{Q}_q^n$. Recall that $\mathbb{Q}_q^n$ contains only countably many lattices and that $GL_n(\mathbb{Q})$ acts transitively on the set of these lattices. (Here a *lattice* is a $\mathbb{Z}_q$-submodule of the form $\mathbb{Z}_q\mathbf{v}_1 \oplus \cdots \oplus \mathbb{Z}_q\mathbf{v}_n$ for some basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ of $\mathbb{Q}_q^n$.) Hence, after slightly adjusting f if necessary, we can suppose that:

(c) $(\widehat{A}_x)_q = \mathbb{Z}_q\mathbf{e}_1 \oplus \cdots \oplus \mathbb{Z}_q\mathbf{e}_n$ for all $x \in X_0$.

From now on, we will let $\Gamma = PSL_n(\mathbb{Z}[1/q])$ and we will work with the action of Γ on X. After replacing X_0 with $\bigcap_{\gamma\in\Gamma}\gamma[X_0]$ if necessary, we can suppose that X_0 is $PSL_n(\mathbb{Z}[1/q])$-invariant. Suppose that $x, y \in X_0$ and that $\gamma \cdot x = y$ for some $\gamma \in \Gamma$. Then there exists $\pi \in GL_n(\mathbb{Q})$ such that $\pi(A_x) = A_y$. Clearly

$$\pi(\mathbb{Z}_q\mathbf{e}_1 \oplus \cdots \oplus \mathbb{Z}_q\mathbf{e}_n) = \pi((\widehat{A}_x)_q) = (\widehat{A}_y)_q = \mathbb{Z}_q\mathbf{e}_1 \oplus \cdots \oplus \mathbb{Z}_q\mathbf{e}_n$$

and so

$$\pi \in GL_n(\mathbb{Q}) \cap GL_n(\mathbb{Z}_q) = GL_n(\mathbb{Z}_{(q)}).$$

Furthermore, if D is the fixed group such that $\mathrm{Aut}(A_x) = D$ for all $x \in X_0$, then $D \leqslant Z(GL_n(\mathbb{Z}_{(q)}))$. Hence we can define a Borel cocycle $\alpha : \Gamma \times X_0 \to GL_n(\mathbb{Z}_{(q)})/D$ by

$$\alpha(\gamma, x) = \text{ the unique element } \pi \in GL_n(\mathbb{Z}_{(q)})/D \text{ such that } \pi(A_x) = A_{\gamma\cdot x}.$$

By Theorem 4.3, there exists a subgroup $\Delta \leqslant \Gamma$ with $[\,\Gamma : \Delta\,] < \infty$ and an ergodic component $Z \subseteq X_0$ for the action of Δ on X_0 such that $\alpha \restriction (\,\Delta \times Z\,)$ is equivalent to either:

(a) an embedding $\varphi : \Delta \to GL_n(\mathbb{Z}_{(q)})/D$; or else
(b) the trivial homomorphism $\varphi : \Delta \to GL_n(\mathbb{Z}_{(q)})/D$ which takes constant value 1.

By Theorem 4.8, there does not exist an embedding $\varphi : \Delta \to GL_n(\mathbb{Z}_{(q)})/D$. But then, arguing as in the proof of Lemma 6.1, it follows that f maps a co-null subset of Z into a single $\cong$-class, which once again contradicts the fact that f is countable-to-one. This completes the proof of Theorem 1.1 for the case when $n \geq 3$.

Appendix A. The proofs of Theorem 4.3 and Theorem 4.4

In this appendix, we will sketch the proofs of Theorem 4.3 and Theorem 4.4. Throughout we will assume that the reader is familiar with the basic ideas of superrigidity theory and our presentation will be terser than in the main body of the paper. It should be stressed that Theorem 4.3 and Theorem 4.4 are immediate consequences of the work of Ioana [16] and Margulis [22].

