Recent progress on lower bounds for arithmetic circuits

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***Abstract*—In recent years there has been much exciting progress on depth reduction of arithmetic circuits and lower bounds for bounded depth arithmetic circuits. We will survey some of these results and highlight some of the main challenges and open questions that remain.**

***Index Terms*—arithmetic circuits, lower bounds, depth reduction**

I. Introduction

The problem of proving lower bounds for arithmetic circuits is one of the most interesting and challenging problems in complexity theory. Arithmetic circuits are a very natural model of computation for many natural algebraic algorithms such as matrix multiplication, computing fast fourier transforms, computing the determinant etc. In a seminal work [Val79], Valiant defined the classes VP and VNP as the algebraic analogs of the classes P and NP. The VP vs VNP is an easier question to answer than the P vs NP question, and moreover the additional structure of arithmetic circuits makes them amenable to anlaysis using a wide variety of techniques. In addition to being extremely interesting from the point of view of proving lower bounds, the VP vs VNP question is also an important question in derandomization. In a beautiful result, Kabanets and Impagliazzo [KI04] showed that proving lower bounds for arithmetic circuits is essentially equivalent to the problem of derandomizing polynomial identity testing.

Although the problem of proving lower bounds for arithmetic circuits has received a great deal of attention, the best lower bounds we know for general arithmetic circuits are barely super linear [Str73], [BS83]. The absence of progress on the general problem has led to a great deal of attention being devoted to proving lower bounds for *restricted classes* of arithmetic circuits. The hope is that understanding restricted classed might shed light on how to approach the much more general and seemingly harder problem. Arithmetic circuits of small depth are one such class that has been intensively studied.

In this survey, we will first see some of the earlier and more classical results related to depth reduction of arithmetic circuits and well as lower bounds for bounded depth arithmetic circuits. We will then outline some of the more recent results on strengthened depth reduction, as well as new lower bounds for richer classes of bounded depth circuits.

II. Classical results

A. Depth reduction

In a very interesting direction of research, Valiant et al [VSBR83] showed that every polynomial of degree n in $\mathsf{poly}(n)$ variables, which can be computed by a $\mathsf{poly}(n)$ sized arithmetic circuit, can also be computed by a $\mathsf{poly}(n)$ sized arithmetic circuit of *depth* $O(\log^2 n)$. In other words, arbitrary depth circuits in VP can be reduced to circuits of depth $O(\log^2 n)$ with only a polynomial blowup in size (i.e. $\mathsf{VP} = \mathsf{VNC}^2$!). Thus, in order to separate VNP from VP, it would suffice to show a super-polynomial lower bound for just circuits of depth $O(\log^2 n)$. This was a very surprising result since nothing like this is believed to be true in the Boolean world.

B. Lower bounds for bounded depth circuits

Lower bounds for depth 2 circuits (computing a sum of products) are trivial. Any polynomial with exponentially many monomials (such as the determinant) needs an exponentially large depth 2 circuit. Already for depth 3 circuits we have much weaker results. The best known lower bounds for general depth 3 circuits (these are circuits computing a sum of products of sums) are only quadratic [SW01]. In particular [SW01] gives a polynomial in n variables and of $O(n)$ degree such that any depth 3 circuit computing it must have size $\Omega(n^2)$. For the case of depth 3 circuits over small constant sized finite fields, Grigoriev and Karpinksi [GK98] and Grigoriev and Razborov [GR98] proved exponential lower bounds. Also, in a very elegant result that influenced many of the later works, Nisan and Wigderson [NW95] proved *exponential* lower bounds for the class of *homogeneous* depth 3 circuits. Several of the results that we will be mentioning later on build upon the Nisan-Widerson result. This result used the *dimension of the space spanned by all partial derivatives* of a polynomial as a measure of of its complexity. By showing that some simple explicit polynomials have a very large complexity and all small homogeneous depth 3 circuits have small complexity, they were able to obtain the lower bounds.

