REALIZING INVARIANT RANDOM SUBGROUPS AS STABILIZER DISTRIBUTIONS

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Abstract. Suppose that ν is an ergodic IRS of a countable group G such that $[\, N_G(H) : H \,] = n < \infty$ for ν-a.e. $H \in \mathrm{Sub}_G$. In this paper, we consider the question of whether ν can be realized as the stabilizer distribution of an ergodic action $G \curvearrowright (\, X, \mu \,)$ on a standard Borel probability space such that the stabilizer map $x \mapsto G_x$ is n-to-one.

1. Introduction

Let G be a countable discrete group and let Sub_G be the compact space of subgroups $H \leqslant G$. Then a Borel probability measure ν on Sub_G which is invariant under the conjugation action of G on Sub_G is called an *invariant random subgroup* or *IRS*. For example, suppose that G acts via measure-preserving maps on the standard Borel probability space $(\, X, \mu \,)$ and let $f : X \to \mathrm{Sub}_G$ be the G-equivariant *stabilizer map* defined by

$$x \mapsto G_x = \{\, g \in G \mid g \cdot x = x \,\}.$$

Then the corresponding *stabilizer distribution* $\nu = f_* \mu$ is an IRS of G. In fact, by a result of Abert-Glasner-Virag [1], every IRS of G can be realized as the stabilizer distribution of a suitably chosen measure-preserving action. Moreover, as pointed out by Creutz-Peterson [3], using the Ergodic Decomposition Theorem, it follows that if ν is an ergodic IRS of G, then ν is the stabilizer distribution of an ergodic action $G \curvearrowright (\, X, \mu \,)$.

If ν is an IRS of a countable group G, then the construction of Abert-Glasner-Virag [1] realizes ν as the stabilizer distribution of a measure-preserving action $G \curvearrowright (\, X, \mu \,)$ such that the set $\{\, x \in X \mid G_x = H \,\}$ is uncountable for ν-a.e. $H \in \mathrm{Sub}_G$. There are many examples of IRSs where this cannot be avoided.

Notation 1.1. Throughout this paper, if $G \curvearrowright X$ is a Borel action of a countable group G on a standard Borel space X, then the corresponding orbit equivalence relation will be denoted by E_G^X.

Theorem 1.2. *Suppose that ν is an ergodic IRS of a countable group G and that $[\, N_G(H) : H \,] = \infty$ for ν-a.e. $H \in Sub_G$. If ν is the stabilizer distribution of a measure-preserving action $G \curvearrowright (\, X, \mu \,)$ on a Borel probability space, then the set $\{\, x \in X \mid G_x = H \,\}$ is uncountable for ν-a.e. $H \in Sub_G$.*

Proof. If not, it follows that the set $\{\, x \in X \mid G_x = H \,\}$ is countable for ν-a.e. $H \in \mathrm{Sub}_G$. Consider the Borel equivalence relation E on X defined by

$$x \, E \, y \quad \Longleftrightarrow \quad G_x = G_y.$$

Then for μ-a.e. $x \in X$, the corresponding E-class $[\, x \,]_E$ is countable. Hence, after restricting to a Borel subset $X_0 \subseteq X$ with $\mu(X_0) = 1$ if necessary, we can suppose that $[\, x \,]_E$ is countable for every $x \in X$. Thus E is a smooth countable Borel equivalence relation on X. Since $E' = E \cap E_G^X \subseteq E$, it follows that E' is also smooth. (This is a straightforward consequence of the Feldman-Moore Theorem [5]. For example, see Thomas [8, Lemma 2.1].) Also, since $G_x = G_{g \cdot x}$ whenever $g \in N_G(G_x)$, it follows that every E'-class is infinite. But then, by Dougherty-Jackson-Kechris [4, Proposition 2.5], since E_G^X contains the smooth aperiodic Borel equivalence relation E', it follows that E_G^X is compressible; and hence, by Dougherty-Jackson-Kechris [4, Theorem 3.5], there does not exist a G-invariant Borel probability measure on X, which is a contradiction. □

