MARTIN'S CONJECTURE AND STRONG ERGODICITY

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Abstract. In this paper, we explore some of the consequences of Martin's Conjecture on degree invariant Borel maps. These include the strongest conceivable ergodicity result for the Turing equivalence relation with respect to the filter on the degrees generated by the cones, as well as the statement that the complexity of a weakly universal countable Borel equivalence relation always concentrates on a null set.

1. Introduction

Recall that if r, $s \in 2^{\mathbb{N}}$, then r is *Turing reducible* to s, written $r \leq_T s$, iff there exists an oracle Turing machine which computes r when its oracle tape contains s. Here an oracle Turing machine is a Turing machine with a second "read only" tape, called the *oracle tape*, upon which we can write any function $s \in 2^{\mathbb{N}}$, which is called the *oracle*. (For more details, see Rogers [17] or Soare [19].) If r, $s \in 2^{\mathbb{N}}$, then r and s are *Turing equivalent*, written $r \equiv_T s$, iff both $r \leq_T s$ and $s \leq_T r$. In this paper, we will study the Turing equivalence relation $\equiv_T$ from the perspective of the theory of countable Borel equivalence equivalence relations.

For each $r \in 2^{\mathbb{N}}$, the corresponding *cone* is the Borel subset $C \subseteq 2^{\mathbb{N}}$ defined by

$$C = \{\, s \in 2^{\mathbb{N}} \mid r \leq_T s \,\}.$$

By Martin [13, 14], if $A \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset, then either A contains a cone or else $2^{\mathbb{N}} \smallsetminus A$ contains a cone. (Here a subset $A \subseteq 2^{\mathbb{N}}$ is said to be $\equiv_T$-*invariant* iff A is a union of $\equiv_T$-classes.) This easily implies that if $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel function which takes a constant value on each $\equiv_T$-class, then there exists a cone C such that $f \restriction C$ is a constant function. As Friedman [7] has pointed out, this can be regarded as an ergodicity result for $\equiv_T$ with respect to the filter on the degrees generated by the cones. In this paper, we will show that if Martin's

Research partially supported by NSF Grant DMS 0600940.

Conjecture [11] on degree invariant Borel maps is true, then $\equiv_T$ satisfies a much stronger ergodicity result.

Before we can give a precise statement of our main result, we first need to recall some of the basic notions of the theory of countable Borel equivalence relations. Suppose that E, F are countable Borel equivalence relations on the standard Borel spaces X, Y respectively. Then the (not necessarily Borel) map $f : X \to Y$ is said to be a *homomorphism* from E to F iff $x \mathrel{E} y$ implies $f(x) \mathrel{F} f(y)$ for all $x, y \in X$. If f is a Borel map and satisfies the stronger condition that $x \mathrel{E} y$ iff $f(x) \mathrel{F} f(y)$ for all $x, y \in X$, then f is said to be a *Borel reduction* and we write $E \leq_B F$. Finally, if there exists a countable-to-one Borel homomorphism $f : X \to Y$ from E to F, then we say that E is *weakly Borel reducible* to F and write $E \leq_B^w F$. In this case, we say that f is a *weak Borel reduction* from E to F.

Definition 1.1. (a) A countable Borel equivalence relation E is said to be *universal* iff $F \leq_B E$ for every countable Borel equivalence relation F.

(b) A countable Borel equivalence relation E is said to be *weakly universal* iff $F \leq_B^w E$ for every countable Borel equivalence relation F.

For example, by Dougherty-Jackson-Kechris [4], the orbit equivalence relation E_∞ arising from the shift action of the free group $\mathbb{F}_2$ on two generators on $2^{\mathbb{F}_2}$ is a universal countable Borel equivalence relation. Of course, if E is a universal countable Borel equivalence relation, then E is weakly universal. It is currently not known whether the converse holds. However, Kechris [22, Corollary 4.9] has pointed out that the Turing equivalence relation $\equiv_T$ is weakly universal; and Dougerty-Kechris [5] have shown that if Martin's Conjecture holds, then $\equiv_T$ is not universal. (The material in Thomas [22, Section 4] is entirely due to Kechris and Miller.)