Sketch Proof of Theorem 4.3. Suppose that G is a countable group and that X is a standard Borel G-space with invariant probability measure μ. Then the action of G on (X, μ) is said to be *profinite* if there exists a directed system of finite G-spaces X_n with invariant probability measures μ_n such that

$$(X, \mu) = \lim_{\leftarrow} (X_n, \mu_n).$$

For example, if $X = PG(n-1, \mathbb{Q}_p)$ and Γ is either $PSL_n(\mathbb{Z})$ or $PSL_n(\mathbb{Z}[1/q])$ for some prime $q \neq p$, then the ergodic action of Γ on (X, μ_p) is profinite. Furthermore, arguing as in Thomas [30, Lemma 6.2], it follows that if

$$X^* = \{x \in X \mid \gamma \cdot x \neq x \text{ for all } 1 \neq \gamma \in \Gamma\}.$$

then $\mu_p(X^*) = 1$ and so the action of Γ on X is also essentially free. Since Γ is a Kazhdan group, it follows that all of the hypotheses of the Ioana cocycle superrigidity theorem [16] are satisfied. Hence if H is any countable group and

$$\alpha : \Gamma \times X \to H$$

is a Borel cocycle, then there exists a subgroup $\Delta \leqslant \Gamma$ with $[\, \Gamma : \Delta \,] < \infty$ and an ergodic component $X_0 \subseteq X$ for the action of Δ on X such that the cocycle $\alpha \restriction (\Delta \times X_0)$ is equivalent to a group homomorphism $\varphi : \Delta \to H$. Suppose that φ is not injective and let $N = \ker \varphi$. Applying Margulis [22, Chapter VIII], since N is a nontrivial normal subgroup of Δ, it follows that $[\Delta : N] < \infty$. Hence, after passing to a subgroup $\Delta' \leqslant \Delta$ of finite index and an ergodic component X_0' for the action of Δ' on X_0, we can suppose that $N = \Delta$ and thus $\alpha \restriction (\Delta \times X_0)$ is equivalent to the trivial homomorphism $\varphi : \Delta \to H$ which takes constant value 1. This completes the proof of Theorem 4.3.

Sketch Proof of Theorem 4.4. Let $n \geq 2$ and let $\Gamma = PSL_n(\mathbb{Z})$. Suppose that $\Delta \leq \Gamma$ with $[\, \Gamma : \Delta \,] < \infty$ and that G is an algebraic $\bar{\mathbb{Q}}$-group such that $\dim G < n^2 - 1$. We have already pointed out that if $n = 2$, then Theorem 4.4 follows from the fact that if G is an algebraic group with $\dim G < 3$, then G is solvable-by-finite, together with the fact that $PSL_2(\mathbb{Z})$ contains a free nonabelian subgroup. Hence we can suppose that $n \geq 3$. With these hypotheses, we will show that if $\varphi : \Delta \to G(\bar{\mathbb{Q}})$ is a homomorphism, then $\varphi(\Delta)$ is finite. First, by Margulis [22, Theorem IX.5.8], it follows that the Zariski closure of $\varphi(\Delta)$ is a semi-simple $\bar{\mathbb{Q}}$-group. Hence, using the fact that every proper normal subgroup of Δ has finite index, it is enough to consider the special case when $G(\bar{\mathbb{Q}})$ is a simple algebraic $\bar{\mathbb{Q}}$-group. Since Δ is finitely generated, there exists an algebraic number field F and a finite set S of valuations of F such that G is an algebraic F-group and $\varphi(\Delta) \leqslant G(F(S))$, where $F(S)$ is the ring of S-integral elements of F. (For example, see Margulis [22, Chapter I].) Clearly we can suppose that S contains the set $\mathcal{R}_\infty$ of archimedean valuations of F. For each $\nu \in S$, let F_ν be the completion of F relative to ν. By Margulis [22, Section I.3.2], if $G(F(S))$ is identified with its image under the diagonal embedding into $G_S = \prod_{\nu \in S} G(F_\nu)$, then $G(F(S))$ is a discrete subgroup of G_S. For each $\nu \in S$, let $\pi_\nu : G_S \to G(F_\nu)$ be the canonical projection and consider the homomorphism $\varphi_\nu = \pi_\nu \circ \varphi : \Delta \to G(F_\nu)$. Applying Margulis [22, Theorem IX.6.16], since $\dim G < n^2 - 1$, it follows that $\varphi_\nu(\Delta)$ is contained in a compact subgroup K_ν of $G(F_\nu)$. Hence $\varphi(\Delta)$ is contained in the compact subgroup $\prod_{\nu \in S} K_\nu$ of G_S. Since $\varphi(\Delta)$ is also contained in the discrete subgroup $G(F(S))$, it follows that $\varphi(\Delta)$ is a finite group. This completes the proof of Theorem 4.4.

