THE FRIEDMAN EMBEDDING THEOREM

SIMON THOMAS

To Harvey on his 60th Birthday

Abstract. In this paper, we will present an explicit construction of Harvey Friedman which to every finitely generated group G associates a 2-generator subgroup $K_G \leqslant \mathrm{Sym}(\mathbb{N})$ such that G embeds into K_G and such that if $G \cong H$, then $K_G = K_H$.

1. Introduction

The classical Higman-Neumann-Neumann Embedding Theorem [5] states that every countable group G can be embedded into a 2-generator group K. In the standard proof of this classical theorem, the construction of the group K involves an enumeration of a set $\{g_n \mid n \in \mathbb{N}\}$ of generators of the group G; and it is clear that the isomorphism type of K usually depends upon both the generating set and the particular enumeration that is used. Consequently, it is natural to ask whether there is a more uniform construction with the property that the isomorphism type of K only depends upon the isomorphism type of G. The main result of Thomas [15] implies that no such construction exists. More precisely, let $\mathcal{G}$ be the Polish space of countably infinite groups and let $\mathcal{G}_2$ be the Polish space of 2-generator groups. (We will recall the definitions of the Polish spaces $\mathcal{G}$ and $\mathcal{G}_2$, together with the notion of a Borel map, in Section 2. For now, we just mention that the notion of a Borel map is intended to capture the intuitive idea of an *explicit* map.)

Theorem 1.1. *There does not exist a Borel map $\varphi : \mathcal{G} \to \mathcal{G}_2$ such that for all countable groups G, $H \in \mathcal{G}$,*

(a) *G embeds into $\varphi(G)$; and*
(b) *if $G \cong H$, then $\varphi(G) \cong \varphi(H)$.*

Key words and phrases. Combinatorial group theory, recursion theory, Borel equivalence relation.

Research partially supported by NSF Grant DMS 0600940.

The proof of Theorem 1.1 is based upon the fact that the isomorphism relation on the space $\mathcal{G}$ of countably infinite groups is much more complex than the isomorphism relation on the space $\mathcal{G}_2$ of 2-generator groups. (The isomorphism relation on $\mathcal{G}$ is complete analytic, while the isomorphism relation on $\mathcal{G}_2$ is a countable Borel equivalence relation.) On the other hand, combining the results of Hjorth [6] and Thomas-Velickovic [16], it follows that the isomorphism relation on the space $\mathcal{G}_{fg}$ of finitely generated groups has precisely the same complexity as the isomorphism relation on the space $\mathcal{G}_2$ of 2-generator groups. This raises the possibility of the existence of a uniform version of the Higman-Neumann-Neumann Embedding Theorem for finitely generated groups. However, when I discussed this question with various group-theorists, they were not even able to find a uniform construction for embedding 3-generator groups into 2-generator groups. It appears that I had been asking the wrong people. In this paper, I will present the proof of the following remarkable theorem of Harvey Friedman.

Theorem 1.2 (The Friedman Embedding Theorem)**.** *There exists a Borel map $\varphi : \mathcal{G}_{fg} \to \mathcal{G}_2$ such that for all G, $H \in \mathcal{G}_{fg}$,*

(a) *G embeds into $\varphi(G)$; and*
(b) *if $G \cong H$, then $\varphi(G) \cong \varphi(H)$.*

In fact, there exists an explicit construction which to each finitely generated group G associates a 2-generator subgroup $K_G \leqslant \mathrm{Sym}(\mathbb{N})$ such that G embeds into K_G and such that if $G \cong H$, then $K_G = K_H$. (Here we should mention that the set of 2-generator subgroups of $\mathrm{Sym}(\mathbb{N})$ cannot be regarded as a Polish space in any natural way. We will return to this point at the end of Section 4.) The key idea behind Friedman's construction is to first associate a recursion-theoretic invariant to each finitely generated group G; namely, the Turing degree $\mathbf{d}_G$ of the word problem for G. It is then easily checked that G embeds into the group R_G of all permutations $\sigma \in \mathrm{Sym}(\mathbb{N})$ such that σ is Turing reducible to $\mathbf{d}_G$; and finally K_G is a suitably constructed 2-generator subgroup of $\mathrm{Sym}(\mathbb{N})$ into which R_G embeds.

This paper is organized as follows. In Section 2, we will recall the definitions of the Polish spaces of countably infinite groups and finitely generated groups. In Section 3, we will recall some basic notions from recursion theory and present some technical results concerning the groups

$$\mathrm{Rec}^A(\mathbb{N}) = \{\, \sigma \in \mathrm{Sym}(\mathbb{N}) \mid \sigma \leq_T A \,\}, \qquad A \in 2^{\mathbb{N}},$$

which will play an important role in the proof of Theorem 1.2. In Section 4, we will present the proof of Theorem 1.2; and in Section 5, we will present an easy but striking application. Finally, in Section 6, we will discuss a number of open questions, including the question of whether there exists a more purely "group-theoretic" approach to the Friedman Embedding Theorem.

Since this paper is intended to be intelligible to non-logicians, it contains detailed explanations of some points which will be obvious to the experts in recursion theory and descriptive set theory.

2. Spaces of Groups

In this section, after first recalling the notion of a Borel map, we will discuss the Polish spaces of countably infinite groups and finitely generated groups.

Suppose that X, Y are Polish spaces; i.e. separable completely metrizable topological spaces. Then a map $f : X \to Y$ is *Borel* if $\mathrm{graph}(f)$ is a Borel subset of $X \times Y$. Equivalently, $f : X \to Y$ is Borel if $f^{-1}(Z)$ is a Borel subset of X for each Borel subset $Z \subseteq Y$. As we mentioned earlier, the notion of a Borel map is intended to capture the intuitive idea of an explicit map.