In this paper, by Martin's Conjecture, we will always mean the following special case of the more general conjecture (also known as the 5th Victoria Delfino Problem) formulated by Martin in Kechris-Moschovakis [11].

Martin's Conjecture (MC)**.** *If $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T$ to $\equiv_T$, then exactly one of the following conditions holds:*

(i) *There exists a cone $C \subseteq 2^{\mathbb{N}}$ such that f maps C into a single $\equiv_T$-class.*

(ii) *There exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $x \leq_T f(x)$ for all $x \in C$.*

Throughout this paper, we will write (MC) to indicate that the (currently known) proof of a given statement makes use of Martin's Conjecture. In Section 4, we will prove the following result.

Theorem 1.2 (MC)**.** *There exist uncountably many weakly universal countable Borel equivalence relations up to Borel bireducibility.*

Theorem 1.2 is a straightforward consequence of the techniques of Thomas [22], together with the following strong ergodicity result for $\equiv_T$. (For the standard measure-theoretic version of strong ergodicity, see Definition 4.1.)

Definition 1.3. Let E be a countable Borel equivalence relation on the standard Borel space X. Then $\equiv_T$ is said to be *E-m-ergodic* iff for every Borel homomorphism $f : 2^{\mathbb{N}} \to X$ from $\equiv_T$ to E, there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that f maps C into a single E-class.

Theorem 1.4 (MC)**.** *If E is any countable Borel equivalence relation, then exactly one of the following conditions holds:*

(a) *E is weakly universal.*

(b) *$\equiv_T$ is E-m-ergodic.*

Of course, if E is a weakly universal countable Borel equivalence relation, then $\equiv_T \leq^w_B E$ and hence $\equiv_T$ is not E-m-ergodic. Thus conditions 1.4(a) and 1.4(b) are mutually exclusive.

As we mentioned earlier, Martin's Theorem on the $\equiv_T$-invariant Borel subsets of $2^{\mathbb{N}}$ easily implies that $\equiv_T$ is $\Delta(2^{\mathbb{N}})$-m-ergodic, where $\Delta(2^{\mathbb{N}})$ denotes the identity relation on $2^{\mathbb{N}}$. On the other hand, there are currently no nonsmooth countable Borel equivalence relations E for which it has been proved that $\equiv_T$ is E-m-ergodic. In particular, it is not known whether $\equiv_T$ is E_0-m-ergodic, where E_0 denotes the Vitali equivalence relation on $2^{\mathbb{N}}$.

This paper is organized as follows. In Section 2, we will point out some simple but useful consequences of Martin's Conjecture. In Section 3, we will prove Theorem 1.4; and in Section 4, we will prove Theorem 1.2. In Section 5, we will present two easy applications of the results of the earlier sections. First, partially answering a question of Boykin-Jackson [3], we will prove that Martin's Conjecture implies that the Turing equivalence relation $\equiv_T$ is not Borel-Bounded. Then, partially answering a question of Thomas [22], we will prove that Martin's Conjecture implies that the complexity of every weakly universal countable Borel equivalence relation concentrates on a null set.

Acknowledgements: I would like to thank Alexander Kechris and John Steel for very helpful discussions concerning the material in this paper. Thanks are also due to the referee for many suggestions that helped to improve the presentation.

2. Some Simple Consequences of Martin's Conjecture

In this section, we will point out some simple but useful consequences of Martin's Conjecture.

