SOME NON-UNIVERSALITY CONJECTURES IN MEASURABLE AND GEOMETRIC GROUP THEORY.

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Abstract. We present a number of non-universality conjectures in measurable and geometric group theory; and, motivated by these conjectures, we study the orbit equivalence problem for the actions $G \curvearrowright (\mathrm{Aut}(\Gamma)/H, \mu)$, where G, H are finitely generated groups with a common Cayley graph Γ.

1. Some non-universality conjectures

Recall that two finitely generated groups G, H are said to be *measure equivalent*, written $G \approx_{ME} H$, if there exist commuting free measure-preserving actions of G and H with finite measure fundamental domains on an infinite measure space (Ω, μ). (For more details, see Furman [7] or Gaboriau [8].) It is a longstanding open question whether the measure equivalence relation $\approx_{ME}$ on the space $\mathcal{G}_{\mathrm{fg}}$ of finitely generated groups is smooth. As we will explain, it turns out that this question is related the question of whether there exists a universal measure equivalence class.

Definition 1.1. A finitely generated group G is *me-universal* if for every finitely generated group K, there exists a finitely generated group H such that $K \hookrightarrow H$ and $H \approx_{ME} G$.

Conjecture 1.2. There does not exist an me-universal group.

Theorem 1.3. *If Conjecture 1.2 holds, then the measure equivalence relation* $\approx_{ME}$ *on the space* $\mathcal{G}_{\mathrm{fg}}$ *of finitely generated groups is not smooth.*

The proofs of nonsmoothness results usually make use of measure theory or Baire category theory. In contrast, the proof of Theorem 1.3 will make use of some notions from recursion theory. Throughout this paper, we will identify the Cantor set $2^{\mathbb{N}}$ with the powerset $\mathcal{P}(\mathbb{N})$ of the natural numbers $\mathbb{N}$. If A, $B \in 2^{\mathbb{N}}$, then A is *Turing reducible* to B, written $A \leq_T B$, if there exists a Turing machine with oracle

B which computes A; and the *Turing equivalence relation* $\equiv_T$ on $2^{\mathbb{N}}$ is defined by

$$A \equiv_T B \iff A \leq_T B \text{ and } B \leq_T A.$$

For each $m \geq 1$, fix an effective enumeration $\{ w_n(x_1, \cdots, x_m) \mid n \in \mathbb{N} \}$ of the words in $x_1, \cdots, x_m$.

Definition 1.4. If $G \in \mathcal{G}_{\text{fg}}$ and $s_1, \cdots, s_m$ is a finite generating set, then

$$R_G = \{ n \in \mathbb{N} \mid w_n(s_1, \cdots, s_m) = 1 \}.$$

Of course, it is well-known that the Turing degree of R_G does not depend on the choice of generating set $\bar{s}$ of G. The proof of Theorem 1.3 makes use of the Friedman Embedding Theorem [28].

Theorem 1.5. *There exists a Borel map $\varphi : 2^{\mathbb{N}} \to \mathcal{G}_{\text{fg}}$ such that:*

- *If $A \equiv_T B$, then $\varphi(A) \cong \varphi(B)$.*
- *If $H \in \mathcal{G}_{fg}$ satisfies $R_H \leq_T A$, then $H \hookrightarrow \varphi(A)$*

A subset $\mathcal{C} \subseteq 2^{\mathbb{N}}$ is a *cone* if there exists $C \in 2^{\mathbb{N}}$ such that

$$\mathcal{C} = \{ B \in 2^{\mathbb{N}} \mid C \leq_T B \}.$$

A cone is the recursion-theoretic analog of a full measure set or a comeager set in measure theory or Baire category theory. The following analog of ergodicity is an easy consequence of Borel Determinacy. (See Martin [17, 18].)

Theorem 1.6. *If $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel map, then there exists a cone $\mathcal{C}$ such that $\theta \restriction \mathcal{C}$ is a constant map.*

We are now ready to present the proof of Theorem 1.3. Assume that Conjecture 1.2 holds; and suppose that the measure equivalence relation $\approx_{ME}$ on the space $\mathcal{G}_{\text{fg}}$ of finitely generated groups is smooth. Then there exists a Borel map $\psi : \mathcal{G}_{\text{fg}} \to 2^{\mathbb{N}}$ such that

$$H \approx_{ME} K \iff \psi(H) = \psi(K).$$

Let $\varphi : 2^{\mathbb{N}} \to \mathcal{G}_{\text{fg}}$ be the Borel map given by the Friedman Embedding Theorem and let $\theta = \psi \circ \varphi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$. Then θ is a $\equiv_T$-invariant Borel map and hence there exists a cone $\mathcal{C}$ such that $\theta \restriction \mathcal{C}$ is a constant map. It follows that there exists $G \in \mathcal{G}_{\text{fg}}$ such that $\varphi(A) \approx_{ME} G$ for all $A \in \mathcal{C}$. If $K \in \mathcal{G}_{\text{fg}}$ is arbitrary, then there exists $A \in \mathcal{C}$ such that $R_K \leq_T A$. It follows that $K \hookrightarrow \varphi(A) \approx_{ME} G$ and so G is an me-universal group, which is a contradiction.

By B. H. Neumann [19], there exist $2^{\aleph_0}$ finitely generated groups up to isomorphism; and it follows that there does not exist a finitely generated group G such that $K \hookrightarrow G$ for every finitely generated group K. Consequently, the above argument adapts to a splendidly elephantine proof that the isomorphism relation $\cong$ on $\mathcal{G}_{\mathrm{fg}}$ is not smooth. (Cf. P. M. Neumann [20, Footnote 2].) In fact, a much stronger result is true: by Thomas-Velickovic [29], the isomorphism relation $\cong$ on $\mathcal{G}_{\mathrm{fg}}$ is a universal countable Borel equivalence relation.

Using the fact that there exist $2^{\aleph_0}$ finitely generated simple groups up to isomorphism, it can easily be shown that there does not exist finitely generated group G such that for every finitely generated group K, there exists a finitely generated group H such that $K \hookrightarrow H$ and H is virtually isomorphic to G. Here two groups G_1, G_2 are said to be *virtually isomorphic* or *commensurable up to finite kernels*, written $G_1 \approx_{VI} G_2$, if there exist subgroups $N_i \leqslant H_i \leqslant G_i$ for $i = 1, 2$ satisfying the following conditions.

(a) $[G_1 : H_1]$, $[G_2 : H_2] < \infty$.
(b) N_1, N_2 are finite normal subgroups of H_1, H_2 respectively.
(c) $H_1/N_1 \cong H_2/N_2$.

