DIAGONAL EMBEDDINGS OF FINITE ALTERNATING GROUPS

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Abstract. We will present an alternative approach to Zalesskiĭ's theorem on diagonal embeddings of finite alternating groups.

1. Introduction

For each positive integer n, let $\Delta_n = \{ 1, 2, \dots, n \}$. If $k < m$, then an embedding $\tau : \mathrm{Alt}(k) \to \mathrm{Alt}(m)$ is said to be *diagonal* if every nontrivial $\tau(\mathrm{Alt}(k))$-orbit on Δ_m is natural. (Here an orbit Ω of $\tau(\mathrm{Alt}(k))$ on Δ_m is said to be *natural* if $|\Omega| = k$ and the action $\tau(\mathrm{Alt}(k)) \curvearrowright \Omega$ is isomorphic to the natural action $\mathrm{Alt}(k) \curvearrowright \Delta_k$.) Of course, it is clear that if $\tau_0 : \mathrm{Alt}(k) \to \mathrm{Alt}(m)$ and $\tau_1 : \mathrm{Alt}(m) \to \mathrm{Alt}(n)$ are diagonal embeddings, then $\tau = \tau_1 \circ \tau_0$ is also a diagonal embedding. In [4], via an ingenious character theoretic argument, Zalesskiĭ proved that the converse also holds.

Theorem 1.1. *Suppose that* $6 \leq k < m < n$ *and that* $\tau_0 : \mathrm{Alt}(k) \to \mathrm{Alt}(m)$ *and* $\tau_1 : \mathrm{Alt}(m) \to \mathrm{Alt}(n)$ *are embeddings of finite alternating groups. If* $\tau = \tau_1 \circ \tau_0$ *is a diagonal embedding, then* τ_0 *and* τ_1 *are both diagonal embeddings.*

In this paper, we will present a more elementary approach which mostly relies on permutation group theoretic arguments.

Remark 1.2. As we will explain later, there are slight inaccuracies in Zalesskiĭ [4] arising from the exceptional properties of $\mathrm{Alt}(6)$. These exceptional properties will also cause some inconvenience in this paper. For example, see Lemma 2.3.

As in Zalesskiĭ [4], our proof of Theorem 1.1 will involve the analysis of various permutation characters. Here if $G \curvearrowright \Omega$ is an action of a finite group G on a finite set Ω, then the corresponding *permutation character* is

$$\chi(g) = |\,\mathrm{Fix}_\Omega(g)|, \qquad g \in G;$$

and the corresponding *normalized permutation character* is

$$\widehat{\chi}(g) = |\,\mathrm{Fix}_\Omega(g)|/|\Omega|, \qquad g \in G.$$

A central role will be played by the permutation character χ_n of the natural action $\mathrm{Alt}(n) \curvearrowright \Delta_n$. Recall that if $n \geq 4$, then there exists an irreducible character η_n

such that $\chi_n = 1_{\mathrm{Alt}(n)} + \eta_n$. Here, if G is any finite group, then 1_G denotes the trivial character. Also recall that if χ is the permutation character of any action $G \curvearrowright \Omega$, then the multiplicity of 1_G in χ is equal to the number of G-orbits on Ω. (For example, see Cameron [1, Section 2.5].))

2. Finite permutation groups

In this section, we will present some results on finite permutation groups which will be used in the proof of Theorem 1.1. The first two results are well-known. (For example, see Dixon-Mortimer [2, Chapter 3].)

Lemma 2.1. *If $H \leqslant \mathrm{Sym}(n)$ is a primitive subgroup which contains a 3-cycle, then $\mathrm{Alt}(n) \leqslant H$.*

Lemma 2.2. *If $n > 6$ and $H \leqslant \mathrm{Sym}(n)$ is a primitive subgroup which contains a 5-cycle, then $\mathrm{Alt}(n) \leqslant H$.*

Proof. If $n \geq 8$, then the result is a special case of Jordan's Theorem on primitive permutation groups. When $n = 7$, the result follows from the fact that the only proper transitive subgroups of $\mathrm{Sym}(7)$ are the cyclic group C_7 of order 7, the Frobenius group F_{21} of order 21, the projective special linear group $\mathrm{PSL}(2,7)$ of order 168 and the alternating group $\mathrm{Alt}(7)$. □

Lemma 2.2 is false when $n = 6$. To see this, let

$$H = \langle\, (\,1\,2\,3\,4\,5\,),\, (\,2\,5\,)(\,3\,4\,)\,\rangle \cong D_{10}.$$

Then $[\,\mathrm{Alt}(5) : H\,] = 6$ and the action $\mathrm{Alt}(5) \curvearrowright \mathrm{Alt}(5)/H$ induces a transitive embedding $\tau : \mathrm{Alt}(5) \to \mathrm{Alt}(6)$. Since $\tau(\mathrm{Alt}(5))$ contains a 5-cycle, it follows that $\tau(\mathrm{Alt}(5))$ is a 2-transitive subgroup of $\mathrm{Alt}(6)$.