The most obvious examples of Polish spaces include the spaces of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ and p-adic numbers $\mathbb{Q}_p$, as well as the Cantor space $2^{\mathbb{N}}$. However, it is also possible to represent the class of countably infinite structures for a countable first order language by the elements of a suitable Polish space. We will illustrate this by describing the space of countably infinite groups. Let $\mathcal{G}$ be the set of countably infinite groups G with underlying set $\mathbb{N}$; and let $2^{\mathbb{N}^3}$ be the Polish space of all 3-ary functions $f : \mathbb{N}^3 \to \{0, 1\}$ with the natural product topology. Then, identifying each group $G \in \mathcal{G}$ with the graph of its multiplication operation $m_G \in 2^{\mathbb{N}^3}$, it is easily checked that $\mathcal{G}$ is a countable intersection of open subsets of $2^{\mathbb{N}^3}$ and hence $\mathcal{G}$ is a Polish subspace of $2^{\mathbb{N}^3}$. This technique can be adapted to construct Polish spaces of countably infinite fields, rings, torsion-free abelian groups, etc. In particular, we can use this technique to construct a Polish space $\tilde{\mathcal{G}}_{fg}$ of infinite finitely generated groups; and this was the approach that was taken in Thomas [15]. (For more details, see Hjorth-Kechris [7] or Thomas-Velickovic [16].) However, in this paper, we will prefer to use an alternative approach due to Grigorchuk [4], which more faithfully reflects various important features of the class of (not necessarily infinite) finitely generated groups.[1]

The Polish space $\mathcal{G}_{fg}$ of (marked) finitely generated groups is defined as follows. A *marked group* $(G, \bar{s})$ consists of a finitely generated group with a distinguished sequence $\bar{s} = (s_1, \cdots, s_m)$ of generators. (Here the sequence $\bar{s}$ is allowed to contain repetitions and we also allow the possibility that the sequence contains the identity element.) Two marked groups $(G, (s_1, \cdots, s_m))$ and $(H, (t_1, \cdots, t_n))$ are said to be *isomorphic* iff $m = n$ and the map $s_i \mapsto t_i$ extends to a group isomorphism between G and H.

Definition 2.1. For each $m \geq 2$, let $\mathcal{G}_m$ be the set of *isomorphism types* of marked groups $(G, (s_1, \cdots, s_m))$ with m distinguished generators.

Let $\mathbb{F}_m$ be the free group on the generators $\{x_1, \cdots, x_m\}$. Then for each marked group $(G, (s_1, \cdots, s_m))$, we can define an associated epimorphism $\theta_{G,\bar{s}} : \mathbb{F}_m \to G$ by $\theta_{G,\bar{s}}(x_i) = s_i$. It is easily checked that two marked groups $(G, (s_1, \cdots, s_m))$ and $(H, (t_1, \cdots, t_m))$ are isomorphic iff $\ker \theta_{G,\bar{s}} = \ker \theta_{H,\bar{t}}$. Thus we can naturally identify $\mathcal{G}_m$ with the set $\mathcal{N}_m$ of normal subgroups of $\mathbb{F}_m$. Note that $\mathcal{N}_m$ is a closed subset of the compact space $2^{\mathbb{F}_m}$ of all subsets of $\mathbb{F}_m$ and so $\mathcal{N}_m$ is also a compact space. Hence, via the above identification, we can regard $\mathcal{G}_m$ as a compact space.

The topologies on $\mathcal{N}_m$ and $\mathcal{G}_m$ can be described more explicitly as follows. For each marked group $(G, \bar{s})$ and integer $\ell \geq 1$, let $B_\ell(G, \bar{s})$ be the closed ball of radius ℓ around the identity element in the (labelled directed) Cayley graph $\mathrm{Cay}(G, \bar{s})$ of G with respect to the generating sequence $\bar{s}$. Then, letting $\bar{x} = (x_1, \cdots, x_m)$, a neighborhood basis in $\mathcal{N}_m$ of the normal subgroup N is given by the collection of open sets

$$U_{N,\ell} = \{\, M \in \mathcal{N}_m \mid M \cap B_\ell(\mathbb{F}_m, \bar{x}) = N \cap B_\ell(\mathbb{F}_m, \bar{x}) \,\}, \quad \ell \geq 1.$$

[1]We should point out that these two approaches are essentially equivalent; namely, there exist Borel maps $\theta : \tilde{\mathcal{G}}_{fg} \to \mathcal{G}_{fg}$ and $\tau : \mathcal{G}_{fg} \to \tilde{\mathcal{G}}_{fg}$ such that $\theta(G) \cong G$ and $\tau(H) \cong H$ for all $G \in \tilde{\mathcal{G}}_{fg}$ and $H \in \mathcal{G}_{fg}$. In particular, Theorem 1.1 remains true when $\tilde{\mathcal{G}}_{fg}$ is replaced by $\mathcal{G}_{fg}$.

If $(G, \bar{s}) \in \mathcal{G}_m$ corresponds to the normal subgroup $N \in \mathcal{N}_m$, then the set of relations $N \cap B_{2\ell+1}(\mathbb{F}_m, \bar{x})$ contains essentially the same information as the closed ball $B_\ell(G, \bar{s})$ in the Cayley graph of $(G, \bar{s})$. It follows that a neighborhood basis in $\mathcal{G}_m$ of the marked group $(G, \bar{s})$ is given by the collection of open sets

$$V_{(G,\bar{s}),\ell} = \{\, (H, \bar{t}\,) \in \mathcal{G}_m \mid B_\ell(H, \bar{t}\,) \cong B_\ell(G, \bar{s})\,\}, \quad \ell \geq 1.$$

For each $m \geq 2$, there is a natural embedding of $\mathcal{N}_m$ into $\mathcal{N}_{m+1}$ defined by

$$N \mapsto \text{ the normal closure of } N \cup \{x_{m+1}\} \text{ in } \mathbb{F}_{m+1}.$$

This enables us to regard $\mathcal{N}_m$ as a clopen subset of $\mathcal{N}_{m+1}$ and to form the locally compact Polish space $\mathcal{N} = \bigcup \mathcal{N}_m$. Note that $\mathcal{N}$ can be identified with the space of normal subgroups N of the free group $\mathbb{F}_\infty$ on countably many generators such that N contains all but finitely many elements of the basis $X = \{x_i \mid i \in \mathbb{N}^+\}$. Similarly, we can form the locally compact Polish space $\mathcal{G}_{fg} = \bigcup \mathcal{G}_m$ of finitely generated groups via the corresponding natural embedding

$$(G, (s_1, \cdots, s_m)) \mapsto (G, (s_1, \cdots, s_m, 1))$$

From now on, we will identify $\mathcal{G}_m$ and $\mathcal{N}_m$ with the corresponding clopen subsets of $\mathcal{G}_{fg}$ and $\mathcal{N}$. If $\Gamma \in \mathcal{G}_{fg}$, then we will write $\Gamma = (G, (s_1, \cdots, s_m))$, where m is the least integer such that $\Gamma \in \mathcal{G}_m$. Following the usual convention, we will completely identify the Polish spaces $\mathcal{G}_{fg}$ and $\mathcal{N}$; and we will work with whichever space is most convenient in any given context.

