350 PRACTICE FIRST MIDTERM QUESTIONS

Question 1.

(i) State the definition of a linearly dependent subset S of a vector space V.

(ii) State the definition of a basis of a vector space V.

(iii) Suppose that V, W are vector spaces over a field F and that $T : V \to W$ is a linear transformation. Give the definitions of $N(T)$ and $R(T)$.

(iv) State the Dimension Theorem.

Question 2.

Let $V = \{ (a_1, a_2) \mid a_1, a_2 \in \mathbb{R} \}$. For (a_1, a_2), $(b_1, b_2) \in V$ and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$

and

$$c(a_1, a_2) = (ca_1, a_2).$$

Is V a vector space over $\mathbb{R}$ with these operations?

Question 3.

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that:

- $T(1, 0, 0) = (2, 3, 4)$;
- $T(0, 1, 0) = (3, 4, 5)$;
- $T(0, 0, 1) = (4, 5, 6)$.

Find bases for $R(T)$ and $N(T)$.

Question 4.

Let $P_3(\mathbb{R})$ be the vector space of polynomials of degree at most 3 over the real numbers $\mathbb{R}$ and let $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be the linear transformation defined by $T(f(x)) = f'(x)$.

(i) Compute the matrix $[\,T\,]_\beta$ with respect to the ordered basis $\beta = \{ 1, x, x^2, x^3 \}$.

(ii) Compute $\operatorname{rank}(T)$ and $\operatorname{nullity}(T)$.

Question 5.

Suppose that V, W are vector spaces over a field F and that $T : V \to W$ is a one-to-one linear transformation. Show that if $\{ v_1, \cdots, v_n \}$ is a linearly independent subset of V, then $\{ T(v_1), \cdots, T(v_n) \}$ is a linearly independent subset of W.