CHARACTERS AND INVARIANT RANDOM SUBGROUPS OF THE FINITARY SYMMETRIC GROUP

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Abstract. We will describe the relationship between the indecomposable characters of $\mathrm{Fin}(\mathbb{N})$ and its ergodic invariant random subgroups; and we will interpret each Thoma character $\chi_{(\beta;\gamma)}$ as an asymptotic limit of a naturally associated sequence of characters induced from linear characters of Young subgroups of finite symmetric groups.

1. Introduction

If Ω is an infinite set, then the corresponding *finitary symmetric group* $\mathrm{Fin}(\Omega)$ is the group of permutations $g \in \mathrm{Sym}(\Omega)$ such that $\mathrm{supp}(g) = \{\, \omega \in \Omega \mid g(\omega) \neq \omega \,\}$ is finite. In his classic paper [7], Thoma classified the indecomposable characters of $\mathrm{Fin}(\mathbb{N})$. More recently, Vershik [11] classified the ergodic invariant random subgroups of $\mathrm{Fin}(\mathbb{N})$; and he pointed out that the indecomposable characters of $\mathrm{Fin}(\mathbb{N})$ are very closely connected with its ergodic invariant random subgroups. In this paper, we will describe the precise relationship between the indecomposable characters of $\mathrm{Fin}(\mathbb{N})$ and its ergodic invariant random subgroups. Before we stating our main result, we will recall Thoma's classification of the the indecomposable characters of $\mathrm{Fin}(\mathbb{N})$ and Vershik's classification[1] of the ergodic invariant subgroups of $\mathrm{Fin}(\mathbb{N})$. Throughout this paper, $D[\,0,1\,]$ will denote the set of sequences

$$\alpha = (\, \alpha_1, \alpha_2, \cdots, \alpha_n, \cdots \,) \in [\,0,1\,]^{\mathbb{N}^+}$$

such that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq \cdots$

First recall that if G is a countable group, then a function $\chi : G \to \mathbb{C}$ is said to be a *character* if the following conditions are satisfied:

(i) $\chi(h\,g\,h^{-1}) = \chi(g)$ for all $g, \in G$.
(ii) $\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j \chi(g_j^{-1} g_i) \geq 0$ for all $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$ and $g_1, \cdots, g_n \in G$.
(iii) $\chi(1_G) = 1$.

A character χ is said to be *indecomposable* or *extremal* if it is impossible to express $\chi = r\chi_1 + (1-r)\chi_2$, where $0 < r < 1$ and $\chi_1 \neq \chi_2$ are distinct characters. By Thoma [7], the indecomposable characters of $\mathrm{Fin}(\mathbb{N})$ are precisely the functions

$$\chi_{(\beta;\gamma)}(g) = \prod_{k=2}^{\infty} (\sum_{i=1}^{\infty} \beta_i^k + (-1)^{k+1} \sum_{i=1}^{\infty} \gamma_i^k \,)^{c_k(g)},$$

where $\beta = (\,\beta_i\,)_{i \in \mathbb{N}^+}$, $\gamma = (\,\gamma_i\,)_{i \in \mathbb{N}^+} \in D[\,0,1\,]$ are such that $\sum_{i=1}^{\infty} \beta_i + \sum_{i=1}^{\infty} \gamma_i \leq 1$, and $c_k(g)$ is the number of cycles of length k in the cyclic decomposition of the permutation g.

[1] We will take this opportunity to correct an inaccuracy in the statement [11] of Vershik's classification theorem.

Next suppose that G is a countably infinite group and let Sub_G be the compact space of subgroups $H \leqslant G$. Then a Borel probability measure ν on Sub_G which is invariant under the conjugation action of G on Sub_G is called an *invariant random subgroup* or *IRS*. For example, suppose that G acts via measure-preserving maps on the Borel probability space (Z, μ) and let $f : Z \to \mathrm{Sub}_G$ be the G-equivariant map defined by

$$z \mapsto G_z = \{ g \in G \mid g \cdot z = z \}.$$

Then the corresponding *stabilizer distribution* $\nu = f_* \mu$ is an IRS of G. In fact, by a result of Abért-Glasner-Virag [1], every IRS of G can be realized as the stabilizer distribution of a suitably chosen measure-preserving action. Moreover, by Creutz-Peterson [2], if ν is an ergodic IRS of G, then ν is the stabilizer distribution of an ergodic action $G \curvearrowright (Z, \mu)$. If ν is an IRS of G, then we can define a corresponding character χ_ν by

$$\chi_\nu(g) = \nu(\{ H \in \mathrm{Sub}_G \mid g \in H \}).$$

Equivalently, $\chi_\nu(g) = \mu(\mathrm{Fix}_Z(g))$, where $G \curvearrowright (Z, \mu)$ is any measure-preserving action with stabilizer distribution ν.

In order to describe the ergodic IRSs of $\mathrm{Fin}(\mathbb{N})$, let $\alpha = (\alpha_i)_{i \in \mathbb{N}^+} \in D[0,1]$ be such that $\sum_{i=1}^\infty \alpha_i \leq 1$ and let $\alpha_0 = 1 - \sum_{i=1}^\infty \alpha_i$. Then we can define a probability measure p_α on $\mathbb{N}$ by $p_\alpha(\{ i \}) = \alpha_i$. Let μ_α be the corresponding product probability measure on $\mathbb{N}^\mathbb{N}$. Then $\mathrm{Fin}(\mathbb{N})$ acts ergodically on $(\mathbb{N}^\mathbb{N}, \mu_\alpha)$ via the shift action $(g \cdot \xi)(n) = \xi(g^{-1}(n))$. For each $\xi \in \mathbb{N}^\mathbb{N}$ and $i \in \mathbb{N}$, let $B_i^\xi = \{ n \in \mathbb{N} \mid \xi(n) = i \}$. Then for μ_α-a.e. $\xi \in \mathbb{N}^\mathbb{N}$, the following statements are equivalent for all $i \in \mathbb{N}$.

(a) $\alpha_i > 0$.
(b) $B_i^\xi \neq \emptyset$.
(c) B_i^ξ is infinite.
(d) $\lim_{n \to \infty} |B_i^\xi \cap \{ 0, 1, \cdots, n-1 \}|/n = \alpha_i > 0$.

In this case, we will say that ξ is *μ_α-generic*. First suppose that $\alpha_0 \neq 1$, so that $I = \{ i \in \mathbb{N}^+ \mid \alpha_i > 0 \} \neq \emptyset$. Let $S_\alpha = \bigoplus_{i \in I} C_i$ be the restricted direct product of the cyclic groups $C_i = \{ \pm 1 \}$ of order 2. (*Warning:* throughout this paper, we will regard S_α as a multiplicative group.) Then for each subgroup $A \leqslant S_\alpha$, we can define a corresponding $\mathrm{Fin}(\mathbb{N})$-equivariant Borel map

$$\begin{aligned} f_\alpha^A : \mathbb{N}^\mathbb{N} &\to \mathrm{Sub}_{\mathrm{Fin}(\mathbb{N})} \\ \xi &\mapsto H_\xi \end{aligned}$$

as follows. If ξ is μ_α-generic, then $H_\xi = s_\xi^{-1}(A)$, where s_ξ is the homomorphism

$$\begin{aligned} s_\xi : \bigoplus_{i \in I} \mathrm{Fin}(B_i^\xi) &\to \bigoplus_{i \in I} C_i \\ (\pi_i) &\mapsto (\mathrm{sgn}(\pi_i)). \end{aligned}$$

Otherwise, if ξ is not μ_α-generic, then we let $H_\xi = 1$. Let $\nu_\alpha^A = (f_\alpha^A)_* \mu_\alpha$ be the corresponding ergodic IRS of $\mathrm{Fin}(\mathbb{N})$. Finally, if $\alpha_0 = 1$, then we define $S_\alpha = 1$ and $\nu_\alpha^{E_\alpha} = \delta_1$.