However, there is a natural non-universality conjecture in geometric group theory which appears to be more challenging. Let $\approx_{QI}$ be the quasi-isometry relation on $\mathcal{G}_{\mathrm{fg}}$. (It is already known [27] that $\approx_{QI}$ is not smooth.)

Definition 1.7. A finitely generated group G is *qi-universal* if for every finitely generated group K, there exists a finitely generated group H such that $K \hookrightarrow H$ and $H \approx_{QI} G$.

Conjecture 1.8. There does not exist an qi-universal group.

The following result is an immediate consequence of the results of Erschler [3, 5]. (Recall that amenability is preserved under quasi-isometry. For example, see de la Harpe [10].)

Theorem 1.9. *There does not exist a finitely generated amenable group G such that for every finitely generated amenable group K, there exists a finitely generated (necessarily amenable) group H such that $K \hookrightarrow H$ and $H \approx_{QI} G$.*

Sketch proof. For each finitely generated amenable group G and finite generating set $S \subseteq G \smallsetminus 1$, let $\mathrm{F}_{G,S}$ be the corresponding Følner function. Then it is well-known that the growth rate of $\mathrm{F}_{G,S}$ is a quasi-isometry invariant of G. Furthermore, by Erschler [3, Lemma 4], if $H \leqslant G$ is a finitely generated subgroup with finite generating set $T \subseteq H \smallsetminus 1$, then the growth rate of $\mathrm{F}_{H,T}$ is less then or equal to that of $\mathrm{F}_{G,S}$. By Erschler [5], if $f : \mathbb{N} \to \mathbb{N}$ is any function, then there exists a finitely generated amenable group G with finite generating set $S \subseteq G \smallsetminus 1$ such that $f(n) \leq \mathrm{F}_{G.S}(n)$ for all but finitely many n. The result follows. □

Unfortunately, no such unbounded hereditary quasi-isometery invariant is known for the class of all finitely generated groups. It seems likely that Conjectures 1.2 and 1.8 are difficult problems. However, there is a common special case of both which might be more approachable. Let G be a finitely generated group and let $S \subseteq G \smallsetminus 1$ be a finite symmetric generating set. Then the corresponding *Cayley graph* $\mathrm{Cay}(G,S)$ is the graph with vertex set G and edge set

$$E = \{\, \{x,y\} \mid y = xs \text{ for some } s \in S \,\}.$$

Definition 1.10. Γ is a *universal Cayley graph* if for every f.g. group K, there exists a f.g. group G with finite generating set S such that $K \hookrightarrow G$ and $\mathrm{Cay}(G,S) \cong \Gamma$.

Clearly if G, H are finitely generated groups with common Cayley graph Γ, then G and H are quasi-isometric; and it is well-known that G and H are also measure equivalent. (We will discuss the latter point in Section 2.)

Conjecture 1.11. There does not exists a universal Cayley graph.

This paper is organized as follows. In Sections 2, 3, 4 and 5, we will discuss Cayley graphs from the point of view of measurable group theory; and, in particular, we will present a number of counterexamples to an initially plausible approach to Conjecture 1.11. Finally, in Section 6, we will construct examples of uncountable collections of finitely generated groups with a common Cayley graph Γ which are pairwise not virtually isomorphic.

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2. Cayley graphs and measure equivalence

Let G be a finitely generated group and let $S \subseteq G \setminus 1$ be a finite symmetric generating set. Then G acts as a regular group of automorphisms of $\mathrm{Cay}(G,S)$ via left multiplication. (Recall that a group action is *regular* if it is both transitive and free.) Conversely, it is well-known that if a finitely generated group H acts as a regular group of automorphism on a connected graph Γ of finite degree, then there exists a finite symmetric generating set $T \subseteq H \setminus 1$ such that $\Gamma \cong \mathrm{Cay}(H,T)$.

Let $\Gamma = \mathrm{Cay}(G,S)$, let $\mathbb{G} = \mathrm{Aut}(\Gamma)$ and let $\mathbb{K} = \mathbb{G}_1$ be the stabilizer of the vertex $1 \in \Gamma$. Then $\mathbb{G}$ is a locally compact group and $\mathbb{K}$ is an open compact subgroup. Let μ be the left-invariant Haar measure on $\mathbb{G}$, normalized so that $\mu(\mathbb{K}) = 1$. Since $\mathbb{G} = \bigsqcup_{g \in G} g\mathbb{K}$, it follows that G is a cocompact lattice in $\mathbb{G}$. Consequently, $\mathbb{G}$ is unimodular and so μ is also right-invariant.

Next suppose that H is any finitely generated group with finite symmetric generating set $T \subseteq H \setminus 1$ such that $\mathrm{Cay}(H,T) \cong \Gamma$. Then G and H are clearly quasi-isometric. Also, identifying $\mathrm{Cay}(H,T)$ and Γ, we can define commuting measure-preserving free actions of G and H on $(\mathbb{G}, \mu)$ by

$$g \cdot x = gx \quad \text{and} \quad h \cdot x = xh^{-1}$$

for $g \in G$, $h \in H$ and $x \in \mathbb{G}$; and clearly $\mathbb{K}$ is a fundamental domain for both actions. Thus G and H are also measure equivalent.

Let $\mathbb{G}/H = \{\, xH \mid x \in \mathbb{G} \,\} = \{\, kH \mid k \in \mathbb{K} \,\}$; and, slightly abusing notation, let μ be the probability measure on $\mathbb{G}/H$ defined by

$$\mu(B) = \mu(\{\, k \in \mathbb{K} \mid kH \in B \,\}).$$

Then the natural action $G \curvearrowright \mathbb{G}/H$ is μ-preserving and is isomorphic to the measure-preserving Borel action $G \curvearrowright (\mathbb{K}, \mu)$ defined for $g \in G$ and $k \in \mathbb{K}$ by $g \cdot k = gkh^{-1}$, where $h \in H$ is the unique element such that $gkh^{-1} \in \mathbb{K}$. Furthermore, it is easily checked that the actions $G \curvearrowright (\mathbb{G}/H, \mu)$ and $H \curvearrowright (\mathbb{G}/G, \mu)$ are orbit equivalent via the map

$$kH \mapsto k^{-1}G, \qquad k \in \mathbb{K}.$$

Thus, identifying the above orbit equivalence relations as (X, E, μ), we see that G and H embed into the corresponding *full group*

$$[\, E\,] = \{\, T \in \mathrm{Aut}(\, X, \mu\,) \mid T(x)\; E\; x \text{ for } \mu\text{-a.e. } x \in X\,\}.$$

At first glance, this seems to suggest that the nonexistence of a universal Cayley graph can be established via an application of the following result of Ozawa [22] (as reformulated by Kechris [13, Section 14(C)]).