On the other hand, suppose that ν is an ergodic IRS of a countable group G such that $[\, N_G(H) : H \,] < \infty$ for ν-a.e. $H \in \mathrm{Sub}_G$. Then there exists an integer $n \geq 1$ such that $[\, N_G(H) : H \,] = n$ for ν-a.e. $H \in \mathrm{Sub}_G$. If $n = 1$, then ν is the stabilizer distribution of the ergodic action $G \curvearrowright (\, \mathrm{Sub}_G, \nu \,)$ and the corresponding stabilizer map $H \mapsto N_G(H)$ is ν-a.e. injective. Now suppose that $n > 1$ and that ν is the stabilizer distribution of the measure-preserving action $G \curvearrowright (\, X, \mu \,)$. If $x \in X$ and $g \in N_G(G_x)$, then $G_x = G_{g \cdot x}$. It follows that for μ-a.e. $x \in X$, the stabilizer map $f : X \to \mathrm{Sub}_G$ is n-to-one on the orbit $G \cdot x$. Consequently, the stabilizer map f is

μ-a.e. n-to-one if and only if the map

$$G \cdot x \mapsto \{ gG_xg^{-1} \mid g \in G \}$$

is μ-a.e. injective. Furthermore, in this case, by restricting to a suitable G-invariant Borel subset $X_0 \subseteq X$ with $\mu(X_0) = 1$, we obtain a measure-preserving action $G \curvearrowright (X_0, \mu)$ with stabilizer distribution ν such that the corresponding stabilizer map is n-to-one.

Question 1.3. Suppose that ν is an ergodic IRS of a countable group G and that $[N_G(H) : H] = n < \infty$ for ν-a.e. $H \in \mathrm{Sub}_G$. Is ν the stabilizer distribution of an ergodic action $G \curvearrowright (X, \mu)$ on a standard Borel probability space such that the stabilizer map $x \mapsto G_x$ is n-to-one?

If we add a suitable extra hypothesis concerning the ergodic IRS ν, then we obtain a positive answer to Question 1.3.

Definition 1.4. If $G \curvearrowright Z$ is a Borel action of a countable group G on a standard Borel space Z, then a Borel map $c : E_G^Z \to G$ is a *cocycle* if whenever $x \, E_G^Z \, y$ and $y \, E_G^Z \, z$, then:

- $c(x, y) \cdot x = y$; and
- $c(x, z) = c(y, z)c(x, y)$.

The Borel action $G \curvearrowright Z$ is said to have the *cocycle property* if there exists a Borel cocycle $c : E_G^Z \to G$.

Remark 1.5. For later use, note that if $c : E_G^Z \to G$ is a cocycle and $x \in Z$, then by taking $x = y = z$, we obtain that $c(x, x) = 1$. It follows that if $x \, E_G^Z \, y$, then $c(y, x) = c(x, y)^{-1}$.

Definition 1.6. A measure-preserving action $G \curvearrowright (Z, \mu)$ on a standard Borel probability space is said to have the *μ-cocycle property* if there exists a G-invariant Borel subset $Z_0 \subseteq Z$ with $\mu(Z_0) = 1$ such that $G \curvearrowright Z_0$ has the cocycle property.

Example 1.7. Let $\mathbb{F}_n$ be the free group on n generators, where $2 \leq n \leq \aleph_0$, and let μ be the usual uniform product probability measure on $2^{\mathbb{F}_n}$. By Hjorth-Kechris [6, Corollary 10.7], the shift action $\mathbb{F}_n \curvearrowright 2^{\mathbb{F}_n}$ does not have the cocycle property.

However, since $\mathbb{F}_n$ acts freely outside a μ-null subset, it follows that the shift action $\mathbb{F}_n \curvearrowright (2^{\mathbb{F}_n}, \mu)$ has the μ-cocycle property.