Theorem 1.1. *If ν is an ergodic IRS of* $\mathrm{Fin}(\mathbb{N})$*, then there exists α, A as above such that $\nu = \nu_\alpha^A$.*

Remark 1.2. There exist examples of sequences α and distinct subgroups A, $A' \leqslant S_\alpha$ such that $\nu_\alpha^A = \nu_\alpha^{A'}$. For example, suppose that $\alpha_1 = \alpha_2 = 1/2$. Then if $A_1 = C_1 \oplus 1$ and $A_2 = 1 \oplus C_2$, then $\nu_\alpha^{A_1} = \nu_\alpha^{A_2}$. However, since for μ_α-a.e. $\xi \in \mathbb{N}^\mathbb{N}$,

$$\lim_{n\to\infty} |B_i^\xi \cap \{\,0, 1, \cdots, n-1\,\}|/n = \alpha_i,$$

it follows that if $\alpha \neq \alpha'$ and A, A' are subgroups of S_α, $S_{\alpha'}$, then $\nu_\alpha^A \neq \nu_{\alpha'}^{A'}$.

Remark 1.3. Suppose that $\alpha = (\,\alpha_i\,)_{i\in\mathbb{N}^+}$ is such that there exist $i \in \mathbb{N}^+$ with $\alpha_i = \alpha_{i+1} > 0$. Let $A \leqslant S_\alpha$ be any subgroup and let $\mathcal{S}_\alpha^A$ be the set of subgroups $H \in \mathrm{Sub}_{\mathrm{Fin}(\mathbb{N}}$ such that there exists a μ_α-generic $\xi \in \mathbb{N}^\mathbb{N}$ with $\xi \overset{f_\alpha^A}{\mapsto} H_\xi = H$. Then the map $\xi \mapsto H_\xi \in \mathcal{S}_\alpha^A$ is not injective; and it is easily seen that there does not exist a Fin($\mathbb{N}$)-equivariant Borel map $H \mapsto \xi_H$ from $\mathcal{S}_\alpha^A$ to $\mathbb{N}^\mathbb{N}$ such that $H = H_{\xi_H}$.

The relationship between the indecomposable characters of Fin($\mathbb{N}$) and its ergodic IRSs is most obvious when $A = S_\alpha$: in this case, it is easily checked that

$$\chi_{\nu_\alpha^{S_\alpha}}(g) = \prod_{k=2}^{\infty} (\sum_{i=1}^{\infty} \alpha_i^k\,)^{c_k(g)} = \chi_{(\alpha;\bar{0})}(g),$$

where $\bar{0} \in D[\,0,1\,]$ denotes the identically zero sequence. In particular, it follows that $\chi_{\nu_\alpha^{S_\alpha}}$ is indecomposable. Conversely, the somewhat ad hoc proof of Thomas-Tucker-Drob [9, Theorem 9.2] shows that if $A \neq S_\alpha$, then $\chi_{\nu_\alpha^A}$ is decomposable. (We will give a more informative proof below.) Of course, this result implies that if $\chi_{(\beta;\gamma)}$ is an indecomposable character with $\gamma \neq \bar{0}$, then there does not exist an ergodic IRS ν of Fin($\mathbb{N}$) such that $\chi_\nu = \chi_{(\beta;\gamma)}$.

In order to understand how arbitrary indecomposable characters $\chi_{(\beta;\gamma)}$ are related to the ergodic IRSs of Fin($\mathbb{N}$), let $\widehat{S}_\alpha$ be the compact group of homomorphisms $\sigma : S_\alpha \to \{\,\pm 1\,\}$. For each $i \in I$, let c_i be the generator of $C_i \leqslant S_\alpha$; and for each homomorphism $\sigma \in \widehat{S}_\alpha$, let $\sigma(i) = \sigma(c_i)$. Then for each $\sigma \in \widehat{S}_\alpha$, we can define an indecomposable character of Fin($\mathbb{N}$) by

$$\chi_\alpha^\sigma(g) = \prod_{k=2}^{\infty} (\sum_{i\in I} \sigma(i)^{k+1}\alpha_i^k\,)^{c_k(g)}. \tag{1.4}$$

Remark 1.5. Suppose that $\sigma \in \widehat{S}_\alpha$. Let $\beta = (\,\beta_j\,)_{j\in\mathbb{N}^+} \in D[\,0,1\,]$ be the list (possibly augmented by a sequence of zeros) in decreasing magnitude of the α_i, $i \in I$, such that $\sigma(i) = 1$; and let $\gamma = (\,\gamma_j\,)_{j\in\mathbb{N}^+} \in D[\,0,1\,]$ be the list (possibly augmented by a sequence of zeros) in decreasing magnitude of the α_i, $i \in I$, such that $\sigma(i) = -1$. Then clearly $\chi_\alpha^\sigma = \chi_{(\beta;\gamma)}$.

Conversely, if $\chi_{(\beta;\gamma)}$ is any indecomposable character of Fin($\mathbb{N}$), then there exists a sequence $\alpha = (\,\alpha_i\,)_{i\in\mathbb{N}^+} \in D[\,0,1\,]$ and a homomorphism $\sigma \in \widehat{S}_\alpha$ such that $\chi_{(\beta;\gamma)} = \chi_\alpha^\sigma$.

Example 1.6. For later use, notice that if $\sigma \in \widehat{S}_\alpha$ is the trivial homomorphism such that $\sigma(s) = 1$ for all $s \in S_\alpha$, then $\chi_\alpha^\sigma = \chi_{(\alpha;\bar{0})} = \chi_{\nu_\alpha^{S_\alpha}}$.

For each subgroup $A \leqslant S_\alpha$, let $(\widehat{S_\alpha/A})$ be the compact subgroup of those $\sigma \in \widehat{S}_\alpha$ such that $\sigma(a) = 1$ for all $a \in A$ and let μ_α^A be the Haar probability measure on $(\widehat{S_\alpha/A})$. The following result describes the relationship between the indecomposable characters of Fin($\mathbb{N}$) and its ergodic invariant random subgroups.

Theorem 1.7. *If α, A are as above, then for each $g \in \mathrm{Fin}(\mathbb{N})$,*

$$\chi_{\nu_\alpha^A}(g) = \int_{\sigma \in \widehat{(S_\alpha/A)}} \chi_\alpha^\sigma(g)\, d\mu_\alpha^A. \tag{1.7}$$

Corollary 1.8. *$\chi_{\nu_\alpha^A}$ is indecomposable if and only if $A = S_\alpha$.*

□

The following corollary shows that for every indecomposable character χ of $\mathrm{Fin}(\mathbb{N})$, there exists a canonically associated ergodic IRS.