Theorem 2.1. *If* (X, E, μ) *is a measure-preserving countable Borel equivalence relation, then there exists a finitely generated group* G *such that* $G \not\hookrightarrow [\, E\,]$.

Unfortunately, as the following result illustrates, if Γ is a graph, then it is not necessarily the case that all of the actions

$$\{\, G \curvearrowright (\, \mathrm{Aut}(\Gamma)/H, \mu\,) \mid G, H \text{ have Cayley graph } \Gamma\,\}$$

are pairwise orbit equivalent.

Theorem 2.2. *If* $G \ncong H$ *are finitely generated groups with common Cayley graph* Γ *and* $\mathrm{Aut}(\Gamma)$ *is countable, then the actions*

$$G \curvearrowright (\, \mathrm{Aut}(\Gamma)/G, \mu\,) \quad \textit{and} \quad H \curvearrowright (\, \mathrm{Aut}(\Gamma)/G, \mu\,)$$

are not orbit equivalent.

Proof. Since $\mathrm{Aut}(\Gamma)$ is countable, it follows that the stabilizer $\mathbb{K}$ of the vertex $1 \in \Gamma$ is finite. Also

$$[\, \mathrm{Aut}(\Gamma) : G\,] = |\mathbb{K}| = [\, \mathrm{Aut}(\Gamma) : H\,];$$

and μ is the uniform probability measure on the finite space $\mathrm{Aut}(\Gamma)/G$. Suppose that the actions $G \curvearrowright (\, \mathrm{Aut}(\Gamma)/G, \mu\,)$ and $H \curvearrowright (\, \mathrm{Aut}(\Gamma)/G, \mu\,)$ are orbit equivalent. Note that the point $G \in \mathrm{Aut}(\Gamma)/G$ is a fixed point for the action $G \curvearrowright (\, \mathrm{Aut}(\Gamma)/G, \mu\,)$. It follows that there exists a fixed point $aG \in \mathrm{Aut}(\Gamma)/G$ for the action $H \curvearrowright (\, \mathrm{Aut}(\Gamma)/G, \mu\,)$. Thus $haG = aG$ for all $h \in H$ and so $a^{-1}Ha \leqslant G$. Since $[\, \mathrm{Aut}(\Gamma) : G\,] = [\, \mathrm{Aut}(\Gamma) : H\,]$, it follows that $a^{-1}Ha = G$, which contradicts the hypothesis that $G \ncong H$. $\square$

Example 2.3. Let $\Gamma = \langle \mathbb{Z}, E \rangle$ be the *standard Cayley graph of* $\mathbb{Z}$; i.e.

$$a \, E \, b \quad \Longleftrightarrow \quad |a - b| = 1.$$

Then Γ is a common Cayley graph of $\mathbb{Z}$ and the infinite dihedral group D_∞. Clearly $\operatorname{Aut}(\Gamma)$ is countable. Hence the actions

$$\mathbb{Z} \curvearrowright (\operatorname{Aut}(\Gamma)/\mathbb{Z}, \mu) \quad \text{and} \quad D_\infty \curvearrowright (\operatorname{Aut}(\Gamma)/\mathbb{Z}, \mu)$$

are not orbit equivalent.

Of course, the actions $G \curvearrowright (\operatorname{Aut}(\Gamma)/H, \mu)$ are of more interest when $\operatorname{Aut}(\Gamma)$ is uncountable; and, in this case, the analog of Theorem 2.2 can fail.

Theorem 2.4. *There exists a graph* Γ *such that there exist* $2^{\aleph_0}$ *groups* $G \in \mathcal{G}_{\mathrm{fg}}$ *up to isomorphism with Cayley graph* Γ *and such that the actions*

$$\{ G \curvearrowright (\operatorname{Aut}(\Gamma)/H, \mu) \mid \Gamma \text{ is a Cayley graph of } G, H \}$$

are pairwise orbit equivalent.

On the other hand, there also examples of Cayley graphs with uncountably many such actions up to orbit equivalence.

Theorem 2.5. *There exists a set* $\mathcal{G} = \{ G_\alpha \mid \alpha < 2^{\aleph_0} \}$ *pairwise nonisomorphic finitely generated groups with common Cayley graph* Γ *such that for each* $\alpha < 2^{\aleph_0}$,

$$\{ G_\beta \curvearrowright (\operatorname{Aut}(\Gamma)/G_\alpha, \mu) \mid \beta < 2^{\aleph_0} \}$$

contains $2^{\aleph_0}$ *actions up to orbit equivalence.*

The proofs of Theorems 2.4 and 2.5 will be presented in Sections 3 and 4.

3. Cayley graphs of amenable groups

In this section, we will present the proof of Theorem 2.4.

Definition 3.1. If (Γ_1, E_1) and (Γ_2, E_2) are graphs, then the *wreath product* $\Gamma_2 \wr \Gamma_1$ is the graph with vertex set $\Gamma_2 \times \Gamma_1$ such that the vertices $(u, v) \neq (x, y)$ are adjacent if either:

(i) v is adjacent to y in Γ_1; or
(ii) $v = y$ and u is adjacent to x in Γ_2.

Let $W = \mathbb{Z} \operatorname{wr} \mathbb{Z}$ be the restricted wreath product of the group $\mathbb{Z}$ with itself. By Leemann and de la Salle [15], there is a finite symmetric generating set S of W such that the corresponding Cayley graph $\Gamma_0 = \operatorname{Cay}(W, S)$ satisfies $\operatorname{Aut}(\Gamma_0) = W$. Let Δ be the complete graph on two vertices and let $\Gamma = \Delta \wr \Gamma_0$ be the wreath product of the graphs Γ_0 and Δ. Then, by Sabidussi [23], we have that $\operatorname{Aut}(\Gamma)$ is the unrestricted wreath product

$$(\prod_{w \in W} C_w) \rtimes W.$$

Here W denotes the canonical copy consisting of those $g \in \operatorname{Aut}(\Gamma)$ such that

$$g \cdot (v, w) = (v, gw), \qquad v \in \Delta \text{ and } w \in \Gamma_0 = W;$$

and C_w is the cyclic group of order 2 generated by the transposition which interchanges the two elements of $\Delta_w = \{ (v, w) \mid v \in \Delta \}$.

