350 FINAL FALL 2019

Question 1. Recall that the set $\mathrm{M}_{2\times 2}(\mathbb{R})$ of 2×2 real matrices, equipped with matrix addition and scalar multiplication

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix} \quad \text{and} \quad r\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix},$$

is a vector space of dimension 4.

(i) Determine whether the set $S = \{\, A \in \mathrm{M}_{2\times 2}(\mathbb{R}) \mid A^t = A \,\}$ of symmetric 2×2 real matrices is a subspace of $\mathrm{M}_{2\times 2}(\mathbb{R})$; and if so, compute its dimension.

(ii) Determine whether the set $S = \{\, A \in \mathrm{M}_{2\times 2}(\mathbb{R}) \mid \det(A) = 0 \,\}$ of non-invertible 2×2 real matrices is a subspace of $\mathrm{M}_{2\times 2}(\mathbb{R})$; and if so, compute its dimension.

(iii) State the Cayley-Hamilton Theorem.

(iv) Prove that if $n \geq 2$, then there does *not* exist an $n \times n$ real matrix $A \in \mathrm{M}_{n\times n}(\mathbb{R})$ such that $\{\, A^\ell \mid 1 \leq \ell \leq n^2 \,\}$ is a basis of $\mathrm{M}_{n\times n}(\mathbb{R})$.

Question 2.

(i) Compute the inverse of the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 4 \\ 1 & 1 & 0 \end{pmatrix} \in \mathrm{M}_{3\times 3}(\mathbb{R})$$

(ii) Solve the following system of linear equations:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 2x_1 + 5x_2 + 4x_3 &= 4 \\ x_1 + x_2 &= 1 \end{aligned}$$

Question 3. Let $A \in \mathrm{M}_{3\times 3}(\mathbb{R})$ be the matrix

$$A = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & 4 \end{pmatrix}$$

(i) Find a diagonal matrix D and an invertible matrix Q such that $D = Q^{-1}AQ$.

(ii) Find a matrix $B \in \mathrm{M}_{3\times 3}(\mathbb{R})$ such that $B^2 = A$. (*Hint:* it is easy to find a matrix $C \in \mathrm{M}_{3\times 3}(\mathbb{R})$ such that $C^2 = D$.)

Question 4. Let $A \in \mathrm{M}_{3\times 3}(\mathbb{C})$ be the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & -1 \\ -6 & -5 & -3 \end{pmatrix}$$

Then A has the characteristic polynomial

$$f(t) = -(t-2)(t+2)^2.$$

Find a Jordan canonical form J of A and an invertible matrix Q such that $Q^{-1}AQ = J$.

Question 5. Recall that if A, $B \in \mathrm{M}_{n\times n}(F)$, then A and B are *similar* if there exists an invertible matrix $Q \in \mathrm{M}_{n\times n}(F)$ such that $Q^{-1}AQ = B$.

(i) Let A, $B \in \mathrm{M}_{3\times 3}(\mathbb{C})$ be the matrices

$$A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{pmatrix}$$

Then A and B both have the characteristic polynomial

$$f(t) = -(t-1)(t+2)^2.$$

Determine whether A and B are similar.

(ii) Suppose that A_1, A_2, $A_3 \in \mathrm{M}_{3\times 3}(\mathbb{C})$ and that all three matrices have the characteristic polynomial

$$f(t) = -(t-1)(t+2)^2.$$

Prove that there exist $i \neq j$ such that A_i is similar to A_j.

Question 6. Let A be an $n \times n$ matrix over a field F.

(i) Give the definition of an eigenvector and eigenvalue of A.

(ii) Prove that if λ is an eigenvalue of A, then λ is a root of the characteristic polynomial $f(t) = \det(A - tI)$.

Let a, b, $c \in F$ be scalars and let

$$B = \begin{pmatrix} a & b & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(iii) Prove that if λ is an eigenvalue of B, then

$$\mathbf{v} = \begin{pmatrix} \lambda^2 \\ \lambda \\ 1 \end{pmatrix}$$

is an eigenvector corresponding to λ.