Theorem 2.1 (*MC*)**.** *If $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T$ to $\equiv_T$, then exactly one of the following conditions holds:*

(i) *There exists a cone $C \subseteq 2^{\mathbb{N}}$ such that f maps C into a single $\equiv_T$-class.*

(ii) *There exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $f \restriction C$ is a weak Borel reduction from $\equiv_T \restriction C$ to $\equiv_T$. Furthermore, in this case, if $D \subseteq 2^{\mathbb{N}}$ is* any *cone, then $[\, f(D)\,]_{\equiv_T}$ contains a cone.*

Proof. Suppose that (i) fails. By Martin's Conjecture, there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $x \leq_T f(x)$ for all $x \in C$. Clearly $f \restriction C$ is countable-to-one and so $f \restriction C$ is a weak Borel reduction. Let $D \subseteq 2^{\mathbb{N}}$ be any cone and let $D_0 = D \cap C$. Since $f \restriction C$ is countable-to-one, it follows that $f(D_0)$ is a Borel subset of $2^{\mathbb{N}}$ and this implies that the $\equiv_T$-saturation $[\, f(D_0)\,]_{\equiv_T}$ is also a Borel subset. By Martin's Theorem [13, 14], since $[\, f(D_0)\,]_{\equiv_T}$ is a $\leq_T$-cofinal $\equiv_T$-invariant Borel subset of $2^{\mathbb{N}}$, it follows that $[\, f(D_0)\,]_{\equiv_T}$ contains a cone. □

Condition 2.1(ii) is reminiscent of the conclusion of the "unique ergodicity argument" first introduced by Adams [1] in the measure-theoretical setting and later exploited by Thomas [20, 21] and Hjorth-Kechris [9]. Of course, the following result is an immediate consequence of Theorem 2.1 and implies that $\equiv_T$ is not countable universal. (Here $\equiv_T \sqcup \equiv_T$ denotes the disjoint union of two copies of the Turing equivalence relation $\equiv_T$.)

Corollary 2.2 (*MC*)**.** $\equiv_T \,<_B\, (\equiv_T \sqcup \equiv_T)$.

Observation 2.3. Suppose that $C = \{x \in 2^{\mathbb{N}} \mid z \leq_T x\}$ is a cone. Then the map, $y \mapsto y \oplus z$, is a weak Borel reduction from $\equiv_T$ to $\equiv_T \restriction C$ and hence $\equiv_T \restriction C$ is also weakly universal. Here $y \oplus z$ denotes the usual *recursive join* of y and z, which is defined by

$$(y \oplus z)(n) = \begin{cases} y(\frac{n}{2}) & \text{if } n \text{ is even;} \\ z(\frac{n-1}{2}) & \text{if } n \text{ is odd.} \end{cases}$$

Corollary 2.4 (MC)**.** *If $A \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset, then $\equiv_T \restriction A$ is weakly universal iff A contains a cone.*

Proof. If $\equiv_T \restriction A$ is weakly universal, then there exists a weak Borel reduction $f : 2^{\mathbb{N}} \to A$ from $\equiv_T$ to $\equiv_T \restriction A$. By Theorem 2.1, it follows that $[\,f(2^{\mathbb{N}})\,]_{\equiv_T}$ contains a cone. $\square$

On the other hand, if $\equiv_T$ is countable universal, then $(\equiv_T \sqcup \equiv_T) \leq_B \equiv_T$ and this easily implies that there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $\equiv_T \restriction (\,2^{\mathbb{N}} \smallsetminus C\,)$ is also countable universal. Consequently, it would be very interesting to obtain lower bounds on the Borel complexity of $\equiv_T \restriction (\,2^{\mathbb{N}} \smallsetminus C\,)$ for cones $C \subseteq 2^{\mathbb{N}}$. In Section 5, we will prove that there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $\equiv_T \restriction (\,2^{\mathbb{N}} \smallsetminus C\,)$ is *not* essentially free.

Remark 2.5. In [8], answering a question of Thomas [22], Hjorth proved that the universal countable Borel equivalence relation E_∞ is not Borel bireducible with a smooth disjoint union of essentially free countable Borel equivalence relations. This can also be seen as follows. Suppose that $E = \bigsqcup_{z \in 2^{\mathbb{N}}} E_z$ is the smooth disjoint union of the essentially free countable Borel equivalence relations $\{E_z \mid z \in 2^{\mathbb{N}}\}$ and that $f : 2^{\mathbb{N}} \to \bigsqcup_{z \in 2^{\mathbb{N}}} X_z$ is a Borel reduction from $\equiv_T$ to E. Then there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $f(C) \subseteq X_z$ for some fixed $z \in 2^{\mathbb{N}}$; and since $\equiv_T \restriction C$ is weakly universal, it follows that E_z is also weakly universal. But this contradicts Thomas [22, Corollary 4.8], which says that weakly universal countable Borel equivalence relations are not essentially free.