In the remaining sections of this paper, the symbol $\cong$ will always denote the usual isomorphism relation on the space $\mathcal{G}_{fg}$ of finitely generated groups; i.e. two marked groups are $\cong$-equivalent iff their underlying groups (obtained by forgetting about the distinguished sequences of generators) are isomorphic. And we will often abuse notation by writing $G \in \mathcal{G}_{fg}$ instead of $\Gamma = (G, (s_1, \cdots, s_m)) \in \mathcal{G}_{fg}$.

3. Some basic recursion theory

In this section, we will recall some basic notions from recursion theory and present two technical lemmas concerning the groups

$$\mathrm{Rec}^A(\mathbb{N}) = \{\, \sigma \in \mathrm{Sym}(\mathbb{N}) \mid \sigma \leq_T A\,\}, \qquad A \in 2^{\mathbb{N}},$$

which will be used in the proof of the Friedman Embedding Theorem. Throughout this paper, we will identify the powerset $\mathcal{P}(\mathbb{N})$ of the natural numbers with the Cantor space $2^{\mathbb{N}}$, by identifying each subset $A \in \mathcal{P}(\mathbb{N})$ with its characteristic function $\chi_A \in 2^{\mathbb{N}}$. The group of recursive permutations of $\mathbb{N}$ will be denoted by $\mathrm{Rec}(\mathbb{N})$.

Recall that if A, $B \in 2^{\mathbb{N}}$, then B is *Turing reducible* to A, written $B \leq_T A$, if there exists an oracle Turing machine which computes χ_B when its oracle tape contains χ_A. Here an oracle Turing machine is a Turing machine with a second "read only" tape, called the oracle tape, upon which we can write the characteristic function of any set $A \in 2^{\mathbb{N}}$, which is called the oracle. (For more details, see Rogers [13].) From now on, we fix an effective enumeration P_0, P_1, ... P_s, ... of the oracle Turing machine programs, chosen so that $P_0 = \emptyset$.

Definition 3.1. If $A \in 2^{\mathbb{N}}$ and $s \in \mathbb{N}$, then φ_s^A denotes the partial function computed by the oracle Turing machine with program P_s and oracle A.

Remark 3.2. We have chosen $P_0 = \emptyset$ to ensure that for every $A \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, the value $\varphi_0^A(n)$ is undefined.

Definition 3.3. If $A \in 2^{\mathbb{N}}$ and $s \in \mathbb{N}$, then ψ_s^A denotes the element of $\mathrm{Rec}^A(\mathbb{N})$ defined by

$$\psi_s^A = \begin{cases} \varphi_s^A & \text{if } \varphi_s^A \in \mathrm{Sym}(\mathbb{N}); \\ \mathrm{id}_{\mathbb{N}} & \text{otherwise.} \end{cases}$$

Remark 3.4. In some of the later proofs in this paper, it will be important to note that $\psi_0^A = \mathrm{id}_{\mathbb{N}}$.

If A, $B \in 2^{\mathbb{N}}$, then A and B are *Turing equivalent*, written $A \equiv_T B$, if both $A \leq_T B$ and $B \leq_T A$. Of course, if $A \equiv_T B$, then $\mathrm{Rec}^A(\mathbb{N}) = \mathrm{Rec}^B(\mathbb{N})$. The following strengthening of this observation will play an essential role in the proof of Theorem 1.2.

Lemma 3.5. *If A, $B \in 2^{\mathbb{N}}$ and $A \equiv_T B$, then there exists a recursive permutation $\sigma \in \mathrm{Rec}(\mathbb{N})$ such that $\psi_s^B = \psi_{\sigma(s)}^A$ for all $s \in \mathbb{N}$.*

Proof. Since $B \leq_T A$, for each $i \in \mathbb{N}$, we can effectively find an integer $j \in \mathbb{N}$ such that $\varphi_j^A = \varphi_i^B$. (Fix some oracle program P_e such that $\varphi_e^A = \chi_B$. Then for each $i \in \mathbb{N}$, we can "combine" the oracle programs P_e and P_i to obtain an oracle program P_j such that $\varphi_j^A = \varphi_i^B$.) Similarly, for each $k \in \mathbb{N}$, we can effectively find an integer $\ell \in \mathbb{N}$ such that $\varphi_\ell^B = \varphi_k^A$. Hence we can construct a recursive permutation $\sigma \in \mathrm{Rec}(\mathbb{N})$ by an inductive back-and-forth argument such that $\varphi_s^B = \varphi_{\sigma(s)}^A$ for all $s \in \mathbb{N}$; and clearly we also have that $\psi_s^B = \psi_{\sigma(s)}^A$ for all $s \in \mathbb{N}$. □

The proof of Theorem 1.2 will also make use of the following technical result.

Lemma 3.6. *For each $A \in 2^{\mathbb{N}}$ and $\ell \in \mathbb{N}$, there exists a recursive permutation $\sigma \in \mathrm{Rec}(\mathbb{N})$ such that*

$$\psi_{\sigma(s)}^A = \begin{cases} \psi_\ell^A & \text{if } s = 0; \\ \psi_s^A & \text{otherwise.} \end{cases}$$

Proof. If $\ell = 0$, then we can let $\sigma = \mathrm{id}_{\mathbb{N}}$. Otherwise, there exist infinite disjoint recursive sets $B = \{ b_n \mid n \in \mathbb{N} \}$ and $C = \{ c_n \mid n \in \mathbb{N} \}$ of natural numbers such that:

(i) $b_0 = 0$ and $\varphi_{b_n}^A = \varphi_0^A$; and
(ii) $c_0 = \ell$ and $\varphi_{c_n}^A = \varphi_\ell^A$ for all $n \in \mathbb{N}$.

(If the oracle program P_ℓ which computes φ_ℓ^A has internal states $\{ q_0, \cdots, q_t \}$, then we can add extraneous instructions which only involve the states q_d for even integers $d > t$ and obtain new oracle programs which also compute φ_ℓ^A. Similarly, we can add extraneous instructions which only involve the states q_d for odd integers $d > t$ to the oracle program $P_0 = \emptyset$.) Clearly the infinite cycle $\sigma = (\ \cdots \ b_2 \ b_1 \ b_0 \ c_0 \ c_1 \ c_2 \ \cdots \ )$ satisfies our requirements. □

4. The proof of the Friedman Embedding Theorem

In [2], while proving some very striking "ergodicity theorems" for countable Borel equivalence relations, Friedman constructed a Borel map $\varphi : \mathcal{G}_{fg} \to \mathcal{G}_4$ such that for all finitely generated groups G, $H \in \mathcal{G}_{fg}$,

(a) G embeds into $\varphi(G)$; and
(b) if $G \cong H$, then $\varphi(G) \cong \varphi(H)$.

In this section, making use of the methods of Galvin [3], we will modify Friedman's original map so that it takes values in $\mathcal{G}_2$.