Corollary 1.9. *If χ is any indecomposable character of $\mathrm{Fin}(\mathbb{N})$, then there exists an $\alpha \in D[\,0,1\,]$ and a subgroup $A \leqslant S_\alpha$ such that*

$$\chi_{\nu_\alpha^A} = \frac{1}{2}\chi_{\nu_\alpha^{S_\alpha}} + \frac{1}{2}\chi.$$

Proof. Applying Thoma's classification [7], let $\beta = (\,\beta_i\,)_{i \in \mathbb{N}^+}$, $\gamma = (\,\gamma_i\,)_{i \in \mathbb{N}^+}$ be such that $\chi = \chi_{(\beta;\gamma)}$. Let $\alpha = (\,\alpha_i\,)_{i \in \mathbb{N}^+}$ be a list in decreasing magnitude of the entries of the sequences $\beta = (\,\beta_i\,)_{i \in \mathbb{N}^+}$ and $\gamma = (\,\gamma_i\,)_{i \in \mathbb{N}^+}$. Let $\sigma \in \widehat{S}_\alpha$ be such that

$$\sigma(i) = \begin{cases} 1, & \text{if } \alpha_i = \beta_j \text{ for some } j \in \mathbb{N}^+; \\ -1, & \text{if } \alpha_i = \gamma_j \text{ for some } j \in \mathbb{N}^+; \end{cases}$$

and let $A = \{\, s \in S_\alpha \mid \sigma(s) = 1 \,\}$. Then

$$\chi_{\nu_\alpha^A} = \frac{1}{2}\chi_{\nu_\alpha^{S_\alpha}} + \frac{1}{2}\chi_\alpha^\sigma = \frac{1}{2}\chi_{\nu_\alpha^{S_\alpha}} + \frac{1}{2}\chi_{(\beta;\gamma)} = \frac{1}{2}\chi_{\nu_\alpha^{S_\alpha}} + \frac{1}{2}\chi.$$

□

The remainder of this paper is organized as follows. In Section 2, we will prove Theorem 1.7 via an easy application of Fubini's Theorem. However, this proof will give no insight into the meaning of equations (1.4) and (1.7). In Section 3, via an application of the Pointwise Ergodic Theorem, we will interpret the integral (1.7) as an asymptotic limit of the Clifford decompositions of a naturally associated sequence of permutation characters of finite symmetric groups; and we will interpret each Thoma character $\chi_{(\beta;\gamma)} = \chi_\alpha^\sigma$ as an asymptotic limit of a naturally associated sequence of characters induced from linear characters of Young subgroups of finite symmetric groups. Finally, in Section 4, slightly correcting the argument of Vershik [11], we will present the proof of Theorem 1.1.

Throughout this paper, we will identify n with the set $\{\, 0, 1, \cdots, n-1 \,\}$ and we will write $S_n = \mathrm{Sym}(n)$.

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2. The proof of Theorem 1.7

In this section, we will present the proof of Theorem 1.7; i.e. that if α, A are as in Section 1, then for each $g \in \mathrm{Fin}(\mathbb{N})$,

$$\chi_{\nu_\alpha^A}(g) = \int_{\sigma \in (\widehat{S_\alpha / A})} \chi_\alpha^\sigma(g)\, d\mu_\alpha^A.$$

Clearly we can suppose that $g \neq 1$. Let $g = g_1 \cdots g_t$ be the decomposition of g into a product of nontrivial cycles; and for each $1 \leq \ell \leq t$, let g_ℓ be an k_ℓ-cycle. Notice that for each $\sigma \in (\widehat{S_\alpha / A})$,

$$|\chi_\alpha^\sigma(g)| = |\prod_{\ell=1}^{t}(\sum_{i \in I} \sigma(i)^{k_\ell+1}\alpha_i^{k_\ell})| \leq \prod_{\ell=1}^{t}(\sum_{i \in I} \alpha_i^{k_\ell}) = \chi_{(\alpha;\bar{0})}(g);$$

and it follows that

$$\int_{\sigma \in (\widehat{S_\alpha / A})} |\chi_\alpha^\sigma(g)|\, d\mu_\alpha^A \leq \chi_{(\alpha;\bar{0})}(g) < \infty.$$

Hence, applying Fubini's theorem, we obtain that

$$\begin{aligned}\int_{\sigma \in (\widehat{S_\alpha / A})} \chi_\alpha^\sigma(g)\, d\mu_\alpha^A &= \int_{\sigma \in (\widehat{S_\alpha / A})} \prod_{\ell=1}^{t}(\sum_{i \in I} \sigma(i)^{k_\ell+1}\alpha_i^{k_\ell})\, d\mu_\alpha^A \\ &= \sum_{\underline{i} \in I^t} \int_{\sigma \in (\widehat{S_\alpha / A})} (\prod_{\ell=1}^{t} \sigma(i_\ell)^{k_\ell+1} \prod_{\ell=1}^{t} \alpha_{i_\ell}^{k_\ell})\, d\mu_\alpha^A \\ &= \sum_{\underline{i} \in I^t} \alpha_{i_1}^{k_1} \cdots \alpha_{i_t}^{k_t} \int_{\sigma \in (\widehat{S_\alpha / A})} \prod_{\ell=1}^{t} \sigma(i_\ell)^{k_\ell+1} d\mu_\alpha^A,\end{aligned}$$

where each $\underline{i} = (\, i_1, \cdots, i_t \,)$.

On the other hand, for each $i \in I$, let $\mathrm{sgn}_i : \mathrm{Fin}(\mathbb{N}) \to C_i \leqslant S_\alpha$ be the homomorphism such that for each $h \in \mathrm{Fin}(\mathbb{N})$ and $j \in I$, the j^{th} component of $\mathrm{sgn}_i(h)$ is given by

$$\mathrm{sgn}_i(h)_j = \begin{cases} \mathrm{sgn}(h) & \text{if } j = i; \\ 1 & \text{if } j \neq i. \end{cases}$$

Then it is clear that

$$\chi_{\nu_\alpha^A}(g) = \sum_{\underline{i} \in I^t} \theta_{\underline{i}}(g) \alpha_{i_1}^{k_1} \cdots \alpha_{i_t}^{k_t},$$

where

$$\theta_{\underline{i}}(g) = \begin{cases} 1, & \text{if } \mathrm{sgn}_{i_1}(g_1) \times \cdots \times \mathrm{sgn}_{i_t}(g_t) \in A; \\ 0, & \text{if } \mathrm{sgn}_{i_1}(g_1) \times \cdots \times \mathrm{sgn}_{i_t}(g_t) \notin A. \end{cases}$$

Thus, in order to prove Theorem 1.7, it is enough to show that for all $\underline{i} \in I^t$,

$$\theta_{\underline{i}}(g) = \int_{\sigma \in (\widehat{S_\alpha / A})} \prod_{\ell=1}^{t} \sigma(i_\ell)^{k_\ell+1} d\mu_\alpha^A. \tag{2.1}$$

Let c_{i_ℓ} be the generator of $C_{i_\ell} = \{\, \pm 1 \,\}$. In the proof of (2.1), we will make use of the observation that if $\sigma \in \widehat{S}_\alpha$, then

$$\sigma(\mathrm{sgn}_{i_\ell}(g_\ell)) = \sigma(c_{i_\ell}^{k_\ell+1}) = \sigma(c_{i_\ell})^{k_\ell+1} = \sigma(i_\ell)^{k_\ell+1};$$

and hence we have that

$$\sigma(i_1)^{k_1+1} \times \cdots \times \sigma(i_t)^{k_t+1} = \sigma(\mathrm{sgn}_{i_1}(g_1)) \times \cdots \times \sigma(\mathrm{sgn}_{i_t}(g_t))$$
$$= \sigma(\mathrm{sgn}_{i_1}(g_1) \times \cdots \times \mathrm{sgn}_{i_t}(g_t))$$

First suppose that $\mathrm{sgn}_{i_1}(g_1) \times \cdots \times \mathrm{sgn}_{i_t}(g_t) \in A$. Then for each $\sigma \in \widehat{(S_\alpha/A)}$,

$$\sigma(i_1)^{k_1+1} \times \cdots \times \sigma(i_t)^{k_t+1} = \sigma(\mathrm{sgn}_{i_1}(g_1) \times \cdots \times \mathrm{sgn}_{i_t}(g_t)) = 1;$$

and so $\int_{\sigma \in \widehat{(S_\alpha/A)}} \prod_{\ell=1}^{t} \sigma(i_\ell)^{k_\ell+1} d\mu_\alpha^A = 1$. Next suppose that

$$s = \mathrm{sgn}_{i_1}(g_1) \times \cdots \times \mathrm{sgn}_{i_t}(g_t) \notin A.$$

For each $\varepsilon \in \{ \pm 1 \}$, let $V_\varepsilon = \{ \sigma \in \widehat{(S_\alpha/A)} \mid \sigma(s) = \varepsilon \}$. Then clearly we have that $\mu_\alpha^A(V_1) = \mu_\alpha^A(V_{-1}) = 1/2$; and if $\sigma \in V_\varepsilon$, then

$$\sigma(i_1)^{k_1+1} \times \cdots \times \sigma(i_t)^{k_t+1} = \sigma(s) = \varepsilon.$$

Hence $\int_{\sigma \in \widehat{(S_\alpha/A)}} \prod_{\ell=1}^{t} \sigma(i_\ell)^{k_\ell+1} d\mu_\alpha^A = 0$. This completes the proof of Theorem 1.7.