Appendix B. The proof of Theorem 1.1 for $n = 2$

In this appendix, we will sketch the proof of Theorem 1.1 for the case when $n = 2$. Once again, we will assume that the reader is familiar with the basic ideas of superrigidity theory.

Suppose that S, T are sets of primes such that $S \nsubseteq T$ and that $(\cong_2^S) \leq_B (\cong_2^T)$. Fix some prime $p \in S \smallsetminus T$. Let $X = PG(1, \mathbb{Q}_p)$ and let E^p be the orbit equivalence relation induced by the action of $PGL_2(\mathbb{Q})$ on X. Then, applying Theorem 3.7, there exists a Borel reduction

$$f : X \to R^T(\mathbb{Q}^2)$$

from E^p to $\cong_2^T$. For each $x \in X$, let $A_x = f(x) \in R^T(\mathbb{Q}^2)$. Since $PSL_2(\mathbb{Z})$ acts ergodically on (X, μ_p), there exists a Borel subset $X_0 \subseteq X$ with $\mu_p(X_0) = 1$ such that for each prime $q \in \mathbb{P}$, there exists a fixed integer $0 \leq d_q \leq 2$ such that $\dim_{\mathbb{Q}_q} V_q^{A_x} = d_q$ for all $x \in X_0$. To simplify notation, we will assume that $X_0 = X$.

Note that if $q \notin T$, then A_x is q-divisible and hence $\dim_{\mathbb{Q}_q} V_q^{A_x} = 2$. In particular, it follows that $d_p = 2$.

Lemma B.1. *For each $q \in \mathbb{P}$, either $d_q = 0$ or $d_q = 2$.*

Proof. Suppose that there exists a prime $q \in \mathbb{P}$ such that $d_q = 1$. Then $q \neq p$ and we can define a Borel homomorphism

$$g : X \to PG(1, \mathbb{Q}_q)$$
$$x \mapsto V_q^{A_x}$$

from E^p to the orbit equivalence relation E^q induced by the action of $PGL_2(\mathbb{Q})$ on $PG(1, \mathbb{Q}_q)$. By Hjorth-Thomas [15, Remark 5.4], there exists a Borel subset $X_0 \subseteq X$ with $\mu_p(X_0) = 1$ such that g maps X_0 to a single E^q-class; and after adjusting f, we can suppose that there is a fixed 1-dimensional subspace $W \leqslant \mathbb{Q}_q^2$ such that $V_q^{A_x} = W$ for all $x \in X_0$. Let $G_{\{W\}}$ be the setwise stabilizer of W in $GL_2(\mathbb{Q})$. Then if $x, y \in X_0$ and $xE^p y$, there exists $\pi \in G_{\{W\}}$ such that $\pi(A_x) = A_y$. Furthermore, Proposition 4.6 implies that $G_{\{W\}}/Z(GL_2(\mathbb{Q}))$ is soluble-by-finite and it follows that $G_{\{W\}}$ is an amenable group.

Let $\Lambda = PSL_2(\mathbb{Z}[1/q])$. Then we can suppose that X_0 is Λ-invariant; and, by Thomas [32, Theorem 6.1], the orbit equivalence relation E_Λ induced by the action of Λ on X_0 is E_0-ergodic. It follows that if F is the orbit equivalence relation induced by the action of the amenable group $G_{\{W\}}$ on $R^T(\mathbb{Q}^2)$, then E_Λ is also F-ergodic. (For example, see Thomas [32, Lemma 4.6].) However, regarding $f \restriction X_0$ as a Borel homomorphism from E_Λ to F, this implies that there exists a Borel subset $X_1 \subseteq X_0$ with $\mu_p(X_1) = 1$ such that f maps X_1 to a single $\cong_2^T$-class, which is a contradiction. □

