350 FIRST MIDTERM SPRING 2020

Question 1.

(i) Let $V = \{\, (a_1, a_2) \mid a_1, a_2 \in \mathbb{R} \,\}$. For (a_1, a_2), $(b_1, b_2) \in V$ and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$$

and

$$c(a_1, a_2) = (ca_1, ca_2).$$

Determine whether V is a vector space over $\mathbb{R}$ with these operations. Justify your answer.

(ii) Determine whether $W = \{\, (a, b, c) \in \mathbb{R}^3 \mid ab + c^2 = 0 \,\}$ is a subspace of $\mathbb{R}^3$. Justify your answer.

Question 2.

(i) Suppose that V, W are vector spaces over a field F and that $T : V \to W$ is a linear transformation. Give the definitions of $N(T)$ and $R(T)$.

(ii) State the Dimension Theorem.

(iii) Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(a_1, a_2, a_3, a_4) = (a_1 + a_2 - 3a_3, a_3 - 2a_4).$$

Find bases for $R(T)$ and $N(T)$.

Question 3.

Let $\beta = \{\, e_1, e_2, e_3 \,\}$ be the standard ordered basis of $\mathbb{R}^3$ and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that

- $T(e_1) = 2e_1$;
- $T(e_2) = -e_1 - e_2 - 2e_3$;
- $T(e_3) = e_1 + 4e_2 + 5e_3$.

Compute $[\,T\,]_\gamma$, where γ is the ordered basis $\{\, e_1 + 2e_2 + e_3, e_1, e_2 + e_3 \,\}$ of $\mathbb{R}^3$.

Question 4.

(i) Let V, W be vector spaces over a field F and let $T : V \to W$ be a linear transformation. Let $\{ w_1, \cdots , w_k \} \subseteq W$ be a set of k linearly independent vectors. Prove that if the vectors $\{ v_1, \cdots , v_k \} \subseteq V$ satisfy $T(v_i) = w_i$ for $1 \le i \le k$, then $\{ v_1, \cdots , v_k \}$ is linearly independent.

(ii) Let V, W be finite-dimensional vector spaces over a field F and let $T : V \to W$ be a linear transformation. Prove that if $\dim(V) < \dim(W)$, then T is not onto.