DIFFUSE INVARIANT RANDOM SUBGROUPS

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Abstract. There exist countable groups G with ergodic invariant random subgroups ν such that $\nu(\{ H \in \mathrm{Sub}_G \mid H \cong K \}) = 0$ for every subgroup $K \leqslant G$.

1. Diffuse invariant random subgroups

Let G be a countable discrete group and let Sub_G be the compact space of subgroups $H \leqslant G$. Then a Borel probability measure ν on Sub_G which is invariant under the conjugation action of G on Sub_G is called an *invariant random subgroup* or *IRS*. If ν is an ergodic IRS of a countable group G, then we obtain a corresponding zero-one law on Sub_G for the class of group-theoretic properties Φ for which the set $\{ H \in \mathrm{Sub}_G \mid H \text{ has property } \Phi \}$ is ν-measurable. These include those properties that can be expressed using the infinitary language $\mathcal{L}_{\omega_1,\omega}$ and thus ν concentrates on a collection of subgroups which are quite difficult to distinguish between. In fact, it seems that all of the examples in the literature have the property that ν concentrates on the subgroups of G of a fixed isomorphism type. For example, the results of Vershik [10][1], Thomas and Tucker-Drob [9], and Bowen, Grigorchuk and Kravchenko [1] imply that if G is either the group $\mathrm{Fin}(\mathbb{N})$ of finitary permutations of $\mathbb{N}$, a diagonal limit of finite alternating groups, or a lamplighter group, and ν is an ergodic IRS of G, then there exists a subgroup $K_\nu \leqslant G$ such that

$$\nu(\{ H \in \mathrm{Sub}_G \mid H \cong K_\nu \}) = 1.$$

(Throughout this paper, $\cong$ denotes the abstract isomorphism relation on the class of groups.) It is well known that if G is a countable group and $K \leqslant G$ is a subgroup, then $\{ H \in \mathrm{Sub}_G \mid H \cong K \}$ is a Borel subset of Sub_G and hence is ν-measurable. (For example, see Kechris [4, Theorem 6.6].) Thus, if ν is an ergodic IRS of G,

[1]There is a slight inaccuracy in Vershik's classification [10] of the ergodic IRSs of $\mathrm{Fin}(\mathbb{N})$. A corrected statement can be found in Thomas [8].

then for each subgroup $K \leqslant G$,

$$\nu(\{ H \in \mathrm{Sub}_G \mid H \cong K \}) \in \{ 0, 1 \}.$$

Definition 1.1. An ergodic IRS ν of a countable group G is said to be *diffuse* if $\nu(\{ H \in \mathrm{Sub}_G \mid H \cong K \}) = 0$ for every subgroup $K \leqslant G$.

In this paper, adapting a technique which was developed in Thomas [7] to construct centerless groups with arbitarily long automorphism towers, we will construct examples of countable groups with diffuse IRSs.

Theorem 1.2. *There exist countable groups with diffuse IRSs.*

Our construction will make use of the following result, which will be proved in Section 2.

Lemma 1.3. *If G is any countable group, then there exists a countable group N and a semidirect product $H = N \rtimes G$ such that for all K_1, $K_2 \in Sub_G$,*

$$N \rtimes K_1 \cong N \rtimes K_2 \quad \Longleftrightarrow \quad (\exists g \in G)\ gK_1g^{-1} = K_2.$$

Proof of Theorem 1.2. Let G be a countable group with an ergodic IRS μ which does not concentrate on a single conjugacy class of subgroups of G. (For example, if G is either $\mathrm{Fin}(\mathbb{N})$, a diagonal limit of finite alternating groups, or a lamplighter group, then G has such an ergodic IRS. See [10, 9, 1].) Let N and $H = N \rtimes G$ be the countable groups given by Lemma 1.3. Let $j : \mathrm{Sub}_G \to \mathrm{Sub}_H$ be the G-equivariant map defined by $j(K) = N \rtimes K$ and let $\nu = j_*\mu$ be the corresponding G-invariant ergodic probability measure on Sub_H. Since N acts trivially on $j(\mathrm{Sub}_G)$, it follows that ν is H-invariant. Thus ν is an ergodic IRS of H. Furthermore, since the isomorphism classes on $j(\mathrm{Sub}_G)$ correspond to the conjugacy classes on Sub_G, it follows that ν is a diffuse IRS of H. □

The following question seems to be open.

Question 1.4. Do there exist "natural" examples of countable groups with diffuse IRSs?

2. The proof of Lemma 1.3

In this section, we will present the proof of Lemma 1.3. The first half of the argument is closely based on the proof of Burnside's Theorem [2] that if S is a simple nonabelian group and $G = \mathrm{Aut}(S)$, then $\mathrm{Aut}(G) = \mathrm{Inn}(G)$.