Alternatively, recall that $\mathrm{Alt}(6)$ has an outer automorphism π which interchanges the conjugacy class of 3-cycles and the conjugacy class of elements of cycle type 3^2. Then we can also realize τ as $\tau = \pi \circ i$, where $i : \mathrm{Alt}(5) \to \mathrm{Alt}(6)$ is the natural inclusion. Note that we can also use π to define a non-natural action of $\mathrm{Alt}(6)$ on Δ_6 via

$$\ell \overset{g}{\mapsto} \pi(g)(\ell), \qquad g \in \mathrm{Alt}(6), \ell \in \Delta_6.$$

We will refer to this as the *nonstandard action* of $\mathrm{Alt}(6)$ on Δ_6.

Lemma 2.3. *If $5 \leq k < m$ and the embedding $\tau : \mathrm{Alt}(k) \to \mathrm{Alt}(m)$ is not diagonal, then either:*

(i) *there exists a $\tau(\mathrm{Alt}(k))$-orbit $\Phi \subseteq \Delta_m$ with $|\Phi| > k$; or*
(ii) *$k = 6$ and there exists $\tau(\mathrm{Alt}(6))$-orbit $\Phi \subseteq \Delta_m$ with $|\Phi| = 6$ such that $\tau(\mathrm{Alt}(6)) \curvearrowright \Phi$ is isomorphic to the nonstandard action of $\mathrm{Alt}(6)$ on Δ_6.*

Proof. Let $\Phi \subseteq \Delta_m$ be a nontrivial non-natural orbit of $\tau(\mathrm{Alt}(k))$. Suppose that $|\Phi| \not> k$. Then $|\Phi| = k$. By Dixon-Mortimer [2, 5.2A], if $k \neq 6$, then any action of $\mathrm{Alt}(k)$ on a set of size k is natural. Hence $k = 6$; and Dixon-Mortimer [2, 5.2A] implies that $\tau(\mathrm{Alt}(6)) \curvearrowright \Phi$ is isomorphic to the nonstandard action of $\mathrm{Alt}(6)$ on Δ_6. □

In [4], Zalesskiĭ stated Theorem 1.1 with the weaker condition that $k \geq 5$. However, the following examples show that it is necessary to strengthen the hypothesis to $k \geq 6$. Let $\tau : \mathrm{Alt}(5) \to \mathrm{Alt}(6)$ be the 2-transitive embedding defined above. Also let $s \geq 1$ and $t \geq 0$ be such that $s + t \geq 2$, and let $\tau'_{s,t} : \mathrm{Alt}(6) \to \mathrm{Alt}(6s + t)$ be an embedding such that $\tau(\mathrm{Alt}(6))$ has s nonstandard actions and t fixed points on Δ_{6s+t}. Then τ and $\tau'_{s,t}$ are both not diagonal and yet $\tau'_{s,t} \circ \tau$ is a diagonal embedding of $\mathrm{Alt}(5)$ into $\mathrm{Alt}(6s + t)$.

Definition 2.4. If $n > 6$, then an embedding $\sigma : \mathrm{Alt}(6) \to \mathrm{Alt}(n)$ is *almost diagonal* if σ is isomorphic to $\tau'_{s,t}$ for some $s, t \geq 0$ with $6s + t = n$.

We can now strengthen Theorem 1.1 as follows.

Theorem 2.5. *Suppose that* $5 \leq k < m < n$ *and that* $\tau_0 : \mathrm{Alt}(k) \to \mathrm{Alt}(m)$ *and* $\tau_1 : \mathrm{Alt}(m) \to \mathrm{Alt}(n)$ *are embeddings of finite alternating groups. If* $\tau = \tau_1 \circ \tau_0$ *is a diagonal embedding. Then either:*

(1) τ_0 *and* τ_1 *are both diagonal embeddings; or*
(2) $k = 5$*,* $m = 6$*, the embedding* $\tau_0 : \mathrm{Alt}(5) \to \mathrm{Alt}(6)$ *is 2-transitive and the embedding* $\tau_1 : \mathrm{Alt}(6) \to \mathrm{Alt}(n)$ *is almost diagonal.*

The proof of Theorem 2.5 will make use of the following observation, a proof of which can be found in Thomas-Tucker-Drob [3, Proposition 2.2].