Lemma B.2. *There exists a Borel subset $X_0 \subseteq X$ with $\mu_p(X_0) = 1$ and a fixed central subgroup $D \leqslant Z(GL_2(\mathbb{Q}))$ such that $\operatorname{Aut}(A_x) = D$ for all $x \in X_0$.*

Proof. Exploiting the E_0-ergodicity of E_Λ as above, where $\Lambda = PSL_2(\mathbb{Z}[1/q])$ for some prime $q \neq p$, the proof of Theorem 5.2 adapts to this setting with only minor changes. The main point is that Proposition 4.5 implies that if S is a $\mathbb{Q}$-subalgebra of $\operatorname{Mat}_2(\mathbb{Q})$ such that the normalizer $N = N_{GL_2(\mathbb{Q})}(S)$ is a proper subgroup of $GL_n(\mathbb{Q})$, then N is an amenable group. This allows us to work directly with the action of N on $\{ A_x \mid x \in X \}$, instead of working with the action of N/S^* on the corresponding set $\{ [A_x] \mid x \in X \}$ of $\approx$-classes. □

Once again, to simplify notation, we will assume that $X_0 = X$. From now on, we will fix a prime $q \neq p$ such that $d_q = 0$. As in the proof of Theorem 1.1 for the case when $n \geq 3$, we can suppose that

$$(\widehat{A}_x)_q = \mathbb{Z}_q \mathbf{e}_1 \oplus \mathbb{Z}_q \mathbf{e}_2$$

for all $x \in X$, where $\mathbf{e}_1$, $\mathbf{e}_2$ is the standard basis of $\mathbb{Q}_q^2$. For the remainder of the proof, we will work with the action of $\Gamma = SL_2(\mathbb{Z}[1/q])$ on X. (The switch from $\Lambda = PSL_2(\mathbb{Z}[1/q])$ to $\Gamma = SL_2(\mathbb{Z}[1/q])$ has been made in order to adapt the arguments of Thomas [32] and Hjorth-Thomas [15] to this setting.) First notice that if x, $y \in X$ and $\gamma \cdot x = y$ for some $\gamma \in \Gamma$, then there exists

$$\sigma \in GL_2(\mathbb{Q}) \cap GL_2(\mathbb{Z}_q) = GL_2(\mathbb{Z}_{(q)})$$

such that $\sigma(A_x) = A_y$. Furthermore, if D is the fixed group such that $\mathrm{Aut}(A_x) = D$ for all $x \in X$, then $D \leqslant Z(GL_2(\mathbb{Z}_{(q)}))$. Hence we can define a Borel cocycle $\alpha : \Gamma \times X \to GL_2(\mathbb{Z}_{(q)})/D$ by

$$\alpha(\gamma, x) = \text{ the unique element } \psi \in GL_2(\mathbb{Z}_{(q)})/D \text{ such that } \psi(A_x) = A_{\gamma \cdot x}.$$

Remark B.3. If the Ioana superrigidity theorem holds for the more general class of groups with Property (τ), then we can conclude the proof at this point via an appeal to Theorem 4.8. Unfortunately it is currently not known whether this is the case and so we must continue for a few more pages.

Let $\theta : GL_2(\mathbb{Z}_{(q)})/D \to PGL_2(\mathbb{Z}_{(q)})$ be the canonical surjective homomorphism and let $\bar{\alpha} : \Gamma \times X \to PGL_2(\mathbb{Z}_{(q)})$ be the Borel cocycle defined by $\bar{\alpha} = \theta \circ \alpha$.

Remark B.4. Let $Y = \{ A \in R^T(\mathbb{Q}^2) \mid \hat{A}_q = \mathbb{Z}_q \mathbf{e}_1 \oplus \mathbb{Z}_q \mathbf{e}_2 \}$ and let $Z = Z(GL_2(\mathbb{Z}_{(q)}))/D$. Then there is a natural action of $PGL_2(\mathbb{Z}_{(q)})$ on the set $\bar{Y}$ of Z-orbits on Y. Furthermore, if we define $\bar{f} : X \to \bar{Y}$ by $\bar{f}(x) = Z \cdot f(x)$, then

$$\bar{\alpha}(\gamma, x) \cdot \bar{f}(x) = \bar{f}(\gamma \cdot x)$$

for all $\gamma \in \Gamma$ and $x \in X$. Of course, $\bar{Y}$ is usually *not* a standard Borel space. However, this will cause no problems since $\bar{Y}$ does not play an essential role in this proof and has only been introduced in order to help the reader to visualize some of the later arguments.