3. An asymptotic interpretation of Theorem 1.7

In this section, via an application of the Pointwise Ergodic Theorem, we will interpret the integral (1.7) as an asymptotic limit of the Clifford decompositions of a naturally associated sequence of permutation characters of finite symmetric groups.

Suppose that $G = \bigcup_{n \in \mathbb{N}} G_n$ is the union of the strictly increasing chain of finite subgroups G_n and that $G \curvearrowright (Z, \mu)$ is an ergodic action on a Borel probability space. Then the following theorem is a special case of more general results of Vershik [10] and Lindenstrauss [5].

The Pointwise Ergodic Theorem. *With the above hypotheses, if $B \subseteq Z$ is a μ-measurable subset, then for μ-a.e. $z \in Z$,*

$$\mu(B) = \lim_{n \to \infty} \frac{1}{|G_n|} |\{ g \in G_n \mid g \cdot z \in B \}|.$$

In particular, the Pointwise Ergodic Theorem applies when B is the μ-measurable subset $\mathrm{Fix}_Z(g) = \{ z \in Z \mid g \cdot z = z \}$ for some $g \in G$. For each $z \in Z$ and $n \in \mathbb{N}$, let $\Omega_n(z) = \{ g \cdot z \mid g \in G_n \}$ be the corresponding G_n-orbit. Then, as pointed out in Thomas-Tucker-Drob [8, Theorem 2.1], the following result is an easy consequence of the Pointwise Ergodic Theorem.

Theorem 3.1. *With the above hypotheses, for μ-a.e. $z \in Z$, for all $g \in G$,*

$$\mu(\mathrm{Fix}_X(g)) = \lim_{n \to \infty} |\, \mathrm{Fix}_{\Omega_n(z)}(g) \,| / |\, \Omega_n(z) \,|.$$

Of course, the permutation group $G_n \curvearrowright \Omega_n(z)$ is isomorphic to $G_n \curvearrowright G_n/H_n$, where G_n/H_n is the set of cosets of $H_n = \{ g \in G_n \mid g \cdot z = z \}$ in G_n; and so

$$\mathrm{Fix}_{\Omega_n(z)}(g) \,| / |\, \Omega_n(z) \,| = 1_{H_n}^{G_n}(g) / [G_n : H_n]. \tag{3.2}$$

Suppose that there exists a subgroup $H_n \leqslant K_n \leqslant G_n$ with $H_n \trianglelefteq K_n$. Then $1_{H_n}^{G_n} = (\, 1_{H_n}^{K_n} \,)^{G_n}$; and, by Clifford's Theorem [3, Theorem 11.5],

$$1_{H_n}^{K_n} = \sum_{\theta \in \mathrm{Irr}_{H_n}(K_n)} \theta(1)\, \theta,$$

where $\mathrm{Irr}_{H_n}(K_n)$ is the set of irreducible characters θ of K_n such that $\theta(h) = \theta(1)$ for all $h \in H_n$. (Thus $\mathrm{Irr}_{H_n}(K_n)$ can be naturally identified with the set of irreducible characters of the quotient group K_n/H_n.) It follows that

$$1_{H_n}^{G_n} = (\, 1_{H_n}^{K_n}\,)^{G_n} = \sum_{\theta \in \mathrm{Irr}_{H_n}(K_n)} \theta(1)\, \theta^{G_n}. \tag{3.3}$$

Now suppose that $\alpha \in D[\,0,1\,]$ and $A \leqslant S_\alpha$ are as in Section 1 and let ν_α^A be the corresponding ergodic IRS of $\mathrm{Fin}(\mathbb{N})$. Applying Creutz-Peterson [2], let ν_α^A be the stabilizer distribution of the ergodic action $\mathrm{Fin}(\mathbb{N}) \curvearrowright (\,Z, \mu\,)$ and let

$$\chi_{\nu_\alpha^A}(g) = \mu(\,\mathrm{Fix}_Z(g)\,) = \nu_\alpha^A(\,\{\,H \in \mathrm{Sub}_{\mathrm{Fin}(\mathbb{N})} \mid g \in H\,\}\,)$$

be the corresponding character. Express $\mathrm{Fin}(\mathbb{N}) = \bigcup_{n\in\mathbb{N}} S_n$, where $S_n = \mathrm{Sym}(n)$; and for each $z \in Z$ and $n \in \mathbb{N}$, let $\Omega_n(z) = \{\, g \cdot z \mid g \in S_n\,\}$ be the corresponding S_n-orbit. (As a matter of convention, we set $\mathrm{Sym}(0) = 1$.) Then for μ-a.e. $z \in Z$, we have that

$$\chi_{\nu_\alpha^A}(g) = \mu(\,\mathrm{Fix}_X(g)\,) = \lim_{n\to\infty} |\,\mathrm{Fix}_{\Omega_n(z)}(g)\,|/|\,\Omega_n(z)\,|.$$

Fix such an element $z \in Z$ and let $H = \{\,h \in \mathrm{Fin}((N) \mid h \cdot z = z\}$ be the corresponding point stabilizer. Then we can suppose that there exists a μ_α-generic point $\xi \in \mathbb{N}^{\mathbb{N}}$ such that $H = s_\xi^{-1}(A)$, where s_ξ is the homomorphism

$$\begin{aligned} s_\xi : \bigoplus_{i\in I} \mathrm{Fin}(B_i^\xi) &\to S_\alpha = \bigoplus_{i\in I} C_i \\ (\,\pi_i\,) &\mapsto (\,\mathrm{sgn}(\pi_i)\,). \end{aligned}$$

For each $n \in \mathbb{N}$, let $H_n = H \cap S_n$ and let $I_n = \{\, i \in I \mid B_i^\xi \cap n \neq \emptyset\,\}$; and for each $i \in I_n$, let $B_i^n = B_i^\xi \cap n$. Let $K_n = \bigoplus_{i\in I_n} \mathrm{Sym}(B_i^n)$ be the corresponding Young subgroup. Then $H_n \trianglelefteq K_n \leqslant S_n$ and

$$K_n/H_n \cong \bigoplus_{i\in I_n} C_i/(\,A \cap \bigoplus_{i\in I_n} C_i\,)$$

is an elementary abelian 2-group; and so each character $\sigma \in \mathrm{Irr}_{H_n}(K_n)$ is linear. Hence, applying equations (3.2) and (3.3), we obtain that for each $g \in S_n$,

$$\begin{aligned} |\,\mathrm{Fix}_{\Omega_n(z)}(g)\,|/|\,\Omega_n(z)\,| &= \frac{1}{[S_n : H_n]} \sum_{\sigma \in \mathrm{Irr}_{H_n}(K_n)} \sigma^{S_n}(g) \\ &= \sum_{\sigma \in \mathrm{Irr}_{H_n}(K_n)} \frac{1}{[K_n : H_n]}\, \frac{\sigma^{S_n}(g)}{\sigma^{S_n}(1)} \\ &= \sum_{\sigma \in \mathrm{Irr}_{H_n}(K_n)} \frac{1}{|\,\mathrm{Irr}_{H_n}(K_n)|}\, \frac{\sigma^{S_n}(g)}{\sigma^{S_n}(1)}. \end{aligned} \tag{3.4}$$