Proposition 2.6. *If* $H \leqslant A$ *are finite groups and* χ_H *is the permutation character corresponding to the action* $A \curvearrowright A/H$*, then for all* $g \in A$*,*

$$\frac{\chi_H(g)}{|A/H|} = \frac{|\, g^A \cap H \,|}{|\, g^A \,|}.$$

The following consequence of Proposition 2.6 implies that when computing upper bounds for the normalized permutation characters of actions $A \curvearrowright A/H$, we can restrict our attention to those coming from maximal subgroups $H < A$.

Corollary 2.7. *If* $H \leqslant H' \leqslant A$ *are finite groups and* χ_H*,* $\chi_{H'}$ *are the permutation characters corresponding to the actions* $A \curvearrowright A/H$ *and* $A \curvearrowright A/H'$*, then for all* $g \in G$*,*

$$\frac{\chi_H(g)}{|A/H|} \leq \frac{\chi_{H'}(g)}{|A/H'|}.$$

The following result, in conjuction with Proposition 2.6, will play a key role in the proof of Theorem 2.5.

Lemma 2.8. *Suppose that $n > 6$. Let $t = \lfloor n/3 \rfloor$, let $s = n - 3t$, and let $g \in \mathrm{Alt}(n)$ be an element with cycle type $1^s 3^t$. If $H \leqslant \mathrm{Alt}(n)$ is a primitive subgroup such that*

$$\frac{|g^{\mathrm{Alt}(n)} \cap H|}{|g^{\mathrm{Alt}(n)}|} \geq \frac{1}{4},$$

then $H = \mathrm{Alt}(n)$.

Once again, Lemma 2.8 is false when $n = 6$. To see this, consider the previously defined 2-transitive embedding $\tau : \mathrm{Alt}(5) \to \mathrm{Alt}(6)$ and let $g \in \mathrm{Alt}(6)$ be a product of two 3-cycles. Then

$$\frac{|g^{\mathrm{Alt}(6)} \cap \tau(\mathrm{Alt}(5))|}{|g^{\mathrm{Alt}(6)}|} = \frac{20}{40} = \frac{1}{2}.$$

The proof of Lemma 2.8 will make use of the following combinatorial result.

Lemma 2.9. *If $\{ g_i \mid 1 \leq i \leq 10 \}$ are distinct elements of $\mathrm{Alt}(6)$ of cycle type 3^2, then there exist $i \neq j$ such that $g_i^{-1} g_j$ is either a 3-cycle or a 5-cycle.*

Proof. Let $\pi : \mathrm{Alt}(6) \to \mathrm{Alt}(6)$ be an outer automorphism which interchanges the conjugacy class of 3-cycles and the conjugacy class of elements of cycle type 3^2; and for each $1 \leq i \leq 10$, let $\tilde{g}_i = \pi(g_i)$. Then $\mathcal{S} = \{ \operatorname{supp} \tilde{g}_i \mid 1 \leq i \leq 10 \}$ contains at least 5 distinct 3-subsets of $\{ 1, 2, \cdots, 6 \}$. If there exist $i \neq j$ such that $\operatorname{supp} \tilde{g}_i \cap \operatorname{supp} \tilde{g}_j = \emptyset$, then $\tilde{g}_i^{-1} \tilde{g}_j$ has cycle type 3^2 and so $g_i^{-1} g_j$ is a 3-cycle. Hence we can suppose that if $S \neq T \in \mathcal{S}$, then $S \cap T \neq \emptyset$. Since $|\mathcal{S}| \geq 5$, there exist $i \neq j$ such that $|\operatorname{supp} \tilde{g}_i \cap \operatorname{supp} \tilde{g}_j| = 1$. Then $\tilde{g}_i^{-1} \tilde{g}_j$ is a 5-cycle and it follows that $g_i^{-1} g_j$ is also a 5-cycle. □

Proof of Lemma 2.8. With the above hypotheses, suppose that $H \leqslant \mathrm{Alt}(n)$ is a primitive subgroup such that

$$\frac{|g^{\mathrm{Alt}(n)} \cap H|}{|g^{\mathrm{Alt}(n)}|} \geq \frac{1}{4}, \tag{2.1}$$