Arguing as in Hjorth-Thomas [15, Section 5], there is a finite ergodic extension $(\widetilde{X}, \widetilde{\mu}_p)$ of (X, μ_p) such that the lift $\widetilde{\alpha} : \Gamma \times \widetilde{X} \to PGL_2(\mathbb{Z}_{(q)})$ of $\bar{\alpha}$ is equivalent to a cocycle $\widetilde{\alpha}'$ taking values in a finitely generated subgroup of $PSL_2(\mathbb{Z}_{(q)})$. Thus there exists a finite subset $T = \{p_1, \cdots, p_t\} \subseteq \mathbb{P} \smallsetminus \{q\}$ such that $\widetilde{\alpha}'$ takes values in $\Lambda_T = PSL_2(\mathbb{Z}[1/p_1, \cdots p_t])$.

Lemma B.5. *$\widetilde{\alpha}'$ is not equivalent to a cocycle taking values in an amenable subgroup of Λ_T.*

Proof. Otherwise, $\widetilde{\alpha}$ is equivalent to a cocycle taking values in an amenable subgroup A of $PGL_2(\mathbb{Z}_{(q)})$. Clearly the action of $PGL_2(\mathbb{Z}_{(q)})$ on the (countable) standard Borel space of finite subsets of the coset space $PGL_2(\mathbb{Z}_{(q)})/A$ is tame. Hence, arguing as in the proof of Adams-Kechris [1, Proposition 2.6], it follows that $\bar{\alpha}$ is also equivalent to a cocycle taking values in an amenable subgroup of $PGL_2(\mathbb{Z}_{(q)})$; and this implies that α is equivalent to a cocycle taking values in an amenable subgroup of $GL_2(\mathbb{Z}_{(q)})/D$. However, using the E_0-ergodicity of the action of Γ on (X, μ_p), this easily leads to a contradiction. □

In order to simplify notation, we will assume that $\widetilde{\alpha}' = \widetilde{\alpha}$. For later use, let $\widetilde{f} : \widetilde{X} \to \bar{Y}$ be the corresponding map such that

$$\widetilde{\alpha}(\gamma, z) \cdot \widetilde{f}(z) = \widetilde{f}(\gamma \cdot z)$$

for all $\gamma \in \Gamma$ and $z \in \widetilde{X}$. From now on, let $G = SL_2(\mathbb{R}) \times SL_2(\mathbb{Q}_q)$ and let $H = PSL_2(\mathbb{R}) \times H_T$, where

$$H_T = PSL_2(\mathbb{Q}_{p_1}) \times \cdots \times PSL_2(\mathbb{Q}_{p_t}).$$

Then if we identify Γ and Λ_T with their images under the diagonal embeddings into G and H respectively, then Γ and Λ_T are irreducible lattices in G and H. Let $\widehat{X} = \widetilde{X} \times G/\Gamma$ be the induced G-space and let $\widehat{\mu} = \widetilde{\mu}_p \times \nu$, where ν is the Haar probability measure on G/Γ. The following two technical results, which are proved in Hjorth-Thomas [15, Section 5], will enable us to apply the arguments of Thomas [32, Section 8] in our setting.

Lemma B.6. $(\widehat{X}, \widehat{\mu})$ *is an irreducible G-space.*

Lemma B.7. Γ *acts E_0-ergodically on* $(\widetilde{X}, \widetilde{\mu}_p)$.

We will also make use of the following result, which is a special case of Hjorth-Thomas [15, Lemma 4.8].