We will begin by associating a recursion-theoretic invariant to each finitely generated group. For each $m \geq 1$, fix an effective enumeration $\{ w_k(x_1, \cdots, x_m) \mid k \in \mathbb{N} \}$ of the free group $\mathbb{F}_m$ on m generators.

Definition 4.1. If $\Gamma = (G, (s_1, \cdots, s_m)) \in \mathcal{G}_{fg}$ is a marked finitely generated group, then $R_\Gamma = \{ k \in \mathbb{N} \mid w_k(s_1, \cdots, s_m) = 1 \}$.

Thus R_Γ is essentially the word problem for the finitely generated group G.

Lemma 4.2 (Folklore)**.** *If $\Gamma = (G, (s_1, \cdots, s_m))$ and $\Delta = (H, (t_1, \cdots, t_n))$ are marked finitely generated groups and $G \cong H$, then $R_\Gamma \equiv_T R_\Delta$.*

Proof. Suppose that $\psi : G \to H$ is an isomorphism; and for each $1 \leq i \leq m$, choose $u_i(x_1, \cdots, x_n) \in \mathbb{F}_n$ such that $\psi(s_i) = u_i(t_1, \cdots, t_n)$. Then for each $k \in \mathbb{N}$,

$$w_k(s_1, \cdots, s_m) = 1 \quad \Longleftrightarrow \quad w_k(u_1(t_1, \cdots, t_n), \cdots, u_n(t_1, \cdots, t_n)) = 1$$

and hence $R_\Gamma \leq_T R_\Delta$. Similarly $R_\Delta \leq_T R_\Gamma$ and so $R_\Gamma \equiv_T R_\Delta$. $\square$

In particular, if G is a finitely generated group, then the Turing degree

$$\mathbf{d}_G = \{ A \in 2^{\mathbb{N}} \mid A \equiv_T R_{(G,(s_1, \cdots, s_m))} \}$$

is independent of the choice of the generating sequence $(s_1, \cdots, s_m)$.

Lemma 4.3. *If $\Gamma = (G, (s_1, \cdots, s_m))$ is a marked finitely generated group, then G embeds into $\mathrm{Rec}^{R_\Gamma}(\mathbb{N})$.*

Proof. Since the result is clearly true if G is finite, we can suppose that G is an infinite group. Using an R_Γ-oracle, we can inductively define an increasing sequence of natural numbers $(\ell(n) \mid n \in \mathbb{N})$ such that $(w_{\ell(n)}(s_1, \cdots, s_m) \mid n \in \mathbb{N})$ lists the distinct elements of G. Then, once again using an R_Γ-oracle, for each $1 \leq i \leq m$ and $n \in \mathbb{N}$, we can find a natural number a_{in} such that

$$s_i \, w_{\ell(n)}(s_1, \cdots, s_m) = w_{\ell(a_{in})}(s_1, \cdots, s_m).$$

Let $g_i : \mathbb{N} \to \mathbb{N}$ be the function defined by $g_i(n) = a_{in}$. Then each $g_i \in \mathrm{Rec}^{R_\Gamma}(\mathbb{N})$ and the map $\theta : \{ s_1, \cdots, s_m \} \to \mathrm{Rec}^{R_\Gamma}(\mathbb{N})$, defined by $\theta(s_i) = g_i$, extends to an embedding of G into $\mathrm{Rec}^{R_\Gamma}(\mathbb{N})$. $\square$

Also notice that if $\Gamma = (G, (s_1, \cdots, s_m))$ and $\Delta = (H, (t_1, \cdots, t_n))$ are marked finitely generated groups and $G \cong H$, then $\mathrm{Rec}^{R_\Gamma}(\mathbb{N}) = \mathrm{Rec}^{R_\Delta}(\mathbb{N})$. But unfortunately $\mathrm{Rec}^{R_\Gamma}(\mathbb{N})$ is not finitely generated. Next, for each $A \in 2^{\mathbb{N}}$, we will construct a suitable 2-generator group K_A into which $\mathrm{Rec}^A(\mathbb{N})$ embeds. The construction proceeds in two steps.

Definition 4.4. For each $A \in 2^{\mathbb{N}}$, define $\pi_A \in \mathrm{Sym}(\mathbb{N} \times \mathbb{N})$ by

$$\pi_A(s, t) = (s, \psi_s^A(t));$$

and let $H_A \leqslant \mathrm{Sym}(\mathbb{N}\times\mathbb{N})$ be the subgroup generated by $\{\, \pi_A \} \cup \{\, \theta_\sigma \mid \sigma \in \mathrm{Rec}(\mathbb{N}) \,\}$, where

$$\theta_\sigma(s, t) = (\sigma(s), t).$$

The next two lemmas imply that that $\mathrm{Rec}^A(\mathbb{N})$ embeds into H_A and that if $A \equiv_T B$, then $H_A = H_B$. But, once again, H_A is not finitely generated.

Definition 4.5. For each $g \in \mathrm{Sym}(\mathbb{N})$, define $\alpha_g \in \mathrm{Sym}(\mathbb{N} \times \mathbb{N})$ by

$$\alpha_g(i, j) = \begin{cases} (\, 0, g(j)\,) & \text{if } i = 0; \\ (\, i, j\,) & \text{otherwise.} \end{cases}$$

Lemma 4.6. *For each $g \in \mathrm{Rec}^A(\mathbb{N})$, there exists $\sigma \in \mathrm{Rec}(\mathbb{N})$ such that*

$$\alpha_g = \theta_\sigma^{-1}\, \pi_A\, \theta_\sigma\, \pi_A^{-1}.$$

Proof. Let $\ell \in \mathbb{N}$ be such that $g = \psi_\ell^A$ and let $\sigma \in \mathrm{Rec}(\mathbb{N})$ be the recursive permutation given by Lemma 3.6. Then it is easily checked that σ satisfies our requirements. (Here it is important to note that $\psi_0^A = \mathrm{id}_{\mathbb{N}}$.) □

Lemma 4.7. *If A, $B \in 2^{\mathbb{N}}$ and $A \equiv_T B$, then there exists $\sigma \in \mathrm{Rec}(\mathbb{N})$ such that $\theta_\sigma^{-1}\, \pi_A\, \theta_\sigma = \pi_B$.*

Proof. Let $\sigma \in \mathrm{Rec}(\mathbb{N})$ be the recursive permutation given by Lemma 3.5. Then it is easily checked that σ satisfies our requirements. □

Finally, using the following slight variant of Galvin's construction [3], we will embed $\mathrm{Rec}^A(\mathbb{N})$ into a suitable 2-generator group K_A.