Lemma 3.2. *If $G \in \mathcal{G}_{\text{fg}}$, then Γ is a Cayley graph of G if and only if G is isomorphic to a central extension of W by a cyclic group C of order 2.*

Proof. Let G be a central extension of W by a cyclic group C of order 2 and let $\pi : G \to W$ be the canonical homomorphism. Let $T = \bigcup_{s \in S} \pi^{-1}(s) \cup (C \smallsetminus 1)$. Then clearly $\operatorname{Cay}(G, T) = \Gamma$. Conversely, suppose that Γ is a Cayley graph of G and let $\pi : G \to \operatorname{Aut}(\Gamma)$ be a corresponding embedding as a regular group of automorphisms. Then $\pi(G)$ permutes the sets $\{ \Delta_w \mid w \in W \}$ and acts as a transitive group of automorphisms on the corresponding quotient graph. It follows that there exists surjective homomorphism $\theta : G \to W$ such that

$$\pi(g)(\Delta_w) = \Delta_{\theta(g)w}, \qquad g \in G, w \in W.$$

Also, if $1 \neq z \in \ker \theta$, then $\pi(z)$ is the involution $\sigma \in \operatorname{Aut}(\Gamma)$ which interchanges interchanges the two elements of Δ_w for each $\in W$. It follows that $\ker \theta = Z(G)$ has order 2. □

The following result is due to Erschler [4, Section 3].

Theorem 3.3. *Up to isomorphism, there exist $2^{\aleph_0}$ central extensions G of $\mathbb{Z} \operatorname{wr} \mathbb{Z}$ by a cyclic group C of order 2.*

Let G, H be finitely generated groups with Cayley graph Γ; and identify G, H with suitable regular subgroups of $\mathrm{Aut}(\Gamma)$. Then

$$Z(G) = Z(H) = Z(\mathrm{Aut}(\Gamma)) = \langle\, \sigma \,\rangle,$$

where σ is the involution which interchanges interchanges the two elements of Δ_z for each $z \in W$. From now on, we write $Z = Z(\mathrm{Aut}(\Gamma))$. Consider the action

$$G \curvearrowright (\, \mathrm{Aut}(\Gamma)/H, \mu\,). \tag{3.1}$$

Note that we can identify the compact subgroup $\mathbb{K} = \prod_{1 \neq w \in W} C_w$ with $(\prod_{w \in W} C_w)/Z$ via the bijection $k \mapsto kZ$. With this identification, the measure preserving action (3.1) is isomorphic to the measure preserving action

$$G \curvearrowright (\, (\prod_{w \in W} C_w)/Z, \mu\,),$$

which is defined as follows. Let $g = ws \in G$, where $w \in W$ and $s \in \prod_{w \in W} C_w$. Then for each $kZ \in (\prod_{w \in W} C_w)/Z$,

$$g \cdot kZ = wskt^{-1}w^{-1}Z,$$

where $t \in \prod_{w \in W} C_w$ is the unique element such that $wt \in H$. Notice that there is a *fixed* element $\tau_g = wst^{-1}w^{-1} \in \prod_{w \in W} C_w$ such that for each $kZ \in (\prod_{w \in W} C_w)/Z$,

$$g \cdot kZ = \tau_g wkw^{-1}Z.$$

When $g = \sigma$, then $w = 1$ and $s = t = \sigma$; and it follows that $\sigma \cdot kZ = kZ$ for all $kZ \in (\prod_{w \in W} C_w)/Z$. Thus we obtain an induced action

$$W \curvearrowright (\, (\prod_{w \in W} C_w)/Z, \mu\,),$$

given by

$$w \cdot kZ = \tau_g wkw^{-1}Z, \tag{3.2}$$

where $g \in G$ is either of the two elements such that $g = ws$ for some $s \in \prod_{w \in W} C_w$.

Proposition 3.4. *The action $W \curvearrowright (\, (\prod_{w \in W} C_w)/Z, \mu\,)$ is strongly mixing.*

Proof. Slightly abusing notation, we will identify $\prod_{w\in W} C_w$ with the Cantor space 2^W. With this identification, passing to the quotient $(\prod_{w\in W} C_w)/Z$ corresponds to identifying each $f \in 2^W$ with the function $\bar{f}$, defined by $\bar{f}(w) = 1 - f(w)$. The basic clopen subsets of $(\prod_{w\in W} C_w)/Z$ are

$$N_\varphi/Z = \{\, f \in 2^W \mid f \restriction \mathrm{dom}(\varphi) = \varphi \text{ or } \bar{f} \restriction \mathrm{dom}(\varphi) = \varphi \,\}/Z,$$

where φ varies over all maps $\varphi : \mathrm{dom}(\varphi) \to 2$ for some finite subset $\mathrm{dom}(\varphi)$ of W. Note that if $\tau \in \prod_{w\in W} C_w$, then there exists $\varphi' : \mathrm{dom}(\varphi) \to 2$ such that $\tau N_\varphi/Z = N_{\varphi'}/Z$. Hence, using Equation (3.2), we see that if $w \in W$, then there exists $\psi : w(\mathrm{dom}(\varphi)) \to 2$ such that $w \cdot N_\varphi/Z = N_\psi/Z$.

Since finite unions of basic clopen sets are dense in the measure algebra of $(\prod_{w\in W} C_w)/Z, \mu\,)$, it is enough to verify the mixing condition when A and B are finite unions of basic clopen sets. Suppose that $A = N_{\varphi_1}/Z \cup \cdots \cup N_{\varphi_n}/Z$ and $B = N_{\theta_1}/Z \cup \cdots \cup N_{\theta_m}/Z$. Then for all but finitely many $w \in W$,

$$w(\mathrm{dom}(\varphi_1) \cup \mathrm{dom}(\varphi_n)) \cap (\mathrm{dom}(\theta_1) \cup \mathrm{dom}(\theta_m)) = \emptyset$$

and so the events $w \cdot A$ and B are independent, which implies that

$$\mu(w \cdot A \cap B) = \mu(w \cdot A)\mu(B) = \mu(A)\mu(B).$$

This completes the proof of Proposition 3.4. □

Next let $1 \neq w \in W$. Then the infinite cyclic subgroup $D = \langle\, w \,\rangle$ acts ergodically on $(\prod_{w\in W} C_w)/Z$. Since $Y = \{\, kZ \in (\prod_{w\in W} C_w)/Z \mid w \cdot k = k \,\}$ is D-invariant, it follows that $\mu(Y) = 0$. Thus W acts μ-a.e. freely on $(\prod_{w\in W} C_w)/Z$. Applying Ornstein-Weiss [21], since W is amenable, it follows that all of its free ergodic actions are orbit equivalent. This completes the proof of Theorem 2.4.