Lemma B.8. *Suppose that Δ is a lattice in $PSL_2(\mathbb{R})$ and that ω is the Haar probability measure on $PSL_2(\mathbb{R})/\Delta$. Let Λ be a lattice in $PSL_2(\mathbb{R})$ and let Λ^+ be a countable group such that $\Lambda \leqslant \Lambda^+ \leqslant PSL_2(\mathbb{R})$. Then $(PSL_2(\mathbb{R})/\Delta, \Lambda^+, \omega)$ is not a quotient of $(\widetilde{X}, \Gamma, \widetilde{\mu}_p)$*

Let $i : \Lambda_T \to H$ be the inclusion map and consider the cocycle

$$i \circ \widetilde{\alpha} : \Gamma \times \widetilde{X} \to H = PSL_2(\mathbb{R}) \times H_T.$$

Let $\sigma : G = SL_2(\mathbb{R}) \times SL_2(\mathbb{Q}_q) \to SL_2(\mathbb{R})$ be the projection map onto the first factor. Then, by the arguments of Thomas [32, Section 8], there exists:

(i) an $\mathbb{R}$-rational surjective homomorphism $\psi : SL_2(\mathbb{R}) \to PSL_2(\mathbb{R})$; and
(ii) a compact subgroup $K \leqslant H_T$ of countable index

such that $i \circ \widetilde{\alpha}$ is equivalent to a cocycle

$$\beta : \Gamma \times \widetilde{X} \to H = PSL_2(\mathbb{R}) \times H_T$$
$$\beta(\gamma, z) = (\varphi(\gamma), \beta_T(\gamma, z)),$$

where $\varphi = \psi \circ \sigma$ and $\beta_T : \Gamma \times \widetilde{X} \to H_T$ is a cocycle taking values in the compact subgroup K. From now on, let $U \subseteq H$ be a Borel transversal for H/Λ_T chosen so that $H_T \subseteq U$ and identify U with H/Λ_T by identifying each $u \in U$ with $u\Lambda_T$. Then the action of H on H/Λ_T induces a corresponding Borel action of H on U, defined by

$$h \cdot u = \text{ the unique element in } U \cap hu\Lambda_T.$$

Let $\rho : H \times U \to \Lambda_T$ be the associated cocycle defined by

$$\begin{aligned}\rho(h, u) &= \text{ the unique } \lambda \in \Lambda_T \text{ such that } (h \cdot u)\lambda = hu \\ &= (h \cdot u)^{-1} hu\end{aligned}$$

Then we can define an induced action of H on $\widehat{Y} = \bar{Y} \times U = \bar{Y} \times (H/\Lambda_T)$ by

$$h \cdot (\, y, u\,) = (\, \rho(h,u) \cdot y, h \cdot u\,).$$

Let $j : \bar{Y} \to \widehat{Y}$ be the Λ_T-equivariant embedding defined by $j(y) = (y, 1)$ and let $\widehat{f} = j \circ \widetilde{f} : \widetilde{X} \to \widehat{Y}$. Then for all $\gamma \in \Gamma$ and $z \in \widetilde{X}$,

$$(i \circ \widetilde{\alpha})(\gamma, z) \cdot \widehat{f}(z) = \widehat{f}(\gamma \cdot z).$$

Let $b : \widetilde{X} \to H$ be a Borel map such that for all $\gamma \in \Gamma$,

$$\beta(\gamma, z) = b(\gamma \cdot z)\,(i \circ \widetilde{\alpha})(\gamma, z)\, b(z)^{-1} \quad \text{for } \widetilde{\mu}_p\text{-a.e. } z \in \widetilde{X};$$

and define $\widehat{g} : \widetilde{X} \to \widehat{Y}$ by $\widehat{g}(z) = b(z) \cdot \widehat{f}(z)$. Then for all $\gamma \in \Gamma$,

$$\beta(\gamma, z) \cdot \widehat{g}(z) = \widehat{g}(\gamma \cdot z) \quad \text{for } \widetilde{\mu}_p\text{-a.e. } z \in \widetilde{X};$$