3. Strong Ergodicity

In this section, we will present the proof of Theorem 1.4 and point out a corollary which appears to also be consistent with Kechris's Conjecture [5] that $\equiv_T$ is countable universal and even with Hjorth's Conjecture [2] that *every* weakly universal countable Borel equivalence relation is countable universal. (I should perhaps point out that Hjorth denies having ever made this conjecture.)

Proof of Theorem 1.4 (MC). As we pointed out earlier, it is clear that conditions (a) and (b) are mutually exclusive. Suppose that $f : 2^{\mathbb{N}} \to X$ witnesses the failure of condition (b). Since $\equiv_T$ is weakly universal, there exists a weak Borel reduction $g : X \to 2^{\mathbb{N}}$ from E to $\equiv_T$. Let $h = g \circ f$. Then h is a Borel homomorphism from $\equiv_T$ to $\equiv_T$. Suppose that there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that h maps C to a single $\equiv_T$-class; say, $[\, x\,]_{\equiv_T}$. Then f maps C into the countable set $Y = g^{-1}([\, x\,]_{\equiv_T})$ and so there exists an element $y \in Y$ such that $Z = f^{-1}(y)$ is a $\leq_T$-cofinal Borel subset of $2^{\mathbb{N}}$. Applying Martin's Theorem, it follows that the $\equiv_T$-saturation $[\, Z\,]_{\equiv_T}$ contains a cone D. But then f maps the cone D into the single E-class $[\, y\,]_E$, which contradicts our assumption that f witnesses the failure of condition (b). By Theorem 2.1, since h does not map any cone into a single $\equiv_T$-class, there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $h \restriction C$ is countable-to-one. This implies that $f \restriction C$ is countable-to-one and hence $(\equiv_T \restriction C) \leq^w_B E$. Finally, since $\equiv_T \restriction C$ is weakly universal, it follows that E is also weakly universal. $\square$

Corollary 3.1 (MC)**.** *Let E, F be countable Borel equivalence relations on the standard Borel spaces X, Y respectively. Suppose that E is weakly universal and that F is not weakly universal. If $f : X \to Y$ is a Borel homomorphism from E to F, then there exists a Borel subset $Z \subseteq X$ such that:*

(i) *$E \restriction Z$ is weakly universal; and*
(ii) *f maps Z into a single F-class.*

Proof. Let $g : 2^{\mathbb{N}} \to X$ be a weak Borel reduction from $\equiv_T$ to E. Then $h = f \circ g$ is a Borel homomorphism from $\equiv_T$ to F. By Theorem 1.4, $\equiv_T$ is F-m-ergodic and hence there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that h maps C to a single F-class. Since g is countable-to-one, it follows that $Z = g(C)$ is a Borel subset of X; and since

$(\equiv_T\restriction C) \leq_B^w (E \restriction Z)$, it follows that $E \restriction Z$ is weakly universal. Thus Z satisfies our requirements. □

4. The Proof of Theorem 1.2

Before we begin the proof of Theorem 1.2, we first need to recall the standard measure-theoretical version of strong ergodicity.

Definition 4.1. Suppose that E, F are countable Borel equivalence relations on the standard Borel spaces X, Y and that μ is an E-invariant probability measure on X. Then E is said to be *F-ergodic with respect to μ* iff for every Borel homomorphism $f : X \to Y$ from E to F, there exists a Borel subset $Z \subseteq X$ with $\mu(Z) = 1$ such that f maps Z into a single F-class.