Definition 4.8. Let Ω be any set, let $(g_3, g_5, g_7, \cdots, g_{2n+1}, \cdots)$ be a sequence of elements of $\mathrm{Sym}(\Omega)$ indexed by the odd integers $m \geq 3$ and let $\pi \in Sym(\Omega)$ be a distinguished permutation. Then the associated permutations

$$a, b_\pi \in \mathrm{Sym}(\mathbb{Z} \times \mathbb{Z} \times \Omega)$$

are defined by $a(\,m, n, \omega\,) = (\,m+1, n, \omega\,)$ and

$$b_\pi(\,m, n, \omega\,) = \begin{cases} (\,0, n+1, \omega\,) & \text{if } m = 0; \\ (\,m, n, g_m(\omega)\,) & \text{if } m \geq 3 \text{ is odd and } n \geq 0; \\ (\,-1, 0, \pi(\omega)\,) & \text{if } m = -1 \text{ and } n = 0; \\ (\,m, n, \omega\,) & \text{otherwise.} \end{cases}$$

For each $i \in \{\,3, 5, 7, \cdots\}$, let $c_i = a^{-i}\, b_\pi\, a^i$ and $d_i = b_\pi\, c_i^{-1}\, b_\pi^{-1}\, c_i$.

Lemma 4.9. *For each* $i \in \{\,3, 5, 7, \cdots\}$,

$$d_i(\,m, n, \omega\,) = \begin{cases} (\,0, 0, g_i(\omega)\,) & \textit{if } m = n = 0; \\ (\,m, n, \omega\,) & \textit{otherwise.} \end{cases}$$

Proof. By a straightforward but tedious calculation. □

From now on, let $\Omega = \mathbb{N} \times \mathbb{N}$ and let $(g_3, g_5, g_7, \cdots, g_{2n+1}, \cdots)$ be the list (with many repetitions) of $\{\,\theta_\sigma \mid \sigma \in \mathrm{Rec}(\mathbb{N})\,\}$ defined by $g_{2n+1} = \theta_{\psi^{\emptyset}_{n-1}}$. (We should point out that *any* list of $\{\,\theta_\sigma \mid \sigma \in \mathrm{Rec}(\mathbb{N})\,\}$ would work equally well in the proof of Theorem 1.2. We have only chosen this particular list so that we can more easily study the complexity of the word problem for K_A in Section 6.) For each $A \in 2^{\mathbb{N}}$, let

$$K_A \leqslant \mathrm{Sym}(\mathbb{Z} \times \mathbb{Z} \times \Omega)$$

be the subgroup generated by the permutations a and b_{π_A}, where $\pi_A \in \mathrm{Sym}(\Omega)$ is the permutation given by Definition 4.4.

Definition 4.10. For each $g \in \mathrm{Sym}(\Omega)$, define $\tilde{g} \in \mathrm{Sym}(\mathbb{Z} \times \mathbb{Z} \times \Omega)$ by

$$\tilde{g}(\,m, n, \omega\,) = \begin{cases} (\,-1, 0, g(\omega)\,) & \text{if } m = -1 \text{ and } n = 0; \\ (\,m, n, \omega\,) & \text{otherwise.} \end{cases}$$

Lemma 4.11. *If* $A \in 2^{\mathbb{N}}$ *and* $\sigma \in \mathrm{Rec}(\mathbb{N})$*, then* $\tilde{\theta}_\sigma \in K_A$.

Proof. By Lemma 4.9, we have that $a\,\tilde{\theta}_\sigma\,a^{-1} \in K_A$ and hence $\tilde{\theta}_\sigma \in K_A$. □

Lemma 4.12. *If $A \in 2^{\mathbb{N}}$ and $g \in \mathrm{Rec}^A(\mathbb{N})$, then $\tilde{\alpha}_g \in K_A$.*

Proof. Let $\sigma \in \mathrm{Rec}(\mathbb{N})$ be the recursive permutation given by Lemma 4.6. Then it follows easily that $\tilde{\alpha}_g = (\tilde{\theta}_\sigma)^{-1}\,b_{\pi_A}\,\tilde{\theta}_\sigma\,b_{\pi_A}^{-1}$. □

In particular, $\mathrm{Rec}^A(\mathbb{N})$ embeds into K_A via the map $g \mapsto \tilde{\alpha}_g$.

Lemma 4.13. *If A, $B \in 2^{\mathbb{N}}$ and $A \equiv_T B$, then $K_A = K_B$.*

Proof. Let $\sigma \in \mathrm{Rec}(\mathbb{N})$ be the recursive permutation given by Lemma 4.7. Then it follows easily that $(\tilde{\theta}_\sigma)^{-1}\,b_{\pi_A}\,\tilde{\theta}_\sigma = b_{\pi_B}$. □

Summing up, for each marked finitely generated group $\Gamma = (G, (s_1, \cdots, s_m))$, we have constructed a 2-generator subgroup $K_{R_\Gamma} \leqslant \mathrm{Sym}(\mathbb{Z} \times \mathbb{Z} \times \mathbb{N} \times \mathbb{N})$ such that:

(a) G embeds into K_{R_Γ}; and

(b) if $\Delta = (H, (t_1, \cdots, t_n)) \in \mathcal{G}_{fg}$ and $G \cong H$, then $K_{R_\Gamma} = K_{R_\Delta}$.

Of course, after fixing a bijection $\mathbb{N} \to \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} \times \mathbb{N}$, we can replace K_{R_Γ} by the corresponding subgroup of $\mathrm{Sym}(\mathbb{N})$.

Remark 4.14. At this point, some readers may be wondering why Theorem 1.2 is not phrased in term of the map $\Gamma \mapsto K_{R_\Gamma}$ from the space $\mathcal{G}_{fg}$ of finitely generated groups into the "space" $\mathcal{S}_2$ of 2-generator subgroups of $\mathrm{Sym}(\mathbb{N})$. However, as we mentioned in Section 1, the set of 2-generator subgroups of $\mathrm{Sym}(\mathbb{N})$ cannot be regarded as a Polish space in any natural way. Of course, using the Axiom of Choice, we can fix a bijection $\pi : \mathbb{R} \to \mathcal{S}_2$ and then use π to define a corresponding Polish topology on $\mathcal{S}_2$. But it is impossible to define a Polish topology on $\mathcal{S}_2$ such that the map $\Gamma \mapsto K_{R_\Gamma}$ is Borel. This follows from the observation that the map $\Gamma \mapsto K_{R_\Gamma}$ is countable-to-one, together with the result of Champetier [1, Section 4] that there does not exist an $\cong$-invariant countable-to-one Borel map from $\mathcal{G}_{fg}$ into any Polish space.