Since K is a compact group, it follows that the set $K\backslash H/\Lambda_T$ of K-orbits on H/Λ_T is a standard Borel space. Furthermore, since the actions of K and $PSL_2(\mathbb{R})$ on H/Λ_T commute, it follows that $PSL_2(\mathbb{R})$ acts on $K\backslash H/\Lambda_T$. Let $\eta : \widehat{Y} \to K\backslash H/\Lambda_T$ be the map defined by $\eta(y, u\Lambda_T) = Ku\Lambda_T$. Then it is easily checked that η is a Borel map. (In fact, by examining the above construction, it is straightforward to define η directly without mentioning the "nonstandard" space $\bar{Y}$.) Let $\omega = (\eta \circ \widehat{g})_* \widetilde{\mu}_p$. Then ω is a $\varphi(\Gamma)$-invariant ergodic probability measure on $K\backslash H/\Lambda_T$. Since K has countable index in H_T, it follows that $PSL_2(\mathbb{R})$ has only countably many orbits on $K\backslash H/\Lambda_T$. Hence, since $\varphi(\Gamma)$ acts ergodically on $K\backslash H/\Lambda_T$, it follows that ω is supported on a single $PSL_2(\mathbb{R})$-orbit Ω on $K\backslash H/\Lambda_T$. A straightforward calculation shows that the stabilizer in $PSL_2(\mathbb{R})$ of each element of $K\backslash H/\Lambda_T$ is countable. Hence we can identify Ω with $PSL_2(\mathbb{R})/\Delta$, where Δ is a suitably chosen discrete subgroup of $PSL_2(\mathbb{R})$. Let $C = \{\, h \in PSL_2(\mathbb{R}) \mid \omega \text{ is } h\text{-invariant}\,\}$. Then C is a (topologically) closed subgroup of $PSL_2(\mathbb{R})$ such that $\varphi(\Gamma) \leqslant C$. Let $\Gamma_0 = SL_2(\mathbb{Z})$. Since Γ_0 is a lattice in $SL_2(\mathbb{R})$, it follows that $\varphi(\Gamma_0)$ is a lattice in $PSL_2(\mathbb{R})$. Also, since $[\,\Gamma : \Gamma_0\,] = \infty$, it follows that $[\,\varphi(\Gamma) : \varphi(\Gamma_0)\,] = \infty$ and hence the topological closure of $\varphi(\Gamma)$ is $PSL_2(\mathbb{R})$. Thus ω is a $PSL_2(\mathbb{R})$-invariant probability measure on $PSL_2(\mathbb{R})/\Delta$, which means that Δ is a lattice in $PSL_2(\mathbb{R})$ and ω is the Haar probability measure. But since $(\, PSL_2(\mathbb{R})/\Delta, \varphi(\Gamma), \omega\,)$ is a quotient of $(\, \widetilde{X}, \Gamma, \widetilde{\mu}_p\,)$ and $\varphi(\Gamma)$ contains the lattice $\varphi(\Gamma_0)$ of $PSL_2(\mathbb{R})$, this contradicts Lemma B.8. This completes the proof of Theorem 1.1 for the case when $n = 2$.

References

[1] S. R. Adams and A. S. Kechris, *Linear algebraic groups and countable Borel equivalence relations*, J. Amer. Math. Soc. **13** (2000), 909–943.

[2] D. M. Arnold, *Abelian Groups and Representations of Finite Partially Ordered Sets*, CMS Books in Mathematics, Springer-Verlag, 2000.

[3] E. Artin, *Geometric Algebra*, Interscience Publishers, New York, 1957.

[4] R. Baer, *Abelian groups without elements of finite order*, Duke Math. Journal **3** (1937), 68–122.

[5] H. Bass, M. Lazard and J.-P. Serre, *Sous-groupes d'indice fini dans* $SL(n, \mathbb{Z})$, Bull. Amer. Math. Soc. **70** (1964), 385–392.

[6] A. Borel, *Linear Algebraic Groups: Second Enlarged Edition*, Graduate Texts in Mathematics **126**, Springer-Verlag, 1991.

[7] S. Coskey, *The classification of torsion-free abelian groups of finite rank up to isomorphism and up to quasi-isomorphism*, to appear in Trans. Amer. Math. Soc.

[8] S. Coskey, *Borel reductions of profinite actions of* $SL_n(\mathbb{Z})$, to appear in Ann. Pure Appl. Logic.

