350 PRACTICE SECOND MIDTERM QUESTIONS

Question 1. Let $A \in \mathrm{M}_{n\times n}(F)$ be an $n \times n$ matrix over the field F.

(i) Give the definition of an eigenvector and eigenvalue of A.

(ii) Give the definition of the characteristic polynomial $f(t)$ of A.

(iii) Prove that if $Q \in \mathrm{M}_{n\times n}(F)$ is an invertible matrix, then $Q^{-1}AQ$ and A have the same characteristic polynomial.

Question 2.

Compute the inverse of the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Question 3.

Find the general solution of the following system of linear equations.

$$x_1 + x_2 - 3x_3 + x_4 = -2$$

$$x_1 + x_2 + x_3 - x_4 = 2$$

$$x_1 + x_2 - x_3 = 0$$

Question 4.

Let $A \in \mathrm{M}_{3\times 3}(\mathbb{R})$ be the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

(i) Find the eigenvalues of A.

(ii) Find the eigenspaces of A.

(iii) Find an invertible matrix $Q \in \mathrm{M}_{3\times 3}(\mathbb{R})$ such that $Q^{-1}AQ$ is a diagonal matrix.

Question 5.

Suppose that A, $B \in \mathrm{M}_{n\times n}(\mathbb{C})$ and that B is invertible. Prove that there exists $c \in \mathbb{C}$ such that $A + cB$ is not invertible. (*Hint:* consider $\det(A + cB)$.)

Question 6.

Let A, $B \in \mathrm{M}_{2\times 2}(\mathbb{R})$ be the matrices:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Prove that $A + rB$ is invertible for all $r \in \mathbb{R}$. (*Hint:* consider $\det(A + rB)$.)