Remark 4.2. More generally, if E is F-ergodic with respect to μ and $f : X \to Y$ is a μ-measurable homomorphism from E to F, then there exists a Borel subset $Z \subseteq X$ with $\mu(Z) = 1$ such that f maps Z into a single F-class. To see this, recall that there exists a Borel map $g : X \to Y$ such that $g(x) = f(x)$ for μ-a.e. x. It is easily checked that

$$W = \{x \in X \mid g(\,[x]_E\,) \text{ is } \textit{not} \text{ contained in a single } F\text{-class }\}$$

is an E-invariant Borel subset of X with $\mu(W) = 0$. Hence, after adjusting the values of g on W, we can suppose that g is a Borel homomorphism from E to F. The result then follows easily.

We will make use of the following result, which was proved in Thomas [22].

Theorem 4.3. *There exists a Borel family $\mathcal{G} = \{G_\alpha \mid \alpha \in 2^{\mathbb{N}}\}$ of finitely generated groups, each with underlying set $\mathbb{N}$, such that the following conditions are satisfied:*

(a) *G_α has an infinite normal subgroup N_α such that $N_\alpha \cong SL_3(\mathbb{Z})$.*
(b) *G_α has no nontrivial finite normal subgroups.*
(c) *If $\alpha \neq \beta$, then G_β does not embed into G_α.*

For each $\alpha \in 2^{\mathbb{N}}$, consider the shift action of G_α on $2^{G_\alpha} = 2^{\mathbb{N}}$. Then the usual product probability measure μ on $2^{\mathbb{N}}$ is G_α-invariant and the *free part* of the action

$$X_\alpha = \{x \in 2^{\mathbb{N}} \mid g \cdot x \neq x \text{ for all } 1 \neq g \in G_\alpha\}$$

has μ-measure 1. Let E_α be the corresponding orbit equivalence relation on X_α. Then the following result is an easy consequence of Popa's Cocycle Superrigidity Theorem [16]. (For example, see Thomas [22, Section 5].)

Theorem 4.4. *If $\alpha \neq \beta$, then E_β is E_α-ergodic with respect to μ.*

Clearly $(\equiv_T \times E_\alpha)$ is weakly universal for each $\alpha \in 2^{\mathbb{N}}$. Hence Theorem 1.2 is an immediate consequence of the following result. (In an earlier version of the proof of Theorem 4.5, I assumed $\boldsymbol{\Sigma}^1_1$-Determinacy in order to obtain the measurability of $\boldsymbol{\Sigma}^1_2$ sets. I am grateful to Alexander Kechris for providing the following elegant method for eliminating the hypothesis of $\boldsymbol{\Sigma}^1_1$-Determinacy.)

Theorem 4.5 (MC)**.** *If $\alpha \neq \beta$, then $(\equiv_T \times E_\beta) \not\leq_B (\equiv_T \times E_\alpha)$.*

Proof. We will first prove Theorem 4.5 under the additional assumption that the universe V also satisfies $MA + 2^{\aleph_0} > \aleph_1$. Notice that Theorem 4.4 implies that E_α is *not* weakly universal. Suppose that

$$f : 2^{\mathbb{N}} \times X_\beta \to 2^{\mathbb{N}} \times X_\alpha$$

is a Borel reduction from $(\equiv_T \times E_\beta)$ to $(\equiv_T \times E_\alpha)$ and let λ, ρ be the Borel functions defined by

$$f(r,x) = (\,\lambda(r,x), \rho(r,x)\,).$$

For each $x \in X_\beta$, let $\rho_x : 2^{\mathbb{N}} \to X_\alpha$ be the map defined by $\rho_x(r) = \rho(r,x)$. Then ρ_x is a Borel homomorphism from $\equiv_T$ to E_α. Since E_α is not weakly universal, Theorem 1.4 implies that there exists a cone $C_x \subseteq 2^{\mathbb{N}}$ such that ρ_x maps C_x to a single E_α-class; say, $\mathbf{d}_x$. If $y E_\beta x$ and $r \in C_x$, then $\rho_y(r) E_\alpha \rho_x(r)$ and so $\rho_y(r) \in \mathbf{d}_x$. Hence if $y\, E_\beta\, x$, then $\mathbf{d}_y = \mathbf{d}_x$. Let $R \subseteq X_\beta \times X_\alpha$ be the $\boldsymbol{\Sigma}^1_2$ subset defined by

$$(x,z) \in R \quad \text{iff} \quad (\,\exists s\,)\,(\,\forall r\,)\,(\,s \leq_T r \text{ implies } \rho(r,x)\, E_\alpha\, z\,).$$