[9] R. Dougherty, S. Jackson and A. S. Kechris, *The structure of hyperfinite Borel equivalence relations*, Trans. Amer. Math. Soc. **341** (1994), 193–225.

[10] J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology and von Neumann algebras, I*, Trans. Amer. Math. Soc. **234** (1977), 289–324.

[11] L. Fuchs, *Infinite Abelian Groups*, Pure and Applied Mathematics Vol. 36, Academic Press, 1970.

[12] A. Furman, *Outer automorphism groups of some ergodic equivalence relations*, Comment. Math. Helv. **80** (2005), 157–196.

[13] G. Hjorth, *Around nonclassifiability for countable torsion-free abelian groups*, in Abelian groups and modules (Dublin, 1998), Trends Math., Birkhäuser, Basel, 1999, 269–292.

[14] G. Hjorth and A.S. Kechris, *Rigidity theorems for actions of product groups and countable Borel equivalence relations*, Mem. Amer. Math. Soc. **177**, no. 833, 2005.

[15] G. Hjorth and S. Thomas, *The classification problem for p-local torsion-free abelian groups of rank two*, J. Math. Logic. **6** (2006), 233-251.

[16] A. Ioana, *Cocycle Superrigidity for Profinite Actions of Property (T) Groups*, preprint, 2008.

[17] S. Jackson, A. S. Kechris, and A. Louveau, *Countable Borel equivalence relations*, J. Math. Logic **2** (2002), 1–80.

[18] V. Jones and K. Schmidt, *Asymptotically invariant sequences and approximate finiteness*, Amer. J. Math. **109** (1987), 91–114.

[19] A. G. Kurosh, *Primitive torsionsfreie abelsche Gruppen vom endlichen Range*, Ann. Math. **38** (1937), 175–203.

[20] A. Lubotzky, *Discrete Groups, Expanding Graphs and Invariant Measures*, Progress in Mathematics **125**, Birkhäuser, 1994.

[21] A. I. Malcev, *Torsion-free abelian groups of finite rank* (Russian), Mat. Sbor. **4** (1938), 45–68.

[22] G. A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Erg. der Math. und ihrer Grenz., **17**, Springer-Verlag, 1991.

[23] J. L. Mennicke, *Finite factor groups of the unimodular group*, Ann. Math. **81** (1965), 31–37.

[24] M. Ratner, *On Raghunathan's measure conjecture*, Ann. Math. **134** (1991), 545-607.

[25] J. D. Reid, *On quasi-decompostions of torsion-free abelian groups*, Proc. Amer. Math. Soc **13** (1962), 550-554.

[26] J. D. Reid, *On the ring of quasi-endomorphisms of a torsion-free group*, in: *Topics in Abelian Groups (Proc. Sympos., New Mexico State Univ., 1962)*, Scott, Foresman and Co., Chicago, 1963, pp. 51–68.

[27] K. Schmidt, *Amenability, Kazhdan's Property T, strong ergodicity and invariant means for ergodic group actions*, Erg. Th. and Dyn. Systems **1** (1981), 223–236.

[28] J.-P. Serre, *Le problème des groupes de congruence pour* $\mathbf{SL}_2$, Ann. Math. **92** (1970), 489–527.

[29] S. Thomas, *The classification problem for torsion-free abelian groups of finite rank*, J. Amer. Math. Soc. **16** (2003), 233–258.

[30] S. Thomas, *Superrigidity and countable Borel equivalence relations*, Ann. Pure Appl. Logic **120** (2003) 237–262.

[31] S. Thomas, *The classification problem for p-local torsion-free abelian groups of finite rank*, unpublished preprint (2002).

[32] S. Thomas, *Property* (τ) *and countable Borel equivalence relations*, J. Math. Logic. **7** (2007), 1-34.

[33] R. Zimmer, *Ergodic Theory and Semisimple Groups*, Birkhäuser, 1984.

Mathematics Department, Rutgers University, 110 Frelinghuysen Road, Piscataway, New Jersey 08854-8019, USA

E-mail address: sthomas@math.rutgers.edu