Question 1. Recall that the set $M_{2 \times 2}(\mathbb{R})$ of $2 \times 2$ real matrices, equipped with matrix addition and scalar multiplication

$$
\begin{pmatrix}
a & b \\
c & d \\
\end{pmatrix} + \begin{pmatrix}
a' & b' \\
c' & d' \\
\end{pmatrix} = \begin{pmatrix}
a + a' & b + b' \\
c + c' & d + d' \\
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
a & b \\
\end{pmatrix} = \begin{pmatrix}
ra & rb \\
rc & rd \\
\end{pmatrix},
$$

is a vector space of dimension 4.

(i) Determine whether the set $S = \{ A \in M_{2 \times 2}(\mathbb{R}) \mid A^t = A \}$ of symmetric $2 \times 2$ real matrices is a subspace of $M_{2 \times 2}(\mathbb{R})$; and if so, compute its dimension.

(ii) Determine whether the set $S = \{ A \in M_{2 \times 2}(\mathbb{R}) \mid \det(A) = 0 \}$ of non-invertible $2 \times 2$ real matrices is a subspace of $M_{2 \times 2}(\mathbb{R})$; and if so, compute its dimension.

(iii) State the Cayley-Hamilton Theorem.

(iv) Prove that if $n \geq 2$, then there does not exist an $n \times n$ real matrix $A \in M_{n \times n}(\mathbb{R})$ such that $\{ A^\ell \mid 1 \leq \ell \leq n^2 \}$ is a basis of $M_{n \times n}(\mathbb{R})$.

Question 2.

(i) Compute the inverse of the following matrix:

$$
A = \begin{pmatrix}
1 & 2 & 1 \\
2 & 5 & 4 \\
1 & 1 & 0 \\
\end{pmatrix} \in M_{3 \times 3}(\mathbb{R})
$$

(ii) Solve the following system of linear equations:

$$
x_1 + 2x_2 + x_3 = 2
$$

$$
2x_1 + 5x_2 + 4x_3 = 4
$$

$$
x_1 + x_2 = 1
$$
Question 3. Let $A \in \mathbb{M}_{3 \times 3}(\mathbb{R})$ be the matrix
\[
A = \begin{pmatrix}
1 & 3 & 3 \\
0 & 1 & 0 \\
0 & 3 & 4
\end{pmatrix}
\]
(i) Find a diagonal matrix $D$ and an invertible matrix $Q$ such that
$D = Q^{-1}AQ$.
(ii) Find a matrix $B \in \mathbb{M}_{3 \times 3}(\mathbb{R})$ such that $B^2 = A$. (Hint: it is easy
to find a matrix $C \in \mathbb{M}_{3 \times 3}(\mathbb{R})$ such that $C^2 = D$.)

Question 4. Let $A \in \mathbb{M}_{3 \times 3}(\mathbb{C})$ be the matrix
\[
A = \begin{pmatrix}
0 & 1 & 1 \\
2 & 1 & -1 \\
-6 & -5 & -3
\end{pmatrix}
\]
Then $A$ has the characteristic polynomial
\[
f(t) = -(t - 2)(t + 2)^2.
\]
Find a Jordan canonical form $J$ of $A$ and an invertible matrix $Q$ such that
$Q^{-1}AQ = J$. 
**Question 5.** Recall that if \( A, B \in M_{n \times n}(F) \), then \( A \) and \( B \) are *similar* if there exists an invertible matrix \( Q \in M_{n \times n}(F) \) such that \( Q^{-1}AQ = B \).

(i) Let \( A, B \in M_{3 \times 3}(\mathbb{C}) \) be the matrices

\[
A = \begin{pmatrix}
1 & 3 & 3 \\
-3 & -5 & -3 \\
3 & 3 & 1
\end{pmatrix} \quad B = \begin{pmatrix}
2 & 4 & 3 \\
-4 & -6 & -3 \\
3 & 3 & 1
\end{pmatrix}
\]

Then \( A \) and \( B \) both have the characteristic polynomial

\[
f(t) = -(t - 1)(t + 2)^2.
\]

Determine whether \( A \) and \( B \) are similar.

(ii) Suppose that \( A_1, A_2, A_3 \in M_{3 \times 3}(\mathbb{C}) \) and that all three matrices have the characteristic polynomial

\[
f(t) = -(t - 1)(t + 2)^2.
\]

Prove that there exist \( i \neq j \) such that \( A_i \) is similar to \( A_j \).

**Question 6.** Let \( A \) be an \( n \times n \) matrix over a field \( F \).

(i) Give the definition of an eigenvector and eigenvalue of \( A \).

(ii) Prove that if \( \lambda \) is an eigenvalue of \( A \), then \( \lambda \) is a root of the characteristic polynomial \( f(t) = \det(A - tI) \).

Let \( a, b, c \in F \) be scalars and let

\[
B = \begin{pmatrix}
a & b & c \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

(iii) Prove that if \( \lambda \) is an eigenvalue of \( B \), then

\[
\mathbf{v} = \begin{pmatrix}
\lambda^2 \\
\lambda \\
1
\end{pmatrix}
\]

is an eigenvector corresponding to \( \lambda \).