Slightly abusing notation, for each character $\sigma \in \mathrm{Irr}_{H_n}(K_n)$, we will also denote the corresponding homomorphism $\bigoplus_{i\in I_n} C_i \to \{\,\pm 1\,\}$ by σ. For each $i \in I_n$, let c_i be the generator of C_i; and for each character $\sigma \in \mathrm{Irr}_{H_n}(K_n)$, let $\sigma(i) = \sigma(c_i)$. If $g \in S_n$ is a k-cycle and $\sigma \in \mathrm{Irr}_{H_n}(K_n)$, then

$$\frac{\sigma^{S_n}(g)}{\sigma^{S_n}(1)} = \frac{1}{|S_n|} \sum_{i\in I_n} \sigma(i)^{k+1} |\{\, s \in S_n \mid sgs^{-1} \in \mathrm{Sym}(B_i^n)\,\}| = \sum_{i\in I_n} \sigma(i)^{k+1} \frac{\binom{|B_i^n|}{k}}{\binom{n}{k}}.$$

If we fix $i \in I$ and let $n \to \infty$, then we have that

$$\binom{|B_i^n|}{k} / \binom{n}{k} \approx (|B_i^n|/n)^k \to \alpha_i^k \qquad \text{as } n \to \infty.$$

Hence we see that

$$\begin{aligned}
\chi_{\nu_\alpha^A}(g) &= \lim_{n\to\infty} |\operatorname{Fix}_{\Omega_n(z)}(g)|/|\Omega_n(z)| \\
&= \lim_{n\to\infty} \sum_{\sigma \in \operatorname{Irr}_{H_n}(K_n)} \frac{1}{|\operatorname{Irr}_{H_n}(K_n)|} \frac{\sigma^{S_n}(g)}{\sigma^{S_n}(1)} \\
&= \int_{\sigma \in \widehat{(S_\alpha/A)}} \Big(\sum_{i \in I} \sigma(i)^{k+1} \alpha_i^k \Big) \, d\mu_\alpha^A \\
&= \int_{\sigma \in \widehat{(S_\alpha/A)}} \chi_\alpha^\sigma(g) \, d\mu_\alpha^A;
\end{aligned}$$

and, more generally, if $g \in \operatorname{Fin}(\mathbb{N})$ has cycle decomposition $g = g_1 \cdots g_t$, where each g_ℓ is a k_ℓ-cycle, then we see that

$$\begin{aligned}
\chi_{\nu_\alpha^A}(g) &= \lim_{n\to\infty} |\operatorname{Fix}_{\Omega_n(z)}(g)|/|\Omega_n(z)| \\
&= \lim_{n\to\infty} \sum_{\sigma \in \operatorname{Irr}_{H_n}(K_n)} \frac{1}{|\operatorname{Irr}_{H_n}(K_n)|} \frac{\sigma^{S_n}(g)}{\sigma^{S_n}(1)} \\
&= \int_{\sigma \in \widehat{(S_\alpha/A)}} \prod_{1 \le \ell \le t} \Big(\sum_{i \in I} \sigma(i)^{k_\ell+1} \alpha_i^{k_\ell} \Big) \, d\mu_\alpha^A \\
&= \int_{\sigma \in \widehat{(S_\alpha/A)}} \chi_\alpha^\sigma(g) \, d\mu_\alpha^A.
\end{aligned}$$

Thus we can interpret Theorem 1.7 as an asymptotic limit of the Clifford decompositions of a naturally associated sequence of permutation characters of finite symmetric groups; and we can interpret each Thoma character $\chi_{(\beta;\gamma)} = \chi_\alpha^\sigma$ as an asymptotic limit of a naturally associated sequence of characters induced from linear characters of Young subgroups of finite symmetric groups.

4. The ergodic invariant random subgroups of $\operatorname{Fin}(\mathbb{N})$

In this section, slightly correcting the argument of Vershik [11], we will present the proof of Theorem 1.1. The classification of the ergodic IRSs of $\operatorname{Fin}(\mathbb{N})$ will be based upon the following two insights of Vershik.

(i) If $H \leqslant \operatorname{Fin}(\mathbb{N})$ is a random subgroup, then the corresponding H-orbit equivalence relation is a random equivalence relation on $\mathbb{N}$; and these have been classified by Kingman [4].

(ii) The induced action of H on each infinite orbit $\Omega \subseteq \mathbb{N}$ can be determined via an application of Wielandt's Theorem [12, Satz 9.4], which states that if Ω is an infinite set, then $\operatorname{Alt}(\Omega)$ and $\operatorname{Fin}(\Omega)$ are the only primitive subgroups of $\operatorname{Fin}(\Omega)$.

The proof of Theorem 1.1 will also make use of an elementary result concerning imprimitive actions of finitary permutation groups. Recall that a transitive subgroup $H \leqslant \operatorname{Sym}(\Omega)$ is said to act *imprimitively* if there exists a nontrivial proper H-invariant equivalence relation E on Ω. In this case, following the usual practice, we will refer to the E-classes as E-*blocks*. Now suppose that Ω is an infinite set and $H \leqslant \mathrm{Fin}(\Omega)$ is an imprimitive subgroup. Of course, in this case, if E is a nontrivial proper H-invariant equivalence relation on Ω, then the E-blocks must be finite. The following result, which is a variant of Neumann [6, Lemma 2.2], implies that we can always make a "canonical" choice of a nontrivial proper H-invariant equivalence relation.[2]

Lemma 4.1. *If Ω is an infinite set and $H \leqslant \mathrm{Fin}(\Omega)$ is an imprimitive subgroup, then at least one of the following two conditions holds:*

(i) *there exists a* unique *maximal proper H-invariant equivalence relation E^{Ω}_{max} on Ω;*

(ii) *there exists a* unique *proper H-invariant equivalence relation E^{Ω}_{min} on Ω which is minimal subject to the condition that E^{Ω}_{min} contains every minimal nontrivial H-invariant equivalence relation on Ω.*

Definition 4.2. If Ω is an infinite set and $H \leqslant \mathrm{Fin}(\Omega)$ is an imprimitive subgroup, then the associated *canonical equivalence relation* $E^{\Omega}_{\mathrm{can}}$ is defined to be $E^{\Omega}_{\mathrm{max}}$ if condition 4.1(i) holds, and is defined to be $E^{\Omega}_{\mathrm{min}}$ otherwise.

The proof of Lemma 4.1 will make use of the following observation.