We can express each permutation $h \in \mathrm{Alt}(n)$ of cycle type $1^s 3^t$ uniquely as a product

$$(a_1) \cdots (a_s) \times c_1 \cdots c_t,$$

where $(a_1), \cdots, (a_s)$ lists the (possibly empty) set of 1-cycles in increasing order and $c_1, \cdots, c_t$ lists the 3-cycles in increasing order with respect to $\min \operatorname{supp} c_i$. Since $\mathrm{Alt}(6)$ has 40 elements of cycle type 3^2, the inequality (2.1) implies that there exists a subset $\Phi \subset \Delta_n$ of cardinality 6 and fixed $(a_1) \cdots (a_s), c_1 \cdots c_{t-2}$ as above such that

$$\{ a_i \mid 1 \leq i \leq s \} \cup \bigcup \{ \operatorname{supp} c_j \mid 1 \leq j \leq t-2 \} = \Delta_n \smallsetminus \Phi$$

and there exist at least 10 distinct permutations $h \in \mathrm{Alt}(\Phi)$ of cycle type 3^2 such that

$$(a_1) \cdots (a_s) \times c_1 \cdots c_{t-2} h \in H.$$

Note that if h, h' are two such permutations, then $h^{-1}h' \in H$. Applying Lemma 2.9, we see that there exist two such permutations h, h' such that $h^{-1}h'$ is either a 3-cycle or a 5-cycle. Applying Lemmas 2.1 and 2.2, since $n > 6$, it follows that $H = \mathrm{Alt}(n)$. □

3. The proof of Theorem 2.5

The following result, which strengthens Zalesskiĭ [4, Lemma 7], will enable us to reduce the proof of Theorem 1.1 to the analysis of finitely many possibilities.

Lemma 3.1. *Suppose that* $5 < k < m$ *and that* $\tau : \mathrm{Alt}(k) \to \mathrm{Alt}(m)$ *is a transitive embedding. If* $4 < d < k$*, then there exists a* $\tau(\mathrm{Alt}(d))$*-orbit* $\Phi \subseteq \Delta_m$ *such that* $|\Phi| > d$*.*

Proof. We will argue by induction on $k-d \geq 1$. First suppose that $k-d = 1$. For the sake of contradiction, suppose that every nontrivial $\tau(\mathrm{Alt}(d))$-orbit on Δ_m has size d. First consider the case when there exists a natural $\tau(\mathrm{Alt}(d))$-orbit $\Psi \subseteq \Delta_m$. Let $x \in \Delta_m$ be the point such that $\tau(\mathrm{Alt}(d))_x = \tau(\mathrm{Alt}(d-1))$; and let $H < \mathrm{Alt}(k)$ be the subgroup such that $\tau(\mathrm{Alt}(k))_x = \tau(H)$. Then $\mathrm{Alt}(k-2) = \mathrm{Alt}(d-1) \leq H < \mathrm{Alt}(k)$. Thus H has an orbit $\Omega \subseteq \Delta_k$ such that $\{1, \cdots, k-2\} \subseteq \Omega$. Let $G \leqslant \mathrm{Sym}(\Omega)$ be group induced by the action of H on Ω. Then G acts primitively on Ω and contains the 3-cycle $(1\,2\,3)$. Applying Lemma 2.1, it follows that $\mathrm{Alt}(\Omega) \leqslant G$. Since

$$[\,\mathrm{Alt}(k) : H\,] = |\Delta_m| = m > k,$$

it follows that $\Omega = \{1, \cdots, k-2\}$. If $G = \mathrm{Alt}(\Omega)$, then H is the pointwise stabilizer of $\{k-1, k\}$ in $\mathrm{Alt}(k)$; and if $G = \mathrm{Sym}(\Omega)$, then H is the setwise stabilizer of $\{k-1, k\}$ in $\mathrm{Alt}(k)$. Consequently, the action of $\tau(\mathrm{Alt}(k))$ on Δ_m is either isomorphic to the action of $\mathrm{Alt}(k)$ on

