350 FIRST MIDTERM FALL 2019

Question 1.

(i) Let $V = \{ (a_1, a_2) \mid a_1, a_2 \in \mathbb{R} \}$. For (a_1, a_2), $(b_1, b_2) \in V$ and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2)$$

and

$$c(a_1, a_2) = (ca_1, ca_2).$$

Determine whether V is a vector space over $\mathbb{R}$ with these operations. Justify your answer.

(ii) Determine whether $W = \{ (a, b) \in \mathbb{R}^2 \mid a^2 - b^2 = 0 \}$ is a subspace of $\mathbb{R}^2$. Justify your answer.

Question 2.

(i) Suppose that V, W are vector spaces over a field F and that $T : V \to W$ is a linear transformation. Give the definitions of $N(T)$ and $R(T)$.

(ii) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(a_1, a_2, a_3) = (a_1 - 2a_2 + a_3, 2a_1 - 3a_2 + a_3).$$

Find bases for $R(T)$ and $N(T)$.

Question 3.

Let $\beta = \{ e_1, e_2, e_3 \}$ be the standard ordered basis of $\mathbb{R}^3$ and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that

- $T(e_1) = 2e_1$;
- $T(e_2) = 2e_1 + 3e_2$;
- $T(e_3) = -2e_1 - e_2 + 2e_3$.

Compute $[T]_\gamma$, where γ is the ordered basis $\{ -e_1, 2e_1 + e_2, e_1 + e_2 + e_3 \}$ of $\mathbb{R}^3$.

Question 4.

(i) State the Dimension Theorem.

(ii) Suppose that V, W are finite dimensional vector spaces over a field F and that there exists a one-to-one linear transformation $T : V \to W$. Prove that $\dim(V) \leq \dim(W)$.