Applying Kondô's Theorem [12], let $h : X_\beta \to X_\alpha$ be a $\boldsymbol{\Sigma}^1_2$ uniformizing function for R. If $U \subseteq X_\alpha$ is an open set, then

$$h^{-1}(U) = \{\, x \in X_\beta \mid (\,\exists y\,)\,(\,y \in U \text{ and } h(x) = y\,\}$$

is a $\boldsymbol{\Sigma}^1_2$ set. By Martin-Solovay [15], since $MA + 2^{\aleph_0} > \aleph_1$ holds, every $\boldsymbol{\Sigma}^1_2$ set is μ-measurable and hence h is μ-measurable. Clearly $h(x) \in \mathbf{d}_x$ for all $x \in X_\beta$ and it follows that h is a μ-measurable homomorphism from E_β to E_α. Since E_β is E_α-ergodic with respect to μ, there exists a Borel subset $X_0 \subseteq X_\beta$ with $\mu(X_0) = 1$ such that h maps X_0 into a single E_α-class; say, $\mathbf{c}$.

For each $x \in X_0$, let $\lambda_x : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the map defined by $\lambda_x(r) = \lambda(r, x)$. Then λ_x is a Borel homomorphism from $\equiv_T$ to $\equiv_T$. If $r,\, s \in C_x$, then $\rho(r, x),\, \rho(s, x) \in \mathbf{c}$ and it follows that

$$r \equiv_T s \quad \text{iff} \quad \lambda_x(r) \equiv_T \lambda_x(s).$$

Thus λ_x induces a Borel reduction from $\equiv_T \restriction C_x$ to $\equiv_T$. Hence, by Theorem 2.1, it follows that $[\operatorname{ran} \lambda_x \restriction C_x]_{\equiv_T}$ contains a cone D_x. In particular, choosing $x,\, y \in X_0$ with $[x]_{E_\beta} \neq [y]_{E_\beta}$, there exist $r \in C_x$ and $s \in C_y$ such that $\lambda_x(r) \equiv_T \lambda_y(s)$. But this means that $f(r, x)\ (\equiv_T \times E_\alpha)\ f(s, y)$, which is a contradiction.

Finally we will explain how to eliminate the additional assumption that V satisfies $MA + 2^{\aleph_0} > \aleph_1$. First note that MC is equivalent to the following Π^1_2 statement.

(MC') If $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T$ to $\equiv_T$, then either:

(a) for all $x \in 2^{\mathbb{N}}$, there exists $x \leq_T y$ such that $f(y) <_T y$; or

(b) for all $x \in 2^{\mathbb{N}}$, there exists $x \leq_T y$ such that $y \leq_T f(y)$.

To see this this, suppose that MC' holds and let $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be a Borel homomorphism from $\equiv_T$ to $\equiv_T$. If (a) holds, then $A = \{\, y \in 2^{\mathbb{N}} \mid f(y) <_T y \,\}$ is a $\leq_T$-cofinal $\equiv_T$-invariant Borel subset of $2^{\mathbb{N}}$; and hence, by Martin's Theorem, A contains a cone C. Applying Slamen-Steel [18], it follows that there exists a cone $D \subseteq C$ such that f maps D into a single $\equiv_T$-class. Similarly, if (b) holds, then there exists a cone C such that $y \leq_T f(y)$ for all $y \in C$. (Of course, it is clear that MC implies MC'.)