Finally we will complete the proof of Theorem 1.2. First we record the following intermediate result, which will be used in Section 5.

Theorem 4.15. *There exists a Borel map $\psi : 2^{\mathbb{N}} \to \mathcal{G}_2$ such that for all A, $B \in 2^{\mathbb{N}}$,*

(a) $\mathrm{Rec}^A(\mathbb{N})$ *embeds into* $\psi(A)$*; and*
(b) *if* $A \equiv_T B$*, then* $\psi(A) \cong \psi(B)$.

Proof. In this case, it is more convenient to work with $\mathcal{N}_2$ rather than with $\mathcal{G}_2$. Let

$$\tau : 2^{\mathbb{N}} \to \mathrm{Sym}(\mathbb{Z} \times \mathbb{Z} \times \mathbb{N} \times \mathbb{N}) \times \mathrm{Sym}(\mathbb{Z} \times \mathbb{Z} \times \mathbb{N} \times \mathbb{N})$$

be the Borel map defined by $\tau(A) = (\, a, b_{\pi_A}\,)$; and let

$$\nu : \mathrm{Sym}(\mathbb{Z} \times \mathbb{Z} \times \mathbb{N} \times \mathbb{N}) \times \mathrm{Sym}(\mathbb{Z} \times \mathbb{Z} \times \mathbb{N} \times \mathbb{N}) \to \mathcal{N}_2$$

be the Borel map defined by $\nu(g,h) = \{\, w(x_1,x_2) \in \mathbb{F}_2 \mid w(g,h) = 1\,\}$. Then $\psi = \nu \circ \tau$ satisfies our requirements. □

Proof of Theorem 1.2. Clearly the Borel map $\varphi : \mathcal{G}_{fg} \to \mathcal{G}_2$, defined by

$$\Gamma = (G, (s_1, \cdots, s_m)) \overset{\varphi}{\mapsto} \psi(R_\Gamma),$$

satisfies our requirements. □

5. An Application

In [2], Friedman proved that if $\theta : \mathcal{G}_4 \to 2^{\mathbb{N}}$ is an $\cong$-invariant Borel map, then there exists an $A \in 2^{\mathbb{N}}$ such that $\theta^{-1}(A)$ is *cofinal*; i.e. such that for every countable group G, there exists a group $K \in \theta^{-1}(A)$ such that G embeds into K. In this section, we will prove a natural strengthening of Friedman's theorem.

Definition 5.1. A Borel subset $\mathcal{B} \subseteq \mathcal{G}_2$ has the *Friedman Embedding Property* if there exists a Borel map $\varphi : \mathcal{G}_{fg} \to \mathcal{B}$ such that for all G, $H \in \mathcal{G}_{fg}$,

(a) G embeds into $\varphi(G)$; and
(b) if $G \cong H$, then $\varphi(G) \cong \varphi(H)$.

It is clear that if $\mathcal{B} \subseteq \mathcal{G}_2$ has the Friedman Embedding Property, then $\mathcal{B}$ is cofinal. However, as we will show at the end of this section, the converse does not hold.

Theorem 5.2. *If* $\theta : \mathcal{G}_2 \to 2^{\mathbb{N}}$ *is an* $\cong$*-invariant Borel map, then there exists an* $A \in 2^{\mathbb{N}}$ *such that* $\theta^{-1}(A)$ *has the Friedman Embedding Property.*

Following Friedman [2], the proof of Theorem 5.2 makes use of the following well-known consequence of Borel Determinacy. Recall that for each $B_0 \in 2^{\mathbb{N}}$, the corresponding *cone* is defined to be $\mathcal{C} = \{ B \in 2^{\mathbb{N}} \mid B_0 \leq_T B \}$.

Theorem 5.3 (Martin [10, 11])**.** *If $\sigma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel map, then there exists an $A \in 2^{\mathbb{N}}$ such that $\sigma^{-1}(A)$ contains a cone.*

Proof of Theorem 5.2. Suppose that $\theta : \mathcal{G}_2 \to 2^{\mathbb{N}}$ is an $\cong$-invariant Borel map. Let $\psi : 2^{\mathbb{N}} \to \mathcal{G}_2$ be the Borel map which is given by Theorem 4.15 and let $\sigma = \theta \circ \psi$. Then $\sigma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel map and hence there exists an $A \in 2^{\mathbb{N}}$ such that $\sigma^{-1}(A)$ contains a cone; say, $\mathcal{C} = \{ B \in 2^{\mathbb{N}} \mid B_0 \leq_T B \}$. Consider the Borel map $\varphi : \mathcal{G}_{fg} \to \theta^{-1}(A)$ defined by

$$\Gamma = (G, (s_1, \cdots, s_m)) \mapsto \psi(B_0 \oplus R_\Gamma),$$

where $B_0 \oplus R_\Gamma = \{ 2n \mid n \in B_0 \} \cup \{ 2n+1 \mid n \in R_\Gamma \}$ is the *recursive join* of B_0 and R_Γ. Then φ witnesses that $\theta^{-1}(A)$ has the Friedman Embedding Property. □

Suppose that $G \mapsto L_G$ is one of the standard "group-theoretic" constructions of a 2-generator group L_G into which the countable group G embeds. Then by restricting our attention to finitely generated groups, we obtain a corresponding Borel map $\lambda : \mathcal{G}_{fg} \to \mathcal{G}_2$. Since each 2-generator group has only countably many finitely generated subgroups, it follows that λ is a countable-to-one map and hence its image $\mathcal{B} = \{ L_\Gamma \mid \Gamma \in \mathcal{G}_{fg} \}$ is a Borel subset of $\mathcal{G}_2$. (For example, see Kechris [8, Exercise 18.14].) It is clear that $\mathcal{B}$ will always be cofinal, but I suspect that $\mathcal{B}$ will never have the Friedman Embedding Property. In the proof of the next theorem, we will check that this is indeed the case for a particular construction due to Miller-Schupp [12].

