Question 1. Let $A \in \text{M}_{n \times n}(F)$ be an $n \times n$ matrix over the field $F$.

(i) Give the definition of an eigenvector and eigenvalue of $A$.

(ii) Give the definition of the characteristic polynomial $f(t)$ of $A$.

(iii) Prove that if $Q \in \text{M}_{n \times n}(F)$ is an invertible matrix, then $Q^{-1}AQ$ and $A$ have the same characteristic polynomial.

Question 2.

Compute the inverse of the following matrix:

$$A = \begin{pmatrix}
1 & 2 & 2 \\
1 & 0 & 1 \\
1 & 1 & 2
\end{pmatrix}$$

Question 3.

Find the general solution of the following system of linear equations.

$$x_1 + x_2 - 3x_3 + x_4 = -2$$
$$x_1 + x_2 + x_3 - x_4 = 2$$
$$x_1 + x_2 - x_3 = 0$$

Question 4.

Let $A \in \text{M}_{3 \times 3}(\mathbb{R})$ be the matrix

$$A = \begin{pmatrix}
3 & 1 & 1 \\
2 & 4 & 2 \\
-1 & -1 & 1
\end{pmatrix}$$

(i) Find the eigenvalues of $A$.

(ii) Find the eigenspaces of $A$.

(iii) Find an invertible matrix $Q \in \text{M}_{3 \times 3}(\mathbb{R})$ such that $Q^{-1}AQ$ is a diagonal matrix.
Question 5.
Suppose that $A, B \in M_{n \times n}(\mathbb{C})$ and that $B$ is invertible. Prove that there exists $c \in \mathbb{C}$ such that $A + cB$ is not invertible. (Hint: consider $\det(A + cB)$.)

Question 6.
Let $A, B \in M_{2 \times 2}(\mathbb{R})$ be the matrices:

\[
A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

Prove that $A + rB$ is invertible for all $r \in \mathbb{R}$. (Hint: consider $\det(A + rB)$.)