Fix some $\alpha \neq \beta$ and let $V^{\mathbb{P}}$ be a generic extension which satisfies $MA + 2^{\aleph_0} > \aleph_1$. Then by the Shoenfield Absoluteness Theorem [10, Theorem 25.20], it follows that $V^{\mathbb{P}}$ also satisfies MC. Furthermore, it is clear that conditions 4.3(a), (b) and (c) are absolute and so E_β remains E_α-ergodic with respect to μ in $V^{\mathbb{P}}$. Hence, by our earlier argument,

$$V^{\mathbb{P}} \models (\equiv_T \times E_\beta) \not\leq_B (\equiv_T \times E_\alpha).$$

By the Shoenfield Absoluteness Theorem, since this is a $\mathbf{\Pi}^1_2$ property of α and β, it follows that

$$V \models (\equiv_T \times E_\beta) \not\leq_B (\equiv_T \times E_\alpha).$$

□

5. Some applications

In this final section, we will present two easy applications of the results of the earlier sections, which answer questions of Boykin-Jackson [3] and Thomas [22], modulo Martin's Conjecture. Throughout this section, if $c, d \in \mathbb{N}^{\mathbb{N}}$, then $c \leq^* d$ iff $c(n) \leq d(n)$ for all but finitely many $n \in \mathbb{N}$; and $c =^* d$ iff both $c \leq^* d$ and $d \leq^* c$. Similarly, we will write $c < d$ iff $c(n) < d(n)$ for all $n \in \mathbb{N}$.

It is well-known that the countable Borel equivalence relation $=^*$ is Borel bireducible with the Vitali equivalence relation E_0. In particular, $=^*$ is not weakly universal.

Definition 5.1 (Boykin-Jackson [3]). Let E be a Borel equivalence relation on the standard Borel space X. Then E is said to be *Borel-Bounded* iff for every Borel map $\varphi : X \to \mathbb{N}^{\mathbb{N}}$, there exists a Borel homomorphism $g : X \to \mathbb{N}^{\mathbb{N}}$ from E to $=^*$ such that $\varphi(x) \leq^* g(x)$ for all $x \in X$

In [3], Boykin-Jackson introduced the notion of Borel-Boundedness and showed that it is closely related to the Union Problem, which asks whether an increasing union of hyperfinite Borel equivalence relations is also hyperfinite. More specifically, Boykin-Jackson proved that every hyperfinite Borel equivalence relation is Borel-Bounded; and they also showed that if the countable Borel equivalence relation E is an increasing union of hyperfinite Borel equivalence relations, then E is hyperfinite iff E is Borel-Bounded. On the other hand, Boykin-Jackson pointed out that, while it is conceivable that every Borel-Bounded countable Borel equivalence relation is hyperfinite, there are currently no examples of countable Borel equivalence relations which are known *not* to be Borel-Bounded.

Theorem 5.2 (*MC*). *The Turing equivalence relation $\equiv_T$ is not Borel-Bounded.*

Proof. Identifying each $r \in 2^{\mathbb{N}}$ with the corresponding subset of $\mathbb{N}$, let $\varphi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the Borel map such that:

- $\varphi(r)$ is the strictly increasing enumeration of $r \cap 2\mathbb{N}$, if $r \cap 2\mathbb{N}$ is infinite;
- $\varphi(r)$ is the zero function, otherwise.

We claim that for each function $h \in \mathbb{N}^{\mathbb{N}}$, the $\equiv_T$-invariant Borel set

$$S_h = \{ r \in 2^{\mathbb{N}} \mid (\exists s \in 2^{\mathbb{N}}) (s \equiv_T r \text{ and } h < \varphi(s)) \}$$

contains a cone. To see this, first fix some strictly increasing function $e \in \mathbb{N}^{\mathbb{N}}$ such that $h < e$. Now suppose that $r \in 2^{\mathbb{N}}$ satisfies $e \leq_T r$ and consider the subset $s \subseteq \mathbb{N}$ defined by

$$s = \{ 2e(n) \mid n \in \mathbb{N} \} \cup \{ 2\ell + 1 \mid \ell \in r \}.$$

Then clearly $s \equiv_T r$ and $h < \varphi(s)$. Thus S_h contains the cone $\{ r \in 2^{\mathbb{N}} \mid e \leq_T r \}$. Now suppose that $g : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T$ to $=^*$ such that $\varphi(r) \leq^* g(r)$ for all $r \in 2^{\mathbb{N}}$. Applying Theorem 1.4, it follows that there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that g maps C into a single $=^*$-class; say, $[h]_{=^*}$. But this means that $S_h \cap C = \emptyset$, which is a contradiction. □