$$\Phi = \{\,(i, j) \mid 1 \leq i \neq j \leq k\,\},$$

or else isomorphic to the action of $\mathrm{Alt}(k)$ on

$$\Phi' = \{\,\{i, j\} \mid 1 \leq i \neq j \leq k\,\}.$$

In the first case, $\tau(\mathrm{Alt}(d)) = \tau(\mathrm{Alt}(k-1))$ has an orbit of size $(k-1)(k-2) > d$; and in the second case, $\tau(\mathrm{Alt}(d))$ has an orbit of size $(k-1)(k-2)/2 > d$, which is a contradiction. Thus it only remains to consider the case when $d = 6$ and every nontrivial $\tau(\mathrm{Alt}(6))$-orbit $\Psi \subseteq \Delta_m$ is isomorphic to the nonstandard action of $\mathrm{Alt}(6)$ on Δ_6. Fix such an orbit Φ and let $x \in \Phi$ be the point such that $\tau(\mathrm{Alt}(6))_x = \tau(\pi(\mathrm{Alt}(5)))$, where π is an outer automorphism of $\mathrm{Alt}(6)$ which interchanges the conjugacy class of 3-cycles and the conjugacy class of elements of cycle type 3^2; and let $H < \mathrm{Alt}(7)$ be the subgroup such that $\tau(\mathrm{Alt}(7))_x = \tau(H)$. Then $\pi(\mathrm{Alt}(5)) \leqslant H$ and so H contains a 5-cycle. Since $\pi(\mathrm{Alt}(5))$ acts 2-transitively on Δ_6, it follows that H has an orbit $\Omega \subseteq \Delta_7$ such that $\{1, 2, \cdots, 6\} \subseteq \Omega$. Suppose that H acts transitively on Δ_7. Then H acts primitively on Δ_7 and contains a 5-cycle; and so by Lemma 2.9, $H = \mathrm{Alt}(7)$, which is a contraction. Thus $\Omega = \{1, 2, \cdots, 6\}$ and $\pi(\mathrm{Alt}(5)) \leqslant H \leqslant \mathrm{Alt}(6)$. Since $\pi(\mathrm{Alt}(5))$ is a maximal proper subgroup of $\mathrm{Alt}(6)$ and $[\mathrm{Alt}(7) : H] = m > 7$, it follows that $H = \pi(\mathrm{Alt}(5))$. Thus the action of $\tau(\mathrm{Alt}(7))$ on Δ_m is isomorphic to $\mathrm{Alt}(7) \curvearrowright \mathrm{Alt}(7)/H = \mathrm{Alt}(7)/\pi(\mathrm{Alt}(5))$. Let $a \in \mathrm{Alt}(7) \smallsetminus \mathrm{Alt}(6)$. If $g \in \mathrm{Alt}(6)$, then $gaH = aH$ if and only if $g \in aHa^{-1} \cap \mathrm{Alt}(6)$. Since $Ha^{-1} \cap \mathrm{Alt}(6) \cong D_{10}$, it follows that $\tau(\mathrm{Alt}(6))$ has an orbit of size 36 on Δ_m, which is a contradiction. This concludes the proof when $k - d = 1$.

Next suppose that $k - d = s \geq 2$ and that the result holds for $s - 1$. Then $\tau(\mathrm{Alt}(d + 1))$ has an orbit $\Psi \subseteq \{1, \cdots, m\}$ such that $m' = |\Psi| > d + 1$. By the previous paragraph, it follows that $\tau(\mathrm{Alt}(d))$ has an orbit $\Psi' \subseteq \Psi$ such that $|\Psi'| > d$. □

Proof of Theorem 2.5. Suppose that $5 \leq k < m < n$ and that $\tau_0 : \mathrm{Alt}(k) \to \mathrm{Alt}(m)$ and $\tau_1 : \mathrm{Alt}(m) \to \mathrm{Alt}(n)$ are embeddings of finite alternating groups such that $\tau = \tau_1 \circ \tau_0$ is a diagonal embedding. If τ_1 is a diagonal embedding, then it is clear that τ_0 must also be a diagonal embedding. So suppose that τ_1 is not a diagonal embedding. By Lemma 2.3, if $\tau_1(\mathrm{Alt}(m))$ has no orbits $\Phi \subseteq \Delta_n$ with $|\Phi| > m$, then $m = 6$ and there exists $\tau_1(\mathrm{Alt}(6))$-orbit $\Phi \subseteq \Delta_n$ with $|\Phi| = 6$ such that $\tau(\mathrm{Alt}(6)) \curvearrowright \Phi$ is isomorphic to the nonstandard action of $\mathrm{Alt}(6)$ on Δ_6. In this case, we must have that $k = 5$ and that $\tau_0 : \mathrm{Alt}(5) \to \mathrm{Alt}(6)$ is the 2-transitive embedding. Since $\tau = \tau_1 \circ \tau_0$ is diagonal, it follows that τ_1 must be almost diagonal. Thus we can suppose that there exists a $\tau_1(\mathrm{Alt}(m))$-orbit $\Phi \subseteq \Delta_n$ with $|\Phi| > m$. To reach a contradiction, it is enough to show that there exists a nontrivial non-natural $\tau(\mathrm{Alt}(k))$-orbit $\Psi \subseteq \Phi$. Consequently, in order to simplify notation, we can suppose that $\Phi = \Delta_n$; i.e. that the embedding $\tau_1 : \mathrm{Alt}(m) \to \mathrm{Alt}(n)$ is transitive.