Remark 1.8. If G is an amenable group, then every measure-preserving action $G \curvearrowright (Z, \mu)$ on a standard Borel probability space has the μ-cocycle property. To see this, recall that by Connes-Feldman-Weiss [2], there exists a G-invariant Borel subset $Z_0 \subseteq Z$ with $\mu(Z_0) = 1$ such that $E_G^{Z_0}$ is hyperfinite; and hence, by Hjorth-Kechris [6, Theorem 8.1], the action $G \curvearrowright Z_0$ has the cocycle property.

The following result will be proved in Section 2.

Theorem 1.9. *Suppose that ν is an ergodic IRS of a countable group G and that:*

(i) $[N_G(H) : H] = n < \infty$ *for ν-a.e.* $H \in \mathit{Sub}_G$*;*

(ii) $G \curvearrowright (\mathit{Sub}_G, \nu)$ *has the ν-cocycle property.*

Then ν is the stabilizer distribution of an ergodic action $G \curvearrowright (X, \mu)$ on a standard Borel probability space such that the stabilizer map $x \mapsto G_x$ is n-to-one.

Corollary 1.10. *If ν is an ergodic IRS of a countable amenable group G such that $[N_G(H) : H] = n < \infty$ for ν-a.e. $H \in \mathit{Sub}_G$, then ν is the stabilizer distribution of an ergodic action $G \curvearrowright (X, \mu)$ on a standard Borel probability space such that the stabilizer map $x \mapsto G_x$ is n-to-one.*

The next result confirms that, as expected, there exist examples of ergodic IRSs which fail to satisfy hypothesis (1.9)(ii).

Theorem 1.11. *There exists a countable group G with an ergodic IRS ν such that $G \curvearrowright (\mathit{Sub}_G, \nu)$ does not have the ν-cocycle property.*

Remark 1.12. We will prove a strengthening of Theorem 1.11 in Section 3.

2. The proof of Theorem 1.9

Clearly we can suppose that $n > 1$. By assumption, there exists a G-invariant Borel subset $Z \subseteq \mathrm{Sub}_G$ with $\nu(Z) = 1$ such that the conjugation action $G \curvearrowright Z$ has the cocycle property. Thus there exists a Borel map $c : E_G^Z \to G$ such that whenever H_1, H_2, $H_3 \in Z$ are conjugate subgroups of G, then:

- $c(H_1, H_2) H_1 c(H_1, H_2)^{-1} = H_2$; and

- $c(H_1, H_3) = c(H_2, H_3)c(H_1, H_2)$.

After slightly shrinking Z if necessary, we can also suppose that $[\, N_G(H) : H\,] = n$ for every $H \in Z$.

Let $X = \{\, aH \mid H \in Z, a \in N_G(H)\,\}$ and let μ be the Borel probability measure on X defined by

$$\mu(B) = \int_Z \frac{|B \cap \{\, aH \mid a \in N_G(H)\,\}|}{n}\, d\nu(H).$$

For each $g \in G$ and $aH \in X$, define

$$g \cdot aH = c(H, gHg^{-1})aHg^{-1}.$$

Let $b \in N_G(H)$ be such that $g = c(H, gHg^{-1})b$. Since $b^{-1}a \in N_G(H)$ and

$$g \cdot aH = gb^{-1}ag^{-1}(gHg^{-1}),$$

it follows that $g \cdot aH$ is a coset of gHg^{-1} in $N_G(gHg^{-1})$ and thus $g \cdot aH \in X$. Also if $g,\, h \in G$ and $aH \in X$, then

$$\begin{aligned} g \cdot (h \cdot aH) &= c(hHh^{-1}, ghHh^{-1}g^{-1})c(H, hHh^{-1})aHh^{-1}g^{-1} \\ &= c(H, ghHh^{-1}g^{-1})aH(gh)^{-1} \\ &= gh \cdot aH. \end{aligned}$$