4. Cayley graphs of Kazhdan groups

In this section, we will present the proof of Theorem 2.5. Let L be a noncyclic torsion-free hyperbolic Kazhdan group. (For example, we can let L be a co-compact lattice in $Sp(n,1)$ for some $n \geq 2$. See de la Harpe-Valette [11].) Let A be a finite generating set for L. Then L can be presented as $\langle\, A \mid S \,\rangle$, where S is the set of all words in A which are equal to 1 in L. By Kulikova [14, Theorem 7], we can extend $\langle\, A \mid S \,\rangle$ to a presentation of a nonabelian torsion-free simple Kazhdan group

$G = \langle\, A \mid S \cup R \,\rangle$ which has a "well-behaved" central extension. In more detail, let F be the free group on the set A and let $\Phi : F \to L = \langle\, A \mid S \,\rangle$ be the canonical homomorphism. Let $N = \ker \Phi$ and let $\bar{N}_R = \Phi(N_R)$, where N_R is the normal closure of R in F. Then $G = L/\bar{N}_R$. Consider the group

$$H = L/[\,\bar{N}_R, L\,] = F/[\,N_R, F\,]N.$$

Then H is torsion-free central extension of G with center

$$\bar{N}_R/[\,\bar{N}_R, L\,] = N_R N/[\,N_R, F\,]N.$$

Furthermore, $\bar{N}_R/[\,\bar{N}_R, L\,]$ is a free abelian group with countably infinite basis $\{\, \bar{r} \mid r \in R \,\}$, where $\bar{r} = rN/[\,N_R, F\,]N$.

Let $C = \langle\, \sigma \,\rangle$ be the cyclic group of order 5; and for each nonempty subset $\emptyset \neq X \subseteq R$, let $\psi_X : \bar{N}_R/[\,\bar{N}_R, L\,] \to C$ be the homomorphism such that

$$\psi_X(\bar{r}) = \begin{cases} \sigma, & \text{if } r \in X; \\ 1, & \text{if } r \notin X. \end{cases}$$

Let $M_X = \ker \psi_X$ and let $H_X = H/M_X$. Then H_X is a central extension of G such that $|Z(H_X)| = 5$; and it follows that H_X is also a Kazhdan group. Note that the word $r \in R$ is equal to 1 in H_X if and only if $r \notin X$. It follows easily that

$$\{\, H_X \mid \emptyset \neq X \subseteq R \,\}$$

contains a collection $\mathcal{G} = \{\, G_\alpha \mid \alpha < 2^{\aleph_0} \,\}$ of pairwise nonisomorphic groups. By Shalom [25, Theorem 6.7], every finitely generated Kazhdan group is a homomorphic image of a finitely presented Kazhdan group. Hence we can suppose that there exists a *fixed* finitely presented Kazhdan group K such that for every $\alpha < 2^{\aleph_0}$, there exists a surjective homomorphism $\pi_\alpha : K \to G_\alpha$.

By Leemann and de la Salle [15], there exists a finite symmetric generating set B of G such that the corresponding Cayley graph $\Gamma_0 = \mathrm{Cay}(G, B)$ satisfies $\mathrm{Aut}(\Gamma_0) = G$. Let Δ be the complete graph on five vertices and let $\Gamma = \Delta \operatorname{wr} \Gamma_0$ be the wreath product of Γ_0 and Δ. Then, by Sabidussi [23], we have that $\mathrm{Aut}(\Gamma)$ is the unrestricted wreath product

$$(\, \prod_{g \in G} S_g \,) \rtimes G.$$

Here G denotes the canonical copy consisting of those $g \in \mathrm{Aut}(\Gamma)$ such that

$$g \cdot (v, h) = (v, gh), \qquad v \in \Delta \text{ and } h \in \Gamma_0 = G;$$

and $S_g = \mathrm{Sym}(\Delta_g) \cong \mathrm{Sym}(5)$ is the symmetric group on $\Delta_g = \{ (v, g) \mid v \in \Delta \}$.

Remark 4.1. Arguing as in the proof of Lemma 3.2, we see that if $E \in \mathcal{G}_{\mathrm{fg}}$, then Γ is a Cayley graph of E if and only if E is isomorphic to a central extension of G by a cyclic group C of order 5. In particular, if Γ is a Cayley graph of E_1 and E_2, then E_1 and E_2 are virtually isomorphic. In Section 6, we will present an example of a collection $\mathcal{G} = \{ G_\alpha \mid \alpha < 2^{\aleph_0} \}$ of finitely generated groups with common Cayley graph Γ which are pairwise not virtually isomorphic.

Fix some $\alpha < 2^{\aleph_0}$ and let $\varphi_\alpha : G_\alpha \to G = G_\alpha / Z(G_\alpha)$ be the canonical homomorphism. Then Γ is the Cayley graph of G_α with respect to the generating set

$$B_\alpha = \{ z \in Z(G_\alpha) \smallsetminus 1 \} \cup \bigcup_{b \in B} \varphi_\alpha^{-1}(b).$$

Notice that $Z(G_\alpha) = \langle z_\alpha \rangle$, where

$$z_\alpha = (\sigma_{\alpha,g} \mid g \in G) \in \prod_{g \in G} S_g$$

for some 5-cycle $\sigma_{\alpha,g} \in S_g = \mathrm{Sym}(\Delta_g)$.

Now fix some $\beta < 2^{\aleph_0}$ and consider the action

$$G_\beta \curvearrowright (\mathrm{Aut}(\Gamma)/G_\alpha, \mu).$$

Note that we can make the identification:

$$\mathrm{Aut}(\Gamma)/G_\alpha = (\prod_{g \in G} S_g)/Z(G_\alpha).$$

Suppose that $z = (z_g \mid g \in G) \in Z(G_\beta) \smallsetminus 1$ and $k = (k_g \mid g \in G) \in \prod_{g \in G} S_g$ satisfy

$$z \cdot kZ(G_\alpha) = zkZ(G_\alpha) = kZ(G_\alpha).$$

Then $k^{-1}zk \in Z(G_\alpha)$; and so $k_g^{-1} z_g k_g \in \langle \sigma_{\alpha,g} \rangle$ for all $g \in G$. Thus there are exactly $|\,\mathrm{Sym}(5)|/6$ possibilities for each component k_g of k; and it follows that $Z(G_\beta)$ acts μ-a.e. freely on $\mathrm{Aut}(\Gamma)/G_\alpha$.