Corollary 5.3 (*MC*)**.** *If E is a weakly universal countable Borel equivalence relation, then E is not Borel-Bounded.*

Proof. If E is weakly universal, then $\equiv_T \leq_B^w E$. Hence, applying Kechris-Miller [22, Theorem 4.4], there exists a countable Borel equivalence relation $S \subseteq E$ such that $\equiv_T \leq_B S$. By Boykin-Jackson[3, Lemmas 10 and 11], it follows that E is not Borel-Bounded. (As we mentioned earlier, the material in Thomas [22, Section 4] is due Kechris-Miller.)

□

The proof of Theorem 5.2 makes use of the E_0-m-ergodicity of the Turing equivalence relation $\equiv_T$. Unfortunately, this argument cannot be carried out within the usual measure-theoretic setting. To see this, suppose that (X, μ) is a standard Borel probability space and that $\theta : X \to \mathbb{N}^{\mathbb{N}}$ is a Borel map. Then the Borel-Cantelli Lemma implies that there exists a function $h \in \mathbb{N}^{\mathbb{N}}$ such that

$$\mu(\{ x \in X \mid \theta(x) \leq^* h \}) = 1.$$

This simple observation has the following striking consequence.

Theorem 5.4 (*MC*)**.** *Let E be a countable Borel equivalence relation on the standard Borel space X and let μ be a (not necessarily E-invariant) Borel probability* *measure on X. Then there exists a Borel subset $Y \subseteq X$ with $\mu(Y) = 1$ such that $E \restriction Y$ is* not *weakly universal.*

Proof. Let $\varphi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the Borel map defined in the proof of Theorem 5.2. By the Feldman-Moore Theorem [6], there exists a countable group $\Gamma = \{ \gamma_m \mid m \in \mathbb{N} \}$ and a Borel action of Γ on $2^{\mathbb{N}}$ such that the corresponding orbit equivalence relation is precisely $\equiv_T$. Let $\psi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the Borel map defined by

$$\psi(x)(n) = \max \{ \varphi(\gamma_m \cdot x)(n) \mid m \leq n \}.$$

Then for all r, $s \in 2^{\mathbb{N}}$, if $s \equiv_T r$, then $\varphi(s) \leq^* \psi(r)$. Let $f : X \to 2^{\mathbb{N}}$ be a weak Borel reduction from E to $\equiv_T$ and let $\theta : X \to \mathbb{N}^{\mathbb{N}}$ be the Borel map defined by $\theta = \psi \circ f$. Then there exists a function $h \in \mathbb{N}^{\mathbb{N}}$ such that the Borel subset

$$Y = \{ x \in X \mid \theta(x) \leq^* h \}$$

satisfies $\mu(Y) = 1$. Let $Z = [\, f(Y) \,]_{\equiv_T}$. Then for each $r \in Z$, we have that $\varphi(s) \leq^* h$ for all $s \equiv_T r$. As in the proof of Theorem 5.2, this implies that $2^{\mathbb{N}} \smallsetminus Z$ contains a cone. Applying Corollary 2.4, if follows that $\equiv_T \restriction Z$ is not weakly universal. Since $(E \restriction Y) \leq^w_B (\equiv_T \restriction Z)$, it follows that $E \restriction Y$ is not weakly universal. □

In particular, assuming Martin's Conjecture, the complexity of a weakly universal countable Borel equivalence relation always concentrates on a null set. This answers Thomas [22, Question 3.22].

Remark 5.5. In [8], Hjorth proved that there exists a countable Borel equivalence relation E on a standard Borel space X with an invariant probability measure μ such that $E \restriction Y$ is not essentially free whenever $Y \subseteq X$ is a Borel subset with $\mu(Y) = 1$. Arguing as in the proof of Theorem 5.4, it follows that there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $\equiv_T \restriction (\, 2^{\mathbb{N}} \smallsetminus C \,)$ is not essentially free.

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