Claim 4.3. *If Ω is an infinite set and $H \leqslant \mathrm{Fin}(\Omega)$ is an imprimitive subgroup, then there exist only finitely many minimal nontrivial H-invariant equivalence relations on Ω.*

Proof of Claim 4.3. Suppose that $\{ E_n \mid n \in \mathbb{N} \}$ are distinct minimal nontrivial H-invariant equivalence relations on Ω. Fix some element $\omega_0 \in \Omega$; and for each $n \in \mathbb{N}$, let Δ_n be the E_n-block such that $\omega_0 \in \Delta_n$. Notice that if $n \neq m$, then $\Delta_n \cap \Delta_m$ is a block; and hence by the minimality of E_n, E_m, we must have that $\Delta_n \cap \Delta_m = \{ \omega_0 \}$. For each $n \in \mathbb{N}$, choose an element $d_n \in \Delta \smallsetminus \{ \omega_0 \}$. Let $\pi \in H$ satisfy $\pi(\omega_0) = d_0$. If $n > 0$, then $d_0 \in \pi(\Delta_n) \neq \Delta_n$ and so $\pi(d_n) \neq d_n$, which contradicts the fact that $\pi \in \mathrm{Fin}(\mathbb{N})$. □

Proof of Lemma 4.1. Once again, fix some element $\omega_0 \in \Omega$. First suppose that there exists a maximal nontrivial proper H-invariant equivalence relation E on Ω. Then the set Ω/E of E-classes is infinite and H acts as a primitive group of finitary permutations on Ω/E. Applying Wielandt's Theorem, it follows that H induces at least $\mathrm{Alt}(\Omega/E)$ on Ω/E. Suppose that $E' \neq E$ is a second maximal proper H-invariant equivalence relation on Ω. Let Δ be the E-block such that $\omega_0 \in \Delta$ and let Δ' be the E'-block such that $\omega_0 \in \Delta'$. Then there exists $d \in \Delta' \smallsetminus \Delta$. Since H acts 2-transitively on Ω/E, it follows that the orbit $H_{\{\Delta\}} \cdot d$ is infinite; and since $[\, H_{\{\Delta\}} : H_{(\Delta)} \,] < \infty$, it follows that the orbit $H_{(\Delta)} \cdot d$ is also infinite. But this means that Δ' is infinite, which is a contradiction.

Next suppose that there does not exist a maximal proper H-invariant equivalence relation on Ω. Then there exists a strictly increasing sequence

$$E_0 \subset E_1 \subset \cdots \subset E_n \subset \cdots$$

of proper H-invariant equivalence relations on Ω. Let $E = \bigcup_{n \in \mathbb{N}} E_n$. Then E is an H-invariant equivalence relation such that every E-class is infinite and it follows

[2] In [11], Vershik suggests choosing the minimal nontrivial H-invariant equivalence relation. However, there exist examples of imprimitive finitary groups H with more than one minimal nontrivial H-invariant equivalence relation. On the other hand, see Claim 4.3.

that $E = \Omega^2$. For each $n \in \mathbb{N}$, let Δ_n be the E_n-block such that $\omega_0 \in \Delta_n$. Then clearly $\bigcup_{n\in\mathbb{N}} \Delta_n = \Omega$.

Applying Claim 4.3, let $R_1, \cdots, R_m$ be the finitely many minimal nontrivial H-invariant equivalence relations on Ω; and for each $1 \leq \ell \leq m$, let Φ_ℓ be the R_ℓ-block such that $\omega_0 \in \Phi_\ell$. Then there exists $n \in \mathbb{N}$ such that $\Phi_\ell \subseteq \Delta_n$ for all $1 \leq \ell \leq m$; and it follows that $R_\ell \subseteq E_n$ for all $1 \leq \ell \leq m$. Finally, letting $E^H_{\min}$ be the intersection of all the proper H-invariant equivalence relations E such that $R_\ell \subseteq E$ for all $1 \leq \ell \leq m$, it is clear that $E^H_{\min}$ satisfies condition 4.1(ii). □

Finally, before beginning the proof of Theorem 1.1, we will present a brief discussion of Kingman's Theorem [4]. Let

$$\mathrm{ER}_{\mathbb{N}} = \{\, E \in 2^{\mathbb{N}\times\mathbb{N}} \mid E \text{ is an equivalence relation on } \mathbb{N} \,\}.$$

Then $\mathrm{ER}_{\mathbb{N}}$ is a compact space and $\mathrm{Fin}(\mathbb{N}) \curvearrowright \mathrm{ER}_{\mathbb{N}}$ via the shift action

$$(\, g \cdot E \,)(n, m) = E(g^{-1}(n), g^{-1}(m)).$$

As expected, a $\mathrm{Fin}(\mathbb{N})$-invariant Borel probability measure m on $\mathrm{ER}_{\mathbb{N}}$ is called an *invariant random equivalence relation*. For example, let $\alpha = (\, \alpha_i \,)_{i\in\mathbb{N}^+} \in D[\,0, 1\,]$ be such that $\sum_{i=1}^{\infty} \alpha_i \leq 1$ and let $\alpha_0 = 1 - \sum_{i=1}^{\infty} \alpha_i$. Let μ_α be the corresponding product measure on $\mathbb{N}^{\mathbb{N}}$, as defined in Section 1; and for each $\xi \in \mathbb{N}^{\mathbb{N}}$ and $i \in \mathbb{N}$, let $B^\xi_i = \{\, n \in \mathbb{N} \mid \xi(n) = i \,\}$. Then we can define a $\mathrm{Fin}(\mathbb{N})$-equivariant map $\xi \overset{\varphi_\alpha}{\mapsto} E_\xi$ from $\mathbb{N}^{\mathbb{N}}$ to $\mathrm{ER}_{\mathbb{N}}$ by letting E_ξ correspond to the partition

$$\mathbb{N} = \bigsqcup_{n\in B^\xi_0} \{\, n \,\} \sqcup \bigsqcup_{i>0} B^\xi_i;$$

and it follows that $m_\alpha = (\varphi_\alpha)_* \mu_\alpha$ is an ergodic random invariant equivalence relation. The following theorem is due to Kingman [4].

Theorem 4.4. *If m is an ergodic random invariant equivalence relation, then there exists α as above such that $m = m_\alpha$.*

Remark 4.5. If $\alpha = (\, \alpha_i \,)_{i\in\mathbb{N}^+}$ is such that there exist $i \in \mathbb{N}^+$ with $\alpha_i = \alpha_{i+1} > 0$, then the map $\xi \mapsto E_\xi$ is not injective, and there does not exist a $\mathrm{Fin}(\mathbb{N})$-equivariant Borel map $E \mapsto \xi_E$ from $\mathrm{ER}_{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$ such that $E = E_{\xi_E}$.

The proof of Theorem 1.1 will also make use of the following easy observation.

Lemma 4.6. *If m is an ergodic random invariant equivalence relation, then m concentrates the equivalence relations $E \in \mathrm{ER}_{\mathbb{N}}$ such that every E-class is either infinite or a singleton.*

Proof. While this result is an immediate consequence of Theorem 4.4, it seems worthwhile to give an elementary proof. So suppose that m is a counterexample. Then, by ergodicity, there exists a fixed integer $k > 1$ such that

$$m(\, \{\, E \in \mathrm{ER}_{\mathbb{N}} \mid \text{ There exists an } E\text{-class of size } k \,\}) = 1.$$

For each $S \in [\,\mathbb{N}\,]^k$, let C_S be the event that S is an E-class. Since $\mathrm{Fin}(\mathbb{N})$ acts transitively on $[\,\mathbb{N}\,]^k$, there exists a fixed real $r > 0$ such that $m(C_S) = r$ for all $S \in [\,\mathbb{N}\,]^k$. But, since the events $\{\, C_S \mid 0 \in S \in [\,\mathbb{N}\,]^k \,\}$ are mutually exclusive, this is impossible. □

We are now ready to begin the proof of Theorem 1.1. So suppose that ν is an ergodic IRS of $\mathrm{Fin}(\mathbb{N})$. Clearly we can suppose that $\nu \neq \delta_1$.