From now on, we will write $X = \mathrm{Aut}(\Gamma)/G_\alpha$. Suppose that there exists an uncountable subset $I \subseteq 2^{\aleph_0}$ such that the actions

$$G_\beta \curvearrowright (X, \mu), \qquad \beta \in I,$$

are pairwise orbit equivalent. For each $\beta \in I$, let $E^X_{G_\beta}$ be the corresponding orbit equivalence relation. Then, after conjugating the actions by suitable elements of $\mathrm{Aut}(X, \mu)$, we can suppose that there exists a fixed measure preserving equivalence relation E such that $E^X_{G_\beta} = E$ for all $\beta \in I$. Recall that there exists a fixed (finitely presented) Kazhdan group K such that for every $\beta \in I$, there exists a surjective homomorphism $\pi_\beta : K \to G_\beta$. Furthermore, since the groups G_β, $\beta \in I$, are nonisomorphic, it follows that the kernels $N_\beta = \ker(\pi_\beta)$ are distinct. Let $A(K, X, \mu)$ be the space of measure preserving actions of K on (X, μ), equipped with the uniform topology. (For more details, see Kechris [13].) For each $\beta \in I$, let $a_\beta \in A(K, X, \mu)$ be the action defined by

$$a_\beta(g) \cdot x = \pi_\beta(g) \cdot x, \qquad g \in K, x \in X.$$

Let $[[E]]$ be the set of all partial Borel automorphisms $\psi : A \to B$ of (X, μ), where A, B are Borel subsets of X, such that $\psi(x)\, E\, x$ for all $x \in A$. Then, applying Kechris [13, Lemma 14.1], the separability of

$$\{\, a \in A(K, X, \mu) \mid a(g) \in [E] \text{ for all } g \in K \,\}$$

in the uniform topology implies that there exist $\beta \neq \gamma \in I$ and $\psi \in [[E]]$ such that

(i) $A = \mathrm{dom}\,\psi$ is a_β-invariant and $B = \mathrm{ran}\,\psi$ is a_γ-invariant;
(ii) $\psi \circ (a_\beta \restriction A) \circ \psi^{-1} = a_\gamma \restriction B$; and
(iii) $\mu(A) > 1/2$.

Since $N_\beta \neq N_\gamma$, without loss of generality, we can suppose that $N_\gamma \nsubseteq N_\beta$. Thus $\pi_\beta(N_\gamma)$ is a nontrivial normal subgroup of G_β. Since the only nontrivial normal subgroups of G_β are $Z(G_\beta)$ and G_β, it follows that $Z(G_\beta) \subseteq \pi_\beta(N_\gamma)$. But if $h \in N_\gamma$, then $a_\gamma(h) \restriction B = \mathrm{id}$ and hence $a_\beta(h) \restriction A = \mathrm{id}$. It follows that $Z(G_\beta)$ fixes A pointwise, which contradicts that fact that that $Z(G_\beta)$ acts μ-a.e. freely on X. This completes the proof of Theorem 2.5.

5. Burger-Mozes groups

In this section, we will present examples of groups G, H with common Cayley graph Γ such that G, H are not virtually isomorphic and the actions

$$G \curvearrowright (\operatorname{Aut}(\Gamma)/G, \mu) \quad \text{and} \quad H \curvearrowright (\operatorname{Aut}(\Gamma)/G, \mu)$$

are not stably orbit equivalent. In preparation, we will first prove an ergodicity result for groups acting on regular trees.

Fix some $n \geq 3$. Let T be the regular tree of degree n and H be a finitely generated group with finite symmetric generating set $S \subseteq H \smallsetminus 1$ such that $\operatorname{Cay}(H, S) = T$. Let $\mathbb{G} = \operatorname{Aut}(T)$ and let μ be the Haar measure on $\mathbb{G}$, normalized so that the stabilizer $\mathbb{K}$ of the vertex $1 \in T$ satisfies $\mu(\mathbb{K}) = 1$.

Theorem 5.1. *If $G \leqslant \mathbb{G}$ is a countable subgroup which acts transitively on T, then the action $G \curvearrowright (\mathbb{G}/H, \mu)$ is strongly mixing.*

Proof. Let d be the path metric on T and let E be the $\mathbb{G}$-invariant equivalence relation on T defined by

$$a \, E \, b \quad \Longleftrightarrow \quad d(a, b) \equiv 0 \pmod 2.$$

Let $\mathbb{G}_0$ be the normal subgroup of index 2 in $\mathbb{G}$ which fixes the two E-classes setwise. Then, by Lubotzky-Mozes [16], $\mathbb{G}_0$ has the Howe-Moore property.

Since H acts transitively on T, it follows that $H \nleqslant \mathbb{G}_0$; and hence if $xH \in \mathbb{G}/H$, then there exists $y \in \mathbb{G}_0$ such that $yH = xH$. It follows that $\mathbb{G}_0$ acts transitively and hence ergodically on $(\mathbb{G}/H, \mu)$. Since G acts transitively on T and $[G : G \cap \mathbb{G}_0] = 2$, it follows that there does not exist a vertex $t \in T$ which is fixed by $G \cap \mathbb{G}_0$; and so $G \cap \mathbb{G}_0$ has a noncompact closure in $\mathbb{G}_0$. Since $\mathbb{G}_0$ has the Howe-Moore property, it follows that the action of $G \cap \mathbb{G}_0$ on $(\mathbb{G}/H, \mu)$ is strongly mixing; and this implies that $G \curvearrowright (\mathbb{G}/H, \mu)$ is also strongly mixing. □

For each $n \geq 2$, let T_{2n} be the regular tree of degree $2n$. Then the following result is an immediate consequence of Hammack-Imrich-Klavžar [9, Corollary 6.12].

Proposition 5.2. *If $m \neq n$, then* $\operatorname{Aut}(T_{2m} \times T_{2n}) = \operatorname{Aut}(T_{2m}) \times \operatorname{Aut}(T_{2n})$.