Suppose that S is a simple nonabelian group. For each $s \in S$, let i_s be the corresponding inner automorphism, defined by $i_s(x) = sxs^{-1}$. It is well-known that the group $\mathrm{Inn}(S)$ of inner automorphisms of S is a normal subgroup of $\mathrm{Aut}(S)$. In fact, if $s \in S$ and $\varphi \in \mathrm{Aut}(S)$, then $\varphi i_s \varphi^{-1} = i_{\varphi(s)}$. In particular, it follows that $C_{\mathrm{Aut}(S)}(\mathrm{Inn}(S)) = 1$.

Now suppose that G is a group such that $\mathrm{Inn}(S) \leqslant G \leqslant \mathrm{Aut}(S)$ and that $1 \neq N \trianglelefteq G$ is a nontrivial normal subgroup of G. Then $[\,\mathrm{Inn}(S), N\,] \leqslant \mathrm{Inn}(S) \cap N$; and since $C_{\mathrm{Aut}(S)}(\mathrm{Inn}(S)) = 1$, it follows that $[\,\mathrm{Inn}(S), N\,] \neq 1$. Thus $\mathrm{Inn}(S) \cap N$ is a nontrivial normal subgroup of $\mathrm{Inn}(S)$; and since S is simple, it follows that $\mathrm{Inn}(S) \leqslant N$. Hence $\mathrm{Inn}(S)$ is the *unique* minimal nontrivial normal subgroup of G.

Lemma 2.1. *Let S be a simple nonabelian group and let G, H be groups such that $\mathrm{Inn}(S) \leqslant G, H \leqslant \mathrm{Aut}(S)$. If $\pi : G \to H$ is an isomorphism, then there exists $\varphi \in \mathrm{Aut}(S)$ such that $\pi(g) = \varphi g \varphi^{-1}$ for all $g \in G$.*

Proof. Since $\mathrm{Inn}(S)$ is the unique minimal nontrivial normal subgroup of both G and H, it follows that $\pi(\mathrm{Inn}(S)) = \mathrm{Inn}(S)$ and hence there exists $\varphi \in \mathrm{Aut}(S)$ such that $\pi(c) = \varphi c \varphi^{-1}$ for all $c \in \mathrm{Inn}(S)$. Now let $g \in G$ be an arbitrary element. Then for all $c \in \mathrm{Inn}(S)$,

$$(\varphi g)c(\varphi g)^{-1} = \varphi(gcg^{-1})\varphi^{-1} = \pi(gcg^{-1}) = \pi(g)\varphi c \varphi^{-1}\pi(g)^{-1} = (\pi(g)\varphi)c(\pi(g)\varphi)^{-1}.$$

Since $C_{\mathrm{Aut}(S)}(\mathrm{Inn}(S)) = 1$, it follows that $\varphi g = \pi(g)\varphi$ and hence $\pi(g) = \varphi g \varphi^{-1}$. □

In the following proof, we will make use of the classical result that if F is a field such that $|F| > 3$, then the projective special linear group $\mathrm{PSL}(2, F)$ is simple. (For example, see Robinson [5, 3.2.9].)

Proof of Lemma 1.3. By Fried and J. Kollár [3], there exists a countably infinite field F such that $\mathrm{Aut}(F) = G$. By Schreier and van der Waerden [6],

$$\mathrm{Aut}(\mathrm{PSL}(2, F)) = \mathrm{P\Gamma L}(2, F) = \mathrm{PGL}(2, F) \rtimes G.$$

Suppose that K_1, $K_2 \in \mathrm{Sub}_G$. Clearly if K_1 and K_2 are conjugate subgroups of G, then $\mathrm{PGL}(2,F) \rtimes K_1$ and $\mathrm{PGL}(2,F) \rtimes K_2$ are conjugate subgroups of $\mathrm{PGL}(2,F) \rtimes G$, and so $\mathrm{PGL}(2,F) \rtimes K_1 \cong \mathrm{PGL}(2,F) \rtimes K_2$. Conversely, suppose that

$$\pi : \mathrm{PGL}(2,F) \rtimes K_1 \to \mathrm{PGL}(2,F) \rtimes K_2$$

is an isomorphism. By Lemma 2.1, there exists an element $h \in \mathrm{PGL}(2,F) \rtimes G$ such that $h(\mathrm{PGL}(2,F) \rtimes K_1)h^{-1} = \mathrm{PGL}(2,F) \rtimes K_2$; and, after factoring by $\mathrm{PGL}(2,F)$, we see that K_1 and K_2 are conjugate subgroups of G. $\square$

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