Lemma 4.7. *For ν-a.e. $H \in \mathrm{Sub}_{\mathrm{Fin}(\mathbb{N})}$, if $\Omega \subseteq \mathbb{N}$ is a nontrivial H-orbit, then Ω is infinite and H induces at least* $\mathrm{Alt}(\Omega)$ *on* Ω.

Proof. Suppose not. Recall that, by Wielandt's Theorem [12], if Ω is an infinite set, then $\mathrm{Alt}(\Omega)$ and $\mathrm{Fin}(\Omega)$ are the only primitive subgroups of $\mathrm{Fin}(\Omega)$. It follows that for ν-a.e. $H \in \mathrm{Sub}_{\mathrm{Fin}(\mathbb{N})}$, either there exists a nontrivial finite H-orbit, or else there exists an infinite H-orbit on which H acts imprimitively. For each such $H \in \mathrm{Sub}_{\mathrm{Fin}(\mathbb{N})}$, let R_H be the equivalence relation on $\mathbb{N}$ such that if $n \neq m \in \mathbb{N}$, then $n\, R_H\, m$ if and only if n, m lie in the same H-orbit Ω and either:

(i) Ω is a nontrivial finite H-orbit; or
(ii) Ω is an infinite imprimitive H-orbit and $n\, E^{\Omega}_{\mathrm{can}}\, m$.

Otherwise, let R_H be the trivial equivalence relation on $\mathbb{N}$. Then clearly the map $H \overset{\psi}{\mapsto} R_H$ is $\mathrm{Fin}(\mathbb{N})$-equivariant, and hence $m = \psi_* \nu$ is an ergodic invariant random equivalence relation. But m concentrates on the $E \in \mathrm{ER}_{\mathbb{N}}$ with a nontrivial finite E-class, which contradicts Lemma 4.6. □

The proof of the following lemma will make use of the notion of a diagonal subgroup, which is defined as follows. Suppose that $r \geq 2$ and that $\Omega_1, \cdots, \Omega_r$ are countably infinite sets. Then $D \leqslant \bigoplus_{1 \leq \ell \leq r} \mathrm{Alt}(\Omega_\ell)$ is said to be a *diagonal subgroup* if there exist isomorphisms $\pi_\ell : \mathrm{Alt}(\Omega_1) \to \mathrm{Alt}(\Omega_\ell)$ for $2 \leq \ell \leq r$ such that

$$D = \{\, (g, \pi_2(g), \cdots, \pi_r(g)) \mid g \in \mathrm{Alt}(\Omega_1) \,\}.$$

Recall that every automorphism of $\mathrm{Alt}(\mathbb{N})$ is the restriction of an inner automorphism of the group $\mathrm{Sym}(\mathbb{N})$ of all permutations of $\mathbb{N}$. It follows that for each $2 \leq \ell \leq r$, there exists a unique bijection $T_\ell : \Omega_1 \to \Omega_\ell$ such that $\pi_\ell(g) = T_\ell g T_\ell^{-1}$. Let T_1 be the identity map on Ω_1; and for each $1 \leq k, \ell \leq r$, let $T_{k,\ell} = T_\ell^{-1} T_k$. Then we will write $D = \mathrm{Diag}(\bigoplus_{1 \leq \ell \leq r} \mathrm{Alt}(\Omega_\ell))$, and say that D is the diagonal subgroup determined by the bijections $\{\, T_{k,\ell} \mid 1 \leq k, \ell \leq r \,\}$. Finally, in the degenerate case when $r = 1$, we will take $\mathrm{Alt}(\Omega_1)$ to be the only diagonal subgroup of $\mathrm{Alt}(\Omega_1)$ and we will take $T_{1,1}$ to be the identity map on Ω_1.

Lemma 4.8. *For ν-a.e. $H \in \mathrm{Sub}_{\mathrm{Fin}(\mathbb{N})}$, if $\{\, \Omega_i \mid i \in I \,\}$ is the set of nontrivial H-orbits, then each Ω_i is infinite and $\bigoplus_{i \in I} \mathrm{Alt}(\Omega_i) \leqslant H$.*

Proof. By Lemma 4.7, for ν-a.e. $H \in \mathrm{Sub}_{\mathrm{Fin}(\mathbb{N})}$, if $\Omega \subseteq \mathbb{N}$ is a nontrivial H-orbit, then Ω is infinite and H induces at least $\mathrm{Alt}(\Omega)$ on Ω. Let H be such a subgroup, let $\{\, \Omega_i \mid i \in I \,\}$ is the set of nontrivial H-orbits, and let $K = H \cap \bigoplus_{i \in I} \mathrm{Alt}(\Omega_i)$.

Claim 4.9. *There exists a partition $\{\, F_j \mid j \in J \,\}$ of I into finite subsets such that*

$$K = \bigoplus_{j \in J} \mathrm{Diag}(\bigoplus_{k \in F_j} \mathrm{Alt}(\Omega_k)),$$

where the diagonal subgroups are determined by unique bijections $T_{k,\ell} : \Omega_k \to \Omega_\ell$ for k, $\ell \in F_j$.

Sketch proof of Claim 4.9. For each $g \in K$, let $g = \prod_{i \in I} g_i$, where $g_i \in \mathrm{Alt}(\Omega_i)$, and let $s(g) = \{\, i \mid g_i \neq 1 \,\}$. Then clearly each $s(g)$ is a finite subset of I. Let $\mathcal{P} = \{\, F_j \mid j \in J \,\}$ be the collection of minimal subsets $A \subseteq I$ such that there exists $1 \neq g \in K$ with $s(g) = A$. Then, using the simplicity of the infinite alternating group and the fact that K projects onto each $\mathrm{Alt}(\Omega_i)$, it is easily checked that $\mathcal{P}$ is a partition of I and that

$$K = \bigoplus_{j \in J} \mathrm{Diag}(\bigoplus_{k \in F_j} \mathrm{Alt}(\Omega_k))$$

for some collection of bijections

$$\{\, T_{k,\ell} : \Omega_k \to \Omega_\ell \mid k, \ell \in F_j \text{ for some } j \in J \,\}.$$

$\square$

Let R_H be the equivalence relation on $\mathbb{N}$ such that if $n \neq m \in \mathbb{N}$, then $n \, R_H \, m$ if and only if there exists $j \in J$ and $k \neq \ell \in F_j$ such that $T_{k,\ell}(n) = m$. Clearly the map $H \overset{\psi}{\mapsto} R_H$ is $\mathrm{Fin}(\mathbb{N})$-equivariant, and hence $m = \psi_* \nu$ is an ergodic invariant random equivalence relation. Applying Lemma 4.6, since every R_H-class is finite, it follows that each $|F_j| = 1$. $\square$

Let $H \overset{p}{\mapsto} E_H$ be the $\mathrm{Fin}(\mathbb{N})$-equivariant map from $\mathrm{Sub}_{\mathrm{Fin}(\mathbb{N}}$ to $\mathrm{ER}_{\mathbb{N}}$ such that E_H is the H-orbit equivalence relation. Then $p_* \nu$ is an ergodic invariant random equivalence relation; and hence, applying Theorem 4.4, it follows that there exists an $\alpha = (\, \alpha_i \,)_{i \in \mathbb{N}^+} \in D[\, 0, 1 \,]$ such that $p_* \nu = m_\alpha$. Since $\nu \neq \delta_1$, it follows that $\alpha_0 \neq 1$. Let $I = \{\, i \in \mathbb{N}^+ \mid \alpha_i > 0 \,\}$. Then for ν-a.e. $H \in \mathrm{Sub}_{\mathrm{Fin}(\mathbb{N})}$, there exists a μ_α-generic $\xi_H \in \mathbb{N}^{\mathbb{N}}$ such that the H-orbit decomposition is given by

$$\mathbb{N} = \bigsqcup_{n \in B_0^{\xi_H}} \{\, n \,\} \sqcup \bigsqcup_{i \in I} B_i^{\xi_H}. \tag{4.10}$$