For each $n \geq 2$, let $\mathbb{G}_n = \operatorname{Aut}(T_{2n})$ and let μ_n be the corresponding Haar measure on $\mathbb{G}_n$, normalized so that the stabilizer $\mathbb{K}_n$ of a vertex satisfies $\mu_n(\mathbb{K}_n) = 1$. Let

$\mathbb{G} = \mathrm{Aut}(T_{2m} \times T_{2n}) = \mathrm{Aut}(T_{2m}) \times \mathrm{Aut}(T_{2n}) = \mathbb{G}_m \times \mathbb{G}_n$ and let $\mu = \mu_m \times \mu_n$ be the canonical Haar measure on $\mathbb{G}$. Note that

$$(\,\mathbb{G}/(F_m \times F_n), \mu\,) = (\,\mathbb{G}_m/F_m \times \mathbb{G}_n/F_n, \mu_m \times \mu_n\,).$$

Lemma 5.3. *If m, $n \geq 2$, then the action*

$$F_m \times F_n \curvearrowright (\,\mathbb{G}_m/F_m \times \mathbb{G}_n/F_n, \mu_m \times \mu_n\,)$$

is ergodic.

Proof. By Theorem 5.1, the actions $F_m \curvearrowright (\,\mathbb{G}_m/F_m, \mu_m\,)$ and $F_n \curvearrowright (\,\mathbb{G}_n/F_n, \mu_n\,)$ are ergodic; and the result follows. □

By Burger-Mozes [2], for suitable values of $m \neq n$, there exists a torsion-free group G with finite symmetric generating set $S \subseteq G \smallsetminus 1$ such that:

(i) $\mathrm{Cay}(G, S) = T_{2m} \times T_{2n}$;

(ii) G has a simple subgroup G_0 of finite index;

(iii) G can be expressed as a nontrivial free product with amalgamation $A *_B C$ of finitely generated free groups A, B and C.

Lemma 5.4. *The action $G \curvearrowright (\,\mathbb{G}_m/F_m \times \mathbb{G}_n/F_n, \mu_m \times \mu_n\,)$ is ergodic.*

Proof. For each $i \in \{\,m, n\,\}$, let $p_i : G \to Aut(T_{2i})$ be the corresponding projection map and let $N_i = \ker p_i$. Suppose that $N_i \neq 1$. Then, since G is torsion-free, it follows that $N_i \cap G_0 \neq 1$ and hence $G_0 \leqslant N_i$. But then G/N_i is finite, which is impossible since G acts transitively on T_{2i}. Thus p_i is an injection for both $i \in \{\,m, n\,\}$. By Theorem 5.1, since each $p_i(G)$ acts transitively on T_{2i}, it follows that the actions

$$p_i(G) \curvearrowright (\,\mathbb{G}_i/F_i, \mu_i\,), \quad i \in \{\,m, n\,\},$$

are strongly mixing; and hence the action $G \curvearrowright (\,\mathbb{G}_m/F_m \times \mathbb{G}_n/F_n, \mu_m \times \mu_n\,)$ is ergodic. □

Lemma 5.5. *The action $G \curvearrowright (\,\mathbb{G}_m/F_m \times \mathbb{G}_n/F_n, \mu_m \times \mu_n\,)$ is essentially free.*

Proof. We will work with the isomorphic action $G \curvearrowright \mathbb{K} = \mathbb{K}_m \times \mathbb{K}_n$. Let $1 \neq g \in G$. Suppose that $k \in \mathbb{K}$ is such that $g \cdot k = k$. Then there exists $h \in F_m \times F_n$ such that

$gkh^{-1} = k$ and so $k^{-1}gk = h \in F_m \times F_n$. It follows that if $\mu(\{ k \in \mathbb{K} | g \cdot k = k \}) > 0$, then $\mu(C_{\mathbb{K}}(g)) > 0$. Thus it is enough to show that $\mu(C_{\mathbb{G}}(g)) = 0$.

For each $i \in \{ m, n \}$, let $p_i : G \to Aut(T_{2i})$ be the corresponding projection map. Since G acts freely on $T_{2m} \times T_{2n}$, it follows that there exists $i \in \{ m, n \}$ such that $h = p_i(g)$ has no fixed points in T_{2i}. Thus h is a hyperbolic automorphism of T_{2i}; and hence there exists a unique doubly infinite path L through T_{2i} such that h fixes L setwise. (For example, see Serre [24, Section 6.4].) Suppose that $a \in C_{\mathbb{G}_i}(h)$. Then $a(L)$ is also h-invariant and so $a(L) = L$. Since $\mathrm{Aut}(T_{2i})$ acts transitively on the set of doubly infinite path through T_{2i}, it follows that $\mu_i(C_{\mathbb{G}_i}(h)) = 0$; and this implies that $\mu(C_{\mathbb{G}}(g)) = 0$. □

Theorem 5.6. *The measure-preserving actions $G \curvearrowright (\mathbb{G}_m/F_m \times \mathbb{G}_n/F_n, \mu_m \times \mu_n)$ and $F_m \times F_n \curvearrowright (\mathbb{G}_m/F_m \times \mathbb{G}_n/F_n, \mu_m \times \mu_n)$ are not stably orbit equivalent.*

Proof. Recall that G can be expressed as a nontrivial free product with amalgamation $A *_B C$ of finitely generated free groups A, B and C. Thus the result follows from Adams [1, Corollary 4.3]. □

6. Uncountably many virtual isomorphism classes

In this section, we will prove the following result.

Theorem 6.1. *There exist uncountable collections $\mathcal{H} = \{ H_\alpha \mid \alpha < 2^{\aleph_0} \}$ of finitely generated groups with a common Cayley graph Γ which are pairwise not virtually isomorphic. Furthermore, $\mathcal{H}$ can be chosen to consist of either amenable groups or nonamenable groups.*

Let G, H be finitely generated groups with Cayley graphs Γ_1, Γ_2. Then the classical wreath product $\Gamma_2 \wr \Gamma_1$, defined in Section 3, satisfies

$$\mathrm{Aut}(\Gamma_2 \wr \Gamma_1) = \mathrm{Aut}(\Gamma_2) \,\mathrm{wr}\, \mathrm{Aut}(\Gamma_1).$$

In [6], Erschler introduced a different graph $\Gamma_2 \,\mathrm{wr}\, \Gamma_1$ which is a Cayley graph of $H \,\mathrm{wr}\, G$.

Definition 6.2. Let (Γ_1, E_1) and (Γ_2, E_2) be graphs and let $v_0 \in \Gamma_2$ be a fixed vertex. If $f : \Gamma_1 \to \Gamma_2$, then we define its *support* to be

$$\mathrm{supp}(f) = \{ u \in \Gamma_1 \mid f(u) \neq v_0 \}.$$

The *generalized wreath product* $\Gamma_2 \operatorname{wr} \Gamma_1$ is the graph with vertices (f, u), where $u \in \Gamma_1$ and $f : \Gamma_1 \to \Gamma_2$ has finite support. The vertices $(f_1, u_1) \neq (f_2, u_2)$ are adjacent if either:

(i) $f_1 = f_2$ and u_1, u_2 are adjacent in Γ_1; or

(ii) $u_1 = u_2$, $f_1(u) = f_2(u)$ for all $u \neq u_1$, and $f_1(u_1)$, $f_2(u_1)$ are adjacent in Γ_2.