Theorem 5.4. *There exists a cofinal Borel subset $\mathcal{B} \subseteq \mathcal{G}_2$ which does not have the Friedman Embedding Property.*

Proof. Let C_5 and C_7 be cyclic groups of orders 5 and 7; and let x, y be generators of C_5 and C_7 respectively. For each marked group $\Gamma = (G, (g_1, g_2)) \in \mathcal{G}_2$, let F_Γ be the free product $G * C_5 * C_7$. Let

$$r_0 = xy\, xy^2\, (xy)^2\, xy^2\, (xy)^3\, xy^2 \cdots (xy)^{80}\, xy^2$$

and for $i = 1, 2$, let

$$r_i = g_i^{-1} \prod_{j=80i+1}^{80(i+2)} (\,(xy)^j\, xy^2\,).$$

Let N_Γ be the normal closure of $R_\Gamma = \{\, r_0, r_1, r_2 \,\}$ in F_Γ and let $L_\Gamma = F_\Gamma / N_\Gamma$. By Lyndon-Schupp [9, Theorem 10.4], G embeds into the 2-generator group L_Γ.

We claim that the groups $\{\, L_\Gamma \mid \Gamma \in \mathcal{G}_2 \,\}$ are pairwise nonisomorphic. To see this, suppose that $\Gamma = (G, (g_1, g_2))$, $\Delta = (H, (h_1, h_2)) \in \mathcal{G}_2$ and that $\psi : L_\Gamma \to L_\Delta$ is an isomorphism. Then the proof of Lyndon-Schupp [9, Theorem 10.4] shows that, after replacing ψ by its composition with a suitably chosen inner automorphism of L_Δ, we can suppose that $\psi(x) = x$ and $\psi(y) = y$. But this means that $\psi(g_1) = h_1$ and $\psi(g_2) = h_2$; and hence the marked groups Γ and Δ are equal.

Clearly the Borel subset $\mathcal{B} = \{\, L_\Gamma \mid \Gamma \in \mathcal{G}_2 \,\} \subseteq \mathcal{G}_2$ is cofinal. To see that $\mathcal{B}$ does not have the Friedman Embedding Property, note that if the Borel map $\varphi : \mathcal{G}_{fg} \to \mathcal{B}$ satisfies conditions 5.1(a) and 5.1(b), then φ is an $\cong$-invariant countable-to-one Borel map from $\mathcal{G}_{fg}$ into the Polish space $\mathcal{G}_2$, which contradicts Champetier [1]. $\square$

6. Some Open Questions

In this section, we will discuss a number of open questions, including the question of whether there exists a more purely "group-theoretic" approach to the Friedman Embedding Theorem. By this, I mean a construction which only involves purely group-theoretic notions such as wreath products, free products with amalgamation, HNN-extensions, etc. In each case that I have considered, such a construction induces a continuous map on the space of marked finitely generated groups. Thus it is natural to ask whether various group-theoretic problems have continuous solutions.

Conjecture 6.1. There does not exist a continuous map $\varphi : \mathcal{G}_3 \to \mathcal{G}_2$ such that for all 3-generator groups G, $H \in \mathcal{G}_3$,

(a) G embeds into $\varphi(G)$; and
(b) if $G \cong H$, then $\varphi(G) \cong \varphi(H)$.

The Borel map $\varphi : \mathcal{G}_{fg} \to \mathcal{G}_2$ constructed in Section 4 is certainly not continuous. To prove this, we will make use of the following observation.

Proposition 6.2. *Suppose that $\tau : \mathcal{G}_{fg} \to \mathcal{G}_{fg}$ is a continuous map. Then there exists a cone $\mathcal{C} \subseteq 2^{\mathbb{N}}$ such that for all $\Gamma = (G, (s_1, \cdots, s_m)) \in \mathcal{G}_{fg}$, if $R_\Gamma \in \mathcal{C}$, then $R_{\tau(\Gamma)} \leq_T R_\Gamma$.*

Proof. Recall that if $\Gamma = (G, (s_1, \cdots, s_m)) \in \mathcal{G}_m \subseteq \mathcal{G}_{fg}$, then a neighborhood basis of Γ in $\mathcal{G}_{fg}$ is given by the collection of open sets

$$V_{(G,\bar{s}),\ell} = \{ (H, \bar{t}\,) \in \mathcal{G}_m \mid B_\ell(H, \bar{t}\,) \cong B_\ell(G, \bar{s}) \}, \quad \ell \geq 1.$$

Let $\{ V_k \mid k \in \mathbb{N} \}$ be an effective enumeration of the countably many open sets which arise as we vary through both m and $\Gamma \in \mathcal{G}_m$. Since $\tau : \mathcal{G}_{fg} \to \mathcal{G}_{fg}$ is continuous, for each $k \in \mathbb{N}$, there exists $f(k) \in \mathbb{N}$ such that $\tau[\,V_{f(k)}\,] \subseteq V_k$. Let $A \in 2^{\mathbb{N}}$ encode the function $f \in \mathbb{N}^{\mathbb{N}}$. Then we claim that the corresponding cone $\mathcal{C} = \{ B \in 2^{\mathbb{N}} \mid A \leq_T B \}$ satisfies our requirements. To see this, suppose that $\Gamma \in \mathcal{G}_{fg}$ with $R_\Gamma \in \mathcal{C}$ and let $\tau(\Gamma) = \Delta$, where $\Delta = (H, \bar{t}\,) \in \mathcal{G}_n$. Let $w(\bar{x}) \in \mathbb{F}_n$ be a word of length ℓ. Then in order to decide whether $w(\bar{t}) = 1$, it is enough to compute the closed ball $B_\ell(H, \bar{t}\,)$ of radius ℓ around the identity element in the (labelled directed) Cayley graph $\mathrm{Cay}(H, \bar{t}\,)$; and this can be computed using an R_Γ-oracle as follows. Working successively through each $r \geq 0$, we first use the R_Γ-oracle to compute the closed ball $B_r(G, \bar{s})$ and let $V_{g(r)}$ be the corresponding open neighborhood of Γ. We then use the R_Γ-oracle to compute the ordered pair $(r, f(r))$. We continue this process until we have obtained natural numbers $i, j \leq r$ such that:

(1) V_i is an open set determined by a closed ball B of radius at least ℓ; and
(2) $(i, g(j)) = (k, f(k))$ for some $k \leq r$.