As we mentioned in Remark 4.5, if $\alpha = (\, \alpha_i \,)_{i \in \mathbb{N}^+}$ is such that there exist $i \in \mathbb{N}^+$ with $\alpha_i = \alpha_{i+1} > 0$, then the map $\xi \mapsto E_\xi$ is not injective. In more detail, let $\equiv$ be the equivalence relation on I defined by

$$k \equiv \ell \quad \Longleftrightarrow \quad \alpha_k = \alpha_\ell;$$

and let $I = \bigsqcup_{j \in J} I_j$ be the decomposition of I into $\equiv$-classes. Then clearly each I_j is finite. Let $P = \prod_{j \in J} \mathrm{Sym}(I_j)$ be the full direct product of the finite groups $\mathrm{Sym}(I_j)$, and let $P \curvearrowright (\, \mathbb{N}^{\mathbb{N}}, \mu_\alpha \,)$ be the measure-preserving action defined by

$$(\, (\pi_j)_{j \in J} \cdot \xi \,)(n) = \begin{cases} \pi_j(\xi(n)), & \text{if } \xi(n) \in I_j; \\ \xi(n), & \text{if } \xi(n) \in \mathbb{N} \smallsetminus I. \end{cases}$$

Then P is a (possibly trivial) compact group; and if ξ, $\xi' \in \mathbb{N}^{\mathbb{N}}$ are μ_α-generic, then $E_\xi = E_{\xi'}$ if and only if there exists $(\pi_j)_{j \in J} \in P$ such that $(\pi_j)_{j \in J} \cdot \xi = \xi'$.

Definition 4.11. A subgroup $H \in \mathrm{Sub}_{\mathrm{Fin}(\mathbb{N})}$ is said to be *ν-generic* if:

(i) there exists a μ_α-generic $\xi_H \in \mathbb{N}^{\mathbb{N}}$ such that the H-orbit decomposition is given by (4.10); and
(ii) H satisfies the conclusion of Lemma 4.8.

Let $H \in \mathrm{Sub}_{\mathrm{Fin}(\mathbb{N})}$ be ν-generic and let $\xi_H \in \mathbb{N}^{\mathbb{N}}$ be the μ_α-generic function chosen so that if $\alpha_i, \alpha_{i+1}, \cdots, \alpha_{i+s}$ is a nontrivial $\equiv$-class, then the corresponding orbits $B_i^{\xi_H}, B_{i+1}^{\xi_H}, \cdots, B_{i+s}^{\xi_H}$ are listed in the order of their least elements. Let

$$s_H : \bigoplus_{i \in I} \mathrm{Fin}(B_i^{\xi_H}) \to S_\alpha = \bigoplus_{i \in I} C_i$$
$$(\,\pi_i\,) \mapsto (\,\mathrm{sgn}(\pi_i)\,)$$

and let $A_H = s_H(H) \leqslant S_\alpha$. Once again, let $P = \prod_{j \in J} \mathrm{Sym}(I_j)$. Then the natural action $P \curvearrowright S_\alpha = \bigoplus_{i \in I} C_i$ induces a corresponding action $P \curvearrowright \mathrm{Sub}_{S_\alpha}$. For each $A \leqslant S_\alpha$, let $[\,A\,]$ be the corresponding P-orbit. Since P is a compact group, it follows that $\mathrm{Sub}_{S_\alpha}/P = \{\,[\,A\,] \mid A \in \mathrm{Sub}_{S_\alpha}\,\}$ is a standard Borel space. Furthermore, the Borel map $H \mapsto [\,A_H\,]$ is clearly $\mathrm{Fin}(\mathbb{N})$-invariant. Hence, by ergodicity, there exists a fixed $A \in \mathrm{Sub}_{S_\alpha}$ such that $[\,A_H\,] = [\,A\,]$ for ν-a.e. $H \in \mathrm{Sub}_{\mathrm{Fin}(\mathbb{N})}$. Let $X_\alpha^A \subseteq \mathrm{Sub}_{\mathrm{Fin}(\mathbb{N})}$ be the set of ν-generic H such that $[\,A_H\,] = [\,A\,]$. Then both ν and ν_α^A concentrate on X_α^A. Hence, in order to complete the proof of Theorem 1.1, it is enough to show that the action $\mathrm{Fin}(\mathbb{N}) \curvearrowright X_\alpha^A$ is uniquely ergodic. As we will explain, this is a straightforward consequence of the Pointwise Ergodic Theorem.

For each pair F_0, F_1 of finite disjoint subsets of $\mathrm{Fin}(\mathbb{N})$, let

$$U_{F_0,F_1} = \{\,H \in \mathrm{Sub}_{\mathrm{Fin}(\mathbb{N})} \mid F_0 \subseteq H \text{ and } F_1 \cap H = \emptyset\,\}.$$

Then the sets U_{F_0,F_1} form a clopen basis of the space $\mathrm{Sub}_{\mathrm{Fin}(\mathbb{N})}$; and thus it is enough to show that $\nu(\,U_{F_0,F_1}\,) = \nu_\alpha^A(\,U_{F_0,F_1}\,)$ for all such F_0, F_1. Hence, by the Pointwise Ergodic Theorem, it is enough to show that if H, $H' \in X_\alpha^A$, then

$$\lim_{n\to\infty} \frac{1}{|S_n|} |\{\,g \in S_n \mid gHg^{-1} \in U_{F_0,F_1}\,\}| = \lim_{n\to\infty} \frac{1}{|S_n|} |\{\,g \in S_n \mid gH'g^{-1} \in U_{F_0,F_1}\,\}|.$$

Equivalently, letting $H_n = H \cap S_n$ and $H'_n = H' \cap S_n$, it is enough to show that

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{|S_n|} |\{\,g \in S_n \mid g^{-1}F_0 g \subseteq H_n \text{ and } g^{-1}F_1 g \cap H_n = \emptyset\,\}| \\ &\qquad = \lim_{n\to\infty} \frac{1}{|S_n|} |\{\,g \in S_n \mid g^{-1}F_0 g \subseteq H'_n \text{ and } g^{-1}F_1 g \cap H'_n = \emptyset\,\}|. \end{aligned} \tag{4.12}$$

To see this, first note that after changing our choice of $\xi_{H'}$ if necessary, we can suppose that $A_H = A_{H'}$. Next fix some $\varepsilon > 0$ and choose an integer $k \in I$ such that $1 - \sum_{i=0}^k \alpha_i \ll \varepsilon$. Let $F_0 \sqcup F_1 \subseteq \mathrm{Sym}(d)$ and let $n \gg d$ be such that $|\,|B_i^{\xi_H} \cap n|/n - \alpha_i| \ll \varepsilon$ and $|\,|B_i^{\xi_{H'}} \cap n|/n - \alpha_i| \ll \varepsilon$ for all $0 \leq i \leq k$. Then it is easily checked that if n is sufficiently large, then

$$\begin{aligned} \Bigg| &\frac{1}{|S_n|} |\{\,g \in S_n \mid g^{-1}F_0 g \subseteq H_n \text{ and } g^{-1}F_1 g \cap H_n = \emptyset\,\}| \\ &\qquad - \frac{1}{|S_n|} |\{\,g \in S_n \mid g^{-1}F_0 g \subseteq H'_n \text{ and } g^{-1}F_1 g \cap H'_n = \emptyset\,\}| \Bigg| < \varepsilon; \end{aligned}$$

and so (4.12) holds. This completes the proof of Theorem 1.1.

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