Proposition 6.3 (Erschler [6])**.** *If Γ_1, Γ_2 are Cayley graphs of G, $H \in \mathcal{G}_{\mathrm{fg}}$, then $\Gamma_2 \operatorname{wr} \Gamma_1$ is a Cayley graph of $H \operatorname{wr} G$.*

The following result was proved in Thomas [26, Theorem 2.5].

Proposition 6.4. *If S is an infinite finitely generated simple group and A, B are arbitrary groups, then*

$$A \cong B \iff (\mathrm{Alt}(5) \operatorname{wr} A) \operatorname{wr} S \approx_{VI} (\mathrm{Alt}(5) \operatorname{wr} B) \operatorname{wr} S.$$

Proof of Theorem 6.1. Applying Theorem 3.3, let $\mathcal{G} = \{ G_\alpha \mid \alpha < 2^{\aleph_0} \}$ be a set of pairwise nonisomorphic central extensions G of $\mathbb{Z} \operatorname{wr} \mathbb{Z}$ by a cyclic group C of order 2. Then, by Lemma 3.2, the elements of $\mathcal{G}$ have a common Cayley graph. Let S be any infinite finitely generated simple group; and for each $\alpha < 2^{\aleph_0}$, let

$$H_\alpha = (\mathrm{Alt}(5) \operatorname{wr} G_\alpha) \operatorname{wr} S.$$

By Proposition 6.3, the elements of $\mathcal{H} = \{ H_\alpha \mid \alpha < 2^{\aleph_0} \}$ have a common Cayley graph; and by Proposition 6.4, the elements of $\mathcal{H}$ are pairwise not virtually isomorphic. Finally note that H_α is amenable if and only if the simple group S is amenable. By Juschenko-Monod [12], there exist infinite amenable finitely generated simple groups; and it is well-known that there also exist nonamenable finitely generated simple groups. □

References

[1] S. Adams, *Indecomposability of treed equivalence relations*, Israel J. Math. **64** (1988), 362–380.

[2] M. Burger and S. Mozes, *Lattices in product of trees*, Inst. Hautes Études Sci. Publ. Math. **92** (2000), 151–194.

[3] A. Erschler, *On isoperimetric profiles of finitely generated groups*, Geom. Dedicata **100** (2003), 157–171.

[4] A. Erschler, *Not residually finite groups of intermediate growth, commensurability and non-geometricity*, J. Algebra **272** (2004), 154–172.

[5] A. Erschler, *Piecewise automatic groups*, Duke Math. J. **134** (2006), 591–613.

[6] A. Erschler, *Generalized wreath products*, Int. Math. Res. Not. 2006, Art. ID 57835, 14 pp.

[7] A. Furman, *A survey of measured group theory*, in: *Geometry, Rigidity, and Group Actions*, Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL, 2011, pp. 296–374.

[8] D. Gaboriau, *On orbit equivalence of measure preserving actions*, in: *Rigidity in Dynamics and Geometry*, Springer, 2002, pp. 167–186.

[9] R. Hammack, W. Imrich and S. Klavžar, *Handbook of Product Graphs*, second ed., CRC Press, Boca Raton, FL, 2011.

[10] P. de la Harpe, *Topics in Geometric Group Theory*, Chicago Lectures in Mathematics Series, The University of Chicago Press, Chicago, 2000.

[11] P. de la Harpe and A. Valette, *La propriété (T) de Kazhdan pour les groupes localement compacts (avec un appendice de Marc Burger)*, Astérisque **175** (1989).

[12] K. Juschenko and N. Monod, *Cantor systems, piecewise translations and simple amenable groups*, Ann. of Math. **178** (2013), 775–787.

[13] A. S. Kechris, *Global aspects of ergodic group actions*, Math. Surveys Monogr. **160**, American Mathematical Society, Providence, RI, 2010.

[14] O. V. Kulikova, *On some quotients of hyperbolic groups*, Math. Notes **114** (2023), 99–107.

[15] P-H. Leemann and M. de la Salle, *Cayley graphs with few automorphisms: the case of infinite groups*, Ann. H. Lebesgue **5** (2022), 73–92.

[16] A. Lubotzky and S. Mozes, *Asymptotic properties of unitary representations of tree automorphisms*, in: *Harmonic analysis and discrete potential theory* (Frascati, 1991), Plenum, New York, 1992, pp. 289–298.

[17] D. A. Martin, *The axiom of determinacy and reduction principles in the analytical hierarchy*, Bull. Amer. Math. Soc. **74** (1968), 687–689.

[18] D. A. Martin, *Borel determinacy*, Ann. Math. **102** (1975), 363–371.

[19] B. H. Neumann, *Some remarks on infinite groups*, J. London Math. Soc. **12** (1937), 120-127.

[20] P. M. Neumann, *On the structure of standard wreath products of groups*, Math. Z. **84** (1964) 343–373.

[21] D. S. Ornstein and B. Weiss, *Ergodic theory of amenable group actions I: The Rohlin lemma*, Bull. Amer. Math. Soc. **2** (1980), 161–164.

[22] N. Ozawa, *There is no separable universal* $\mathbf{II}_1$*-factor*, Proc. Amer. Math. Soc. **132** (2004), 487–490.

[23] G. Sabidussi, *The composition of graphs*, Duke Math. J. **26** (1959), 693–696.

[24] J. P. Serre, *Trees*, Springer Monogr. Math., Springer, Berlin, 2003.

[25] Y. Shalom, *Rigidity of commensurators and irreducible lattices*, Invent. Math. **141** (2000), 1–54.

[26] S. Thomas, *The virtual isomorphism problem for finitely generated groups*, Bull. London Math. Soc. **35** (2003), 777–784.

[27] S. Thomas, *Cayley graphs of finitely generated groups*, Proc. Amer. Math. Soc. **134** (2006), 289–294.

[28] S. Thomas, *The Friedman embedding theorem*, J. Algebra 332 (2011), 322–333.

[29] S. Thomas and B. Velickovic, *On the complexity of the isomorphism relation for finitely generated groups*, J. Alg. **217** (1999), 352–373.

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