Next note that $\tau \restriction \mathrm{Alt}(5)$ is also a diagonal embedding. Let $\Omega \subseteq \Delta_m$ be a nontrivial $\tau_0(\mathrm{Alt}(5))$-orbit. Then, by Lemma 3.1, there exists a $\tau_1(\mathrm{Alt}(\Omega))$-orbit $\Phi \subseteq \Delta_n$ such that $|\Phi| > |\Omega|$. In order to simplify notation, we will suppose that $k = 5$, that $\Omega = \Delta_m$ and that $\Phi = \Delta_n$. Thus $\tau_0 : \mathrm{Alt}(5) \to \mathrm{Alt}(m)$ and $\tau_1 : \mathrm{Alt}(m) \to \mathrm{Alt}(n)$ are transitive embeddings with $5 \leq m < n$ such that $\tau = \tau_1 \circ \tau_0$ is a diagonal embedding. If $m = 5$, then $\tau : \mathrm{Alt}(5) \to \mathrm{Alt}(n)$ is a transitive embedding with $n > 5$ and so τ is not diagonal. Thus we can suppose that $m > 5$. Let χ be the permutation character of $\mathrm{Alt}(m)$ arising from the transitive action $\tau_1(\mathrm{Alt}(m)) \curvearrowright \Delta_n$.

Claim 3.2. *There exist integers a, $b \geq 0$ such that $\chi \restriction \tau_0(\mathrm{Alt}(5)) = a\eta_5 + b1_5$.*

Proof of Claim 3.2. Since $\tau = \tau_1 \circ \tau_0$ is a diagonal embedding, there exist integers c, $d \geq 0$ such that $\tau(\mathrm{Alt}(5))$ has c natural orbits and d trivial orbits on Δ_n. It follows that $\chi \restriction \tau_0(\mathrm{Alt}(5)) = c\chi_5 + d1_{\mathrm{Alt}(5)} = c\eta_5 + (c+d)1_{\mathrm{Alt}(5)}$. □

Note that there exists a proper subgroup H of Alt(5) such that the action $\tau_0(\mathrm{Alt}(5)) \curvearrowright \Delta_m$ is isomorphic to $\mathrm{Alt}(5) \curvearrowright \mathrm{Alt}(5)/H$. Consequently, the possibilities for m are 6,10,12,15,20,30,60. From now on, let $g \in \mathrm{Alt}(5)$ be a 3-cycle. Then the cycle structure of $\tau_0(g)$ is easily computed, using the observation that if $a \in \mathrm{Alt}(5)$, then $gaH = aH$ if and only if $a^{-1}ga \in H$. Thus $\tau_0(g)$ has 0 fixed points if $m = 6, 12, 15, 30, 60$, and has 1 fixed point if $m = 10$, and has 2 fixed points if $m = 20$. In other words, $\tau_0(g)$ is a product of $t = \lfloor m/3 \rfloor$ 3-cycles and $s = m - 3t$ 1-cycles.

We will first consider the case when $m = 6$. Then $\tau_0(g)$ is a product of two 3-cycles; and if $h \in \mathrm{Alt}(5)$ is not a 3-cycle, then the cycle structure of $\tau_0(h)$ is obtained from that of h by adding one more 1-cycle. Consider the following character table of Alt(6). (Recall that Alt(6) has two conjugacy classes of 5-cycles.)

class	1^6	2^21^2	3^2	3^11^3	4^12^1	5^11^1A	5^11^1B
$1_{\mathrm{Alt}(6)}$	1	1	1	1	1	1	1
η_6	5	1	-1	2	-1	0	0
θ	5	1	2	-1	-1	0	0
ψ_1	8	0	-1	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
ψ_2	8	0	-1	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
ψ_3	9	1	0	0	1	-1	-1
ψ_4	10	-2	1	1	0	0	0

The character table of Alt(6)

Here $\theta = \eta_6 \circ \pi$, where π is any outer automorphism of Alt(6) which interchanges the conjugacy class of 3-cycles and the conjugacy class of elements of cycle type 3^2. Note that $\theta \restriction \tau_0(\mathrm{Alt}(5)) = \varphi_5 + 1_5$.[1] Also, examining the above character table, we see that if $\psi \neq \theta$ is any other nontrivial irreducible representation of Alt(6), then there do not exist integers c, $d \geq 0$ such that $\psi \restriction \tau_0(\mathrm{Alt}(5)) = c\varphi_5 + d1_5$. It follows that there must exist an integer s, $t \geq 0$ such that $\chi = s\theta + t1_6$. Since the embedding $\tau_1 : \mathrm{Alt}(6) \to \mathrm{Alt}(n)$ is transitive, it follows that $t = 1$ and hence $s > 1$. But then if $h \in \mathrm{Alt}(6)$ is a 3-cycle, then $\chi(h) = 1 - s < 0$, which is impossible since χ is a permutation character.