Thus the maps $aH \mapsto g \cdot aH$ define an action of G on X, which is easily seen to be μ-preserving. Furthermore, for each $aH \in X$, the corresponding G-orbit is $G \cdot aH = \{\, b\, gHg^{-1} \mid g \in G, b \in N_G(gHg^{-1})\,\}$; and it follows that the action $G \curvearrowright (\, X, \mu\,)$ is ergodic. Finally suppose that $g \in G$ and $aH \in X$ are such that $g \cdot aH = aH$. Then clearly $g \in N_G(H)$ and thus $aH = c(H, H)aHg^{-1} = ag^{-1}H$. It follows that $g \in H$ and hence H is the stabilizer of aH under the action $G \curvearrowright (\, X, \mu\,)$. Thus the stabilizer map $aH \overset{f}{\mapsto} G_{aH}$ is n-to-one. Also if $T \subseteq \mathrm{Sub}_G$ is a Borel subset, then $(f_* \mu)(T) = \mu(\, \{\, aH \mid H \in T \cap Z, a \in N_G(H)\,\}\,) = \nu(T)$ and so ν is the stabilizer distribution of $G \curvearrowright (\, X, \mu\,)$. This completes the proof of Theorem 1.9.

3. The weak cocycle property

Suppose that ν is an ergodic IRS of a countable group G with the property that $[\, N_G(H) : H\,] = n < \infty$ for ν-a.e. $H \in \mathrm{Sub}_G$. Then, in the statement of Theorem

1.9, we can weaken the hypothesis that $G \curvearrowright (\mathrm{Sub}_G, \nu)$ has the ν-cocycle property, as follows.

Definition 3.1. An IRS ν of a countable group G is said to have the *weak cocycle property* if there exists a G-invariant Borel subset $Z \subseteq \mathrm{Sub}_G$ with $\nu(Z) = 1$ and a Borel map $c : E_G^Z \to G$ such that whenever H_1, H_2, $H_3 \in Z$ are conjugate subgroups of G, then:

- $c(H_1, H_2) H_1 c(H_1, H_2)^{-1} = H_2$; and
- $c(H_1, H_3)^{-1} c(H_2, H_3) c(H_1, H_2) \in H_1$.

In this case, we say that c is a *weak cocycle*.

Theorem 3.2. *If ν is an ergodic IRS of a countable group G with the property that $[\, N_G(H) : H \,] = n < \infty$ for ν-a.e. $H \in Sub_G$, then the following conditions are equivalent:*

(i) *ν has the weak cocycle property.*
(ii) *ν is the stabilizer distribution of an ergodic action $G \curvearrowright (X, \mu)$ on a standard Borel probability space such that the stabilizer map $x \mapsto G_x$ is n-to-one.*

Proof. It is easily checked that the construction in Theorem 1.9 goes through under the hypothesis that ν has the weak cocycle property. Conversely, suppose that the ergodic IRS ν is the stabilizer distribution of an ergodic action $G \curvearrowright (X, \mu)$ on a standard Borel probability space such that the stabilizer map $x \overset{f}{\mapsto} G_x$ is n-to-one. Then we can suppose that $[\, N_G(G_x) : G_x \,] = n$ for all $x \in X$; and, as we explained in Section 1, it follows that the map

$$G \cdot x \mapsto \{\, g G_x g^{-1} \mid g \in G \,\}$$

is injective. Let $Z = \{\, G_x \mid x \in X \,\}$. Then $\nu(Z) = 1$; and for all $H \in Z$, the n-set $f^{-1}(H) = \{\, x \in X \mid G_x = H \,\}$ lies in a single G-orbit. Let $\prec$ be a Borel linear ordering of X and let $\varphi : Z \to X$ be the Borel map defined by

$$\varphi(H) = \text{ the } \prec\text{-least } x \in f^{-1}(H).$$

Finally let $c : E_G^Z \to G$ be any Borel map such that if H_1, $H_2 \in Z$ are conjugate subgroups, then