Since $\Gamma \in V_{g(j)}$ and $\tau[\,V_{g(j)}\,] = \tau[\,V_{f(k)}\,] \subseteq V_k = V_i$, it follows that $\Delta \in V_i$ and hence $B_\ell(H, \bar{t}) \subseteq B$. □

In order to prove that the Borel map $\varphi : \mathcal{G}_{fg} \to \mathcal{G}_2$ constructed in Section 4 is not continuous, it is thus enough to show that if $\Gamma \in \mathcal{G}_{fg}$ and $A = R_\Gamma$, then $A <_T R_{K_A}$. To see this, let

$$S = \{ \ell \in \mathbb{N} \mid \varphi_\ell^A \in \mathrm{Sym}(\mathbb{N}) \smallsetminus \{ \mathrm{id}_{\mathbb{N}} \} \}.$$

Then we can define a Turing reduction from S to the word problem for K_A as follows. Examining the proof of Lemma 3.6, we see that if $\ell \in \mathbb{N}$, then we can effectively find an integer $k \in \mathbb{N}$ such that the recursive permutation $\sigma = \varphi_k^{\emptyset} = \psi_k^{\emptyset}$ satisfies

$$\psi^A_{\sigma(s)} = \begin{cases} \psi^A_\ell & \text{if } s = 0; \\ \psi^A_s & \text{otherwise.} \end{cases}$$

Using Lemmas 4.9 and 4.11, we can then effectively find a word $w(x_1, x_2) \in \mathbb{F}_2$ such that $w(a, b_{\pi_A}) = \tilde{\theta}_\sigma$. Applying Lemma 4.12, we have that

$$\ell \in S \iff (\tilde{\theta}_\sigma)^{-1}\, b_{\pi_A}\, \tilde{\theta}_\sigma\, b_{\pi_A}^{-1} \neq 1.$$

and hence $S \leq_T R_{K_A}$. Finally a routine modification of Soare [14, 4.3.2] shows that $S \equiv_T A''$, where A'' denotes the double Turing jump of A. (Recall that if $A \in 2^{\mathbb{N}}$, then the *Turing jump* of A is $A' = \{\, s \in \mathbb{N} \mid \varphi^A_s(s) \text{ is defined} \,\}$ and that $A <_T A'$. For example, see Rogers [13].) It follows that $A <_T A' <_T A'' \leq_T R_{K_A}$.

There are many other situations in which it is known that there exists a Borel map $\varphi : \mathcal{G}_{fg} \to \mathcal{G}_{fg}$ satisfying certain properties and it is unknown whether there exists a continuous such map. For example, consider the following Borel equivalence relation on the space $\mathcal{G}_{fg}$ of finitely generated groups.

Definition 6.3. The finitely generated groups G_1, $G_2 \in \mathcal{G}_{fg}$ are *bi-embeddable*, written $G_1 \approx_E G_2$, if G_1 embeds into G_2 and G_2 embeds into G_1.

Since each finitely generated group has only countably many finitely generated subgroups, it follows that $\approx_E$ is a *countable* Borel equivalence relation; i.e. that every $\approx_E$-class is countable. Since Hjorth [6] has shown that the isomorphism relation $\cong$ on $\mathcal{G}_2$ is a universal countable Borel equivalence relation, it follows that there exists a *Borel reduction* $\varphi : \mathcal{G}_{fg} \to \mathcal{G}_2$ from the bi-embeddability relation $\approx_E$ to the isomorphism relation $\cong$; i.e. a Borel map φ such that for all G_1, $G_2 \in \mathcal{G}_{fg}$,

$$G_1 \approx_E G_2 \iff \varphi(G_1) \cong \varphi(G_2).$$

However, I do not know how to explicitly define an example of such a Borel reduction φ and it seems unlikely that there exists a purely group-theoretic reduction. (The only known proof of the existence of such a Borel reduction ultimately relies on the Lusin-Novikov Uniformization Theorem [8, 18.10] and this proof does not provide an explicit example of such a Borel reduction.)

Conjecture 6.4. There does not exist a continuous Borel reduction $\varphi : \mathcal{G}_{fg} \to \mathcal{G}_2$ from the bi-embeddability relation $\approx_E$ to the isomorphism relation $\cong$.

In a similar vein, consider the commensurability relation on the space $\mathcal{G}_{fg}$ of finitely generated groups.

Definition 6.5. The finitely generated groups G_1, $G_2 \in \mathcal{G}_{fg}$ are *commensurable*, written $G_1 \approx_C G_2$, if there exist subgroups $H_i \leqslant G_i$ of finite index such that $H_1 \cong H_2$.

Once again, $\approx_C$ is a countable Borel equivalence relation and hence there exists a Borel reduction $\varphi : \mathcal{G}_{fg} \to \mathcal{G}_2$ from the commensurability relation $\approx_C$ to the isomorphism relation $\cong$. And once again, I do not know how to explicitly define an example of such a Borel reduction φ.

Conjecture 6.6. There does not exist a continuous Borel reduction $\varphi : \mathcal{G}_{fg} \to \mathcal{G}_2$ from the commensurability relation $\approx_C$ to the isomorphism relation $\cong$.

References

[1] C. Champetier, *L'espace des groupes de type fini*, Topology **39** (2000), 657–680.

[2] H. Friedman, *New Borel independence results*, preprint (2007).

[3] F. Galvin, *Embedding Countable Groups in 2-generator Groups*, Amer. Math. Monthly **100** (1993), 578–580.

[4] R. I. Grigorchuk, *Degrees of growth of finitely generated groups and the theory of invariant means*, Math. SSSR. Izv. **25** (1985), 259–300.

[5] G. Higman, B. H. Neumann and H. Neumann, *Embedding theorems for groups*, J. London Math. Soc. **24** (1949), 247–254.

[6] G. Hjorth, *Two-generated groups are universal*, preprint (2008).

[7] G. Hjorth and A. S. Kechris, *Borel equivalence relations and classification of countable models*, Ann. Pure Appl. Logic **82** (1996), 221–272.

[8] A. S. Kechris *Classical Descriptive Set Theory*, Graduate Texts in Mathematics **156**, Springer-Verlag, 1995.

[9] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Springer-Verlag, Berlin 1977.

[10] D. A. Martin, *The axiom of determinacy and reduction principles in the analytical hierarchy*, Bull. Amer. Math. Soc. **74** (1968), 687–689.

[11] D. A. Martin, *Borel determinacy*, Ann. Math. **102** (1975), 363–371.

[12] C. F. Miller III and P. E. Schupp, *Embeddings into Hopfian groups*, J. Algebra **17** (1971), 171–176.

[13] Hartley Rogers Jr., *Theory of Recursive Functions and Effective Computability*, 2nd edition, MIT Press, Cambridge, 1987.

[14] R. I. Soare, *Recursively Enumerable Sets and Degrees*, Springer-Verlag, New York, 1987.

[15] S. Thomas, *A remark on the Higman-Neumann-Neumann Embedding Theorem*, J. Group Theory **12** (2009), 561–565.

[16] S. Thomas and B. Velickovic, *On the complexity of the isomorphism relation for finitely generated groups*, J. Algebra **217** (1999), 352–373.

Mathematics Department, Rutgers University, 110 Frelinghuysen Road, Piscataway, New Jersey 08854-8019, USA

E-mail address: `sthomas@math.rutgers.edu`