[1] In particular, θ is a counterexample to Zalesskiĭ [4, Lemma 8].

Thus we can suppose that $m = 10, 12, 15, 20, 30, 60$. Since $\chi \restriction \tau_0(\mathrm{Alt}(5)) = a\eta_5 + b1_5$, it follows that $\chi(\tau_0(g)) = a + b$ and $|\Phi| = \chi(1) = 4a + b$; and so

$$\frac{|\operatorname{Fix}_{\Delta_n}(\tau(g))|}{|\Delta_n|} = \frac{a+b}{4a+b} \geq \frac{1}{4}.$$

There exists a proper subgroup K of $\mathrm{Alt}(m)$ such that $\tau_1(\mathrm{Alt}(m)) \curvearrowright \Delta_n$ is isomorphic to $\mathrm{Alt}(m) \curvearrowright \mathrm{Alt}(m)/K$. Suppose that M is a maximal proper subgroup of $\mathrm{Alt}(m)$ such that $K \leqslant M$ and let $\Omega = \mathrm{Alt}(m)/M$. Then Corollary 2.7 implies that

$$\frac{|\operatorname{Fix}_{\Omega}(\tau_0(g))|}{|\Omega|} \geq \frac{|\operatorname{Fix}_{\Delta_n}(\tau(g))|}{|\Delta_n|} \geq \frac{1}{4}. \tag{3.1}$$

We will obtain a contradiction from (3.1) by considering the various possibilities for the action of the maximal subgroup M on Δ_m.

Case 1: Suppose that M acts intransitively on Δ_m. Since M is a maximal proper subgroup, it follows that there exists a subset $S \subseteq \Delta_m$ with $1 \leq \ell = |S| < m/2$ such that $M = \mathrm{Alt}(m)_{\{S\}}$ is the setwise stabilizer of S in $\mathrm{Alt}(m)$. Thus $\mathrm{Alt}(m) \curvearrowright \Omega$ is isomorphic to the action of $\mathrm{Alt}(m)$ on the set $[\,\Delta_m\,]^\ell$ of ℓ-subsets of Δ_m. If $\ell = 1$, then

$$\frac{1}{4} \leq \frac{|\operatorname{Fix}_{\Omega}(\tau_0(g))|}{|\Omega|} = \frac{|\operatorname{Fix}_{\Delta_m}(\tau_0(g))|}{|\Delta_m|} \leq \frac{2}{m}$$

and so $m \leq 8$, which is a contradiction. Also, since $\tau_0(g)$ fixes at most one 2-subset of Δ_m, it follows that $\ell \neq 2$. Thus $3 \leq \ell < m/2$. Clearly if $T \in [\,\Delta_m\,]^\ell$, then T is fixed setwise by $\tau_0(g)$ if and only if T is a union of $\tau_0(g)$)-orbits. Let $\mathcal{F} = \operatorname{Fix}_{[\Delta_m]^\ell}(\tau_0(g))$; and for each $S \in \mathcal{F}$, let $\alpha(S) = \min\{\, s \in S \mid \tau_0(g) \cdot s \neq s \,\}$. Then the sets

$$\mathcal{F} \cup \{\, (S \smallsetminus \{\, \alpha(S) \,\}) \cup \{\, t \,\} \mid S \in \mathcal{F}, t \in \Delta_m \smallsetminus (S \cup \operatorname{Fix}_{\Delta_m}(\tau_0(g)) \,\}$$

are distinct. Note that if $S \in \mathcal{F}$, then

$$|\Delta_m \smallsetminus (S \cup \operatorname{Fix}_{\Delta_m}(\tau_0(g))| \geq m - (\ell + 2) > \frac{m}{2} - 2 \geq 3.$$

It follows that

$$|\operatorname{Fix}_{\Omega}(\tau_0(g))| = |\operatorname{Fix}_{[\Delta_m]^\ell}(\tau_0(g))| \leq \frac{1}{5}|[\Delta_m]^\ell| = \frac{1}{5}|\Omega|,$$

which contradicts (3.1).

Case 2: Suppose that M acts transitively but imprimitively on Δ_m. Then there exists an M-invariant partition $\mathcal{P}$ of Δ_m into ℓ-subsets for some divisor ℓ of m with $2 \leq \ell \leq m/2$; and, by the maximality of M, we can suppose that $M = \mathrm{Alt}(m)_{\mathcal{P}}$ is the stabilizer of $\mathcal{P}$ in $\mathrm{Alt}(m)$. Hence, letting Π be the set of partitions of Δ_m into ℓ-subsets, $\mathrm{Alt}(m) \curvearrowright \Omega$ is isomorphic to the action of $\mathrm{Alt}(m)$ on Π. If $\mathcal{Q} \in \operatorname{Fix}_{\Pi}(\tau_0(g))$, then we define the integer $\alpha(\mathcal{Q})$ as follows.