$$c(H_1, H_2) \cdot \varphi(H_1) = \varphi(H_2).$$

Clearly if H_1, $H_2 \in Z$ are conjugate subgroups, then

$$c(H_1, H_2) H_1 c(H_1, H_2)^{-1} = H_2.$$

Also if H_1, H_2, $H_3 \in Z$ are conjugate subgroups of G, then

$$c(H_2, H_3) c(H_1, H_2) \cdot \varphi(H_1) = \varphi(H_3) = c(H_1, H_3) \cdot \varphi(H_1)$$

and so

$$c(H_1, H_3)^{-1} c(H_2, H_3) c(H_1, H_2) \in G_{\varphi(H_1)} = H_1.$$

Thus $c : E_G^Z \to G$ is a weak cocyle. □

The remainder of this section is devoted to the proof of the following strengthening of Theorem 1.11.

Theorem 3.3. *There exist a countable group G with an ergodic IRS ν which does not have the weak cocycle property.*

The proof of Theorem 3.3 makes use of Popa's Cocycle Superrigidity Theorem [7], which involves a slightly different formulation of the notion of a Borel cocycle.

Definition 3.4. If $G \curvearrowright (X, \mu)$ is a measure-preserving action of a countable group on a standard Borel probability space and H is a countable group, then a Borel function $\alpha : G \times X \to H$ is called a *cocycle* if for all g, $h \in G$,

$$\alpha(hg, x) = \alpha(h, g \cdot x) \alpha(g, x) \quad \text{for } \mu\text{-a.e. } x \in X.$$

Proof of Theorem 1.11. Most of our effort will go into showing that there exists an ergodic probability measure μ on $2^{\mathbb{F}_2}$ such that the shift action $\mathbb{F}_2 \curvearrowright (2^{\mathbb{F}_2}, \mu)$ does not have the μ-cocycle property. (Of course, Example 1.7 shows that μ is not the usual uniform product probability measure.) First recall that $\Gamma = SL(3, \mathbb{Z})$ is a 2-generator Kazhdan group. (For example, Zimmer [9, Chapter 7].) Let $\pi : \mathbb{F}_2 \to \Gamma$ be a surjective homomorphism, let m be the uniform product probability measure on 2^Γ and let $\mathbb{F}_2 \curvearrowright (2^\Gamma, m)$ be the ergodic action defined by $g \cdot x = \pi(g) \cdot x$.

Claim 3.5. $\mathbb{F}_2 \curvearrowright (2^\Gamma, m)$ *does not have the m-cocycle property.*

Proof of Claim 3.5. Suppose that $Z \subseteq 2^\Gamma$ is an $\mathbb{F}_2$-invariant Borel subset with $m(Z) = 1$ and that $c : E_{\mathbb{F}_2}^Z \to \mathbb{F}_2$ is a Borel cocycle. Then we can define a Borel cocycle $\alpha : \Gamma \times Z \to \mathbb{F}_2$ by $\alpha(\gamma, z) = c(z, \gamma \cdot z)$. By Popa's Cocycle Superrigidity Theorem [7], after deleting an m-null subset of Z if necessary, there exists a Borel map $b : Z \to \mathbb{F}_2$ and a homomorphism $\varphi : \Gamma \to \mathbb{F}_2$ such that for all $\gamma \in \Gamma$ and $z \in Z$,

$$\varphi(\gamma) = b(\gamma \cdot z)\alpha(\gamma, z)b(z)^{-1}.$$

Since Γ does not embed into $\mathbb{F}_2$, it follows that $N = \ker \varphi \neq 1$; and this implies that $[\,\Gamma : N\,] < \infty$. (For example, Zimmer [9, Chapter 8].) In particular, N is an infinite subgroup of Γ. Since the action $\Gamma \curvearrowright (\,2^{\Gamma}, m\,)$ is strongly mixing, it follows that N acts ergodically on $(\,2^{\Gamma}, m\,)$. Note that if $\gamma \in N$ and $z \in Z$, then