(a) If $\mathcal{Q}$ contains a $\tau_0(g)$-invariant block B such that $\tau_0(g) \restriction B \neq \mathrm{id}_B$, then $\alpha(\mathcal{Q})$ is the least $s \in \Delta_m$ such that $[\,s\,]_{\mathcal{Q}}$ is $\tau_0(g)$-invariant and $\tau_0(g) \cdot s \neq s$.

(b) Otherwise, $\alpha(\mathcal{Q})$ is the least $s \in \Delta_m$ such that $\tau_0(g) \cdot s \neq s$.

For each $t \in \Delta \smallsetminus [\,\alpha(\mathcal{Q})\,]_{\mathcal{Q}}$, we define $\mathcal{Q}(t) \in \Pi$ to be the partition obtained from $\mathcal{Q}$ by replacing the block $[\,\alpha(\mathcal{Q})\,]_{\mathcal{Q}}$ by $([\,\alpha(\mathcal{Q})\,]_{\mathcal{Q}} \smallsetminus \{\,\alpha(\mathcal{Q})\,\}) \cup \{\,t\,\}$ and the block $[\,t\,]_{\mathcal{Q}}$ by $([\,t\,]_{\mathcal{Q}} \smallsetminus \{\,t\,\}) \cup \{\,\alpha(\mathcal{Q})\,\}$.

Claim 3.3. $\mathcal{Q}(t) \notin \mathrm{Fix}_\Pi(\tau_0(g))$.

Proof of Claim 3.3. First suppose that $\mathcal{Q}$ contains a $\tau_0(g)$-invariant block B such that $\tau(g) \restriction B \neq \mathrm{id}_B$. Then clearly $\tau_0(g) \cdot [\,t\,]_{\mathcal{Q}(t)} \neq [\,t\,]_{\mathcal{Q}(t)}$. Also, since $\ell \geq 3$, it follows that $\tau_0(g) \cdot [\,t\,]_{\mathcal{Q}(t)} \cap [\,t\,]_{\mathcal{Q}(t)} \neq \emptyset$. Hence $\mathcal{Q}(t) \notin \mathrm{Fix}_\Pi(\tau_0(g))$.

Thus we can suppose that $\mathcal{Q}$ does not contain a $\tau_0(g)$-invariant block B such that $\tau_0(g) \restriction B \neq \mathrm{id}_B$. For each $0 \leq i < 3$, let $S_i = \tau_0(g)^i \cdot [\,\alpha(\mathcal{Q})\,]_{\mathcal{Q}}$. Then there exists $0 < i < 2$ such that $S_i \in \mathcal{Q}(t)$. Since $S_0 = \tau_0(g)^{3-i} \cdot S_i \notin \mathcal{Q}(t)$, it follows that $\mathcal{Q}(t) \notin \mathrm{Fix}_\Pi(\tau_0(g))$. □

If $\mathcal{Q}$, $\mathcal{Q}' \in \mathrm{Fix}_\Pi(\tau_0(g))$ and $\mathcal{Q}(t) = \mathcal{Q}'(t')$, then it is easily checked that $\mathcal{Q} = \mathcal{Q}'$ and $t = t'$. Thus

$$|\Pi| \geq (1 + m - \ell)|\,\mathrm{Fix}_\Pi(\tau_0(g))| \geq (1 + m/2)|\,\mathrm{Fix}_\Pi(\tau_0(g))|.$$

It follows that

$$|\,\mathrm{Fix}_\Omega(\tau_0(g))| = |\,\mathrm{Fix}_\Pi(\tau_0(g))| \leq \frac{1}{6}|[\Pi| = \frac{1}{6}|\Omega|,$$

which contradicts (3.1).

Case 3: Finally suppose that M acts primitively on Δ_m. Applying inequality (3.1) and Proposition 2.6, we see that

$$\frac{1}{4} \leq \frac{|\,\mathrm{Fix}_\Omega(\tau_0(g))|}{|\Omega|} = \frac{|\,\tau_0(g)^{\mathrm{Alt}(m)} \cap M\,|}{|\,\tau_0(g)^{\mathrm{Alt}(m)}\,|}.$$

But then Lemma 2.8 implies that $M = \mathrm{Alt}(m)$, which is a contradiction. This completes the proof of Theorem 2.5. □

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