$$c(z, \gamma \cdot z) = \alpha(\gamma, z) = b(\gamma \cdot z)^{-1} b(z);$$

and hence

$$\begin{aligned} b(\gamma \cdot z) \cdot (\gamma \cdot z) &= b(z)c(z, \gamma \cdot z)^{-1} \cdot (\gamma \cdot z) \\ &= b(z)c(\gamma \cdot z, z) \cdot (\gamma \cdot z) \\ &= b(z) \cdot z. \end{aligned}$$

But then, since the action $N \curvearrowright (\,2^{\Gamma}, m\,)$ is ergodic, it follows that the Borel map $z \mapsto b(z) \cdot z$ is m-a.e. constant, which is a contradiction. □

Hence, letting $j : 2^{\Gamma} \to 2^{\mathbb{F}_2}$ be the Borel injection defined by $j(x)(g) = x(\pi(g))$ and $\mu = j_* m$, it follows that the shift action $\mathbb{F}_2 \curvearrowright (\,2^{\mathbb{F}_2}, \mu\,)$ does not have the μ-cocycle property. Next let $B = \bigoplus_{h \in \mathbb{F}_2} C_h$, where each C_h is a cyclic group of order 2. Then the wreath product $G = C_2 \operatorname{wr} \mathbb{F}_2$ is defined to be the semidirect product $B \rtimes \mathbb{F}_2$, where $gC_h g^{-1} = C_{gh}$ for each g, $h \in \mathbb{F}_2$. Let $\theta : 2^{\mathbb{F}_2} \to \mathrm{Sub}_G$ be the injective $\mathbb{F}_2$-equivariant map defined by

$$x \mapsto B_x = \bigoplus \{\, C_h \mid h \in \mathbb{F}_2, x(h) = 1 \,\}$$

and let $\nu = \theta_* \mu$ be the corresponding $\mathbb{F}_2$-invariant ergodic probability measure on Sub_G. Since B acts trivially on $\theta(2^{\mathbb{F}_2})$, it follows that ν is G-invariant and thus ν is an ergodic IRS of G. We claim that ν does not have the weak cocycle property. To see this, suppose that $Z \subseteq \mathrm{Sub}_G$ is a G-invariant Borel subset with $\nu(Z) = 1$ and that the Borel map $c : E_G^Z \to G$ is a weak cocycle. Then we can suppose that $Z \subseteq \theta(2^{\mathbb{F}_2})$. Let $Y \subseteq 2^{\Gamma}$ be the $\mathbb{F}_2$-invariant Borel subset with $m(Y) = 1$ such that

$Z = (\theta \circ j)(Y)$. Let $\bar{c} : E^Y_{\mathbb{F}_2} \to \mathbb{F}_2$ be the Borel map such that if $y_1 \, E^Y_{\mathbb{F}_2} \, y_2$ and $H_i = (\theta \circ j)(y_i)$ for $i = 1, 2$, then

$$c(H_1, H_2) = b(H_1, H_2) \, \bar{c}(y_1, y_2),$$

where $b(H_1, H_2) \in B$. Since B acts trivially on Z, it follows that $\bar{c}(y_1, y_2) \cdot y_1 = y_2$. Also if $y_2 \, E^Y_{\mathbb{F}_2} \, y_3$ and $H_3 = (\theta \circ j)(y_3)$, then

$$c(H_1, H_3)^{-1} c(H_2, H_3) c(H_1, H_2) \in H_1 \leqslant B;$$

and it follows that

$$\bar{c}(y_1, y_3)^{-1} \bar{c}(y_2, y_3) \bar{c}(y_1, y_2) = 1.$$

But this means that $\bar{c} : E^Y_{\mathbb{F}_2} \to \mathbb{F}_2$ is a cocycle, which contradicts Claim 3.5. This completes the proof of Theorem 3.3. □

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