For several years thereafter, there were no improved lower bounds - even for the case of depth 4 homogeneous circuits, the best lower bounds known were just mildly super-linear [Raz10]. This is contrary to what is known for Boolean circuits, where we know exponential lower bounds for constant depth circuits. Indeed this seemed quite surprising until in later works on depth reduction (which we discuss next) it was shown that depth 4 circuits are already quite complex.

C. Terminology

Since we will be focussing on depth 4 arithmetic circuits and some of its variants, we formally give a definition of the model.

Arithmetic Circuits: An arithmetic circuit over a field $\mathbb{F}$ and a set of variables $x_1, x_2, \ldots, x_N$ is a directed acyclic graph with internal nodes labelled by the field operations and the leaf nodes labelled by input variables or field elements. By the *size* of the circuit, we mean the total number of nodes in the underlying graph and by the *depth* of the circuit, we mean the length of the longest path from the output node to a leaf node. A circuit is said to be *homogeneous* if the polynomial computed at every node is a homogeneous polynomial. By a $\Sigma\Pi\Sigma\Pi$ circuit or a depth 4 circuit, we mean a circuit of depth 4 with the top layer and the third layer only have sum gates and the second and the bottom layer have only product gates. Observe that a *depth 4 circuit* can be converted into a *depth 4 formula* with only a polynomial blow up in size. We will therefore, use the term formula or circuit for a depth 4 circuit interchangeably. A homogeneous polynomial P of degree n in N variables, which is computed by a homogeneous $\Sigma\Pi\Sigma\Pi$ circuit can be written as

$$P(x_1, x_2, \ldots, x_N) = \sum_{i=1}^{T} \prod_{j=1}^{d_i} Q_{i,j}(x_1, x_2, \ldots, x_N) \quad (1)$$

Here, T is the top fan-in of the circuit. Since the circuit is homogeneous, we know that for every $i \in \{1, 2, 3, \ldots, T\}$,

$$\sum_{j=i}^{d_i} \deg(Q_{i,j}) = n$$

The homogeneous $\Sigma\Pi\Sigma\Pi$ circuit in Equation 1, is said to be a $\Sigma\Pi\Sigma\Pi^{[a]}$ circuit, if each $Q_{i,j}$ is a polynomial of degree at most a. In this case we say that the *bottom fan-in is bounded by* a.

III. Recent results

A. Depth reduction

In a surprising work, Agrawal and Vinay [AV08] gave an "explanation" of why proving lower bounds for bounded depth arithmetic circuits seems so difficult. By building upon the results of Valiant et al [VSBR83], Agrawal and Vinay showed that (in a certain sense) much stronger depth reductions are possible. Every homogeneous polynomial which can be computed by a polynomial sized (or even $2^{o(n)}$ sized) arithmetic circuit *of arbitrary depth* in $\mathsf{poly}(n)$ variables and of degree n, can also be computed by a *depth 4* $\Sigma\Pi\Sigma\Pi$ circuit of size $2^{o(n)}$. Thus in order to prove exponential ($2^{\Omega(n)}$) lower bounds for general arithmetic circuits (in particular in order to separate VNP form VP), it would suffice to prove strong enough ($2^{\Omega(n)}$) lower bounds for just *homogeneous depth 4 circuits*. Thus in some sense, homogeneous depth 4 circuits *capture* the inherent complexity of general arithmetic circuits.

In a recent sequence of works, Koiran [Koi12] and Tavenas [Tav13] built upon the results of Valiant et al [VSBR83] and Agrawal-Vinay [AV08] and showed that if one starts with a polynomial sized homogeneous arithmetic circuit (instead of a subexponential one), then one can depth reduce to an even more restricted class of circuits. One can in fact reduce to a $2^{O(\sqrt{n}\log n)}$ sized homogeneous $\Sigma\Pi\Sigma\Pi^{[\sqrt{n}]}$. More formally, the following result is true:

Theorem 3.1 (Koiran[Koi12], Tavenas [Tav13]): Every arithmetic circuit of size $\mathsf{poly}(n)$, computing a polynomial of degree n in $N = n^{O(1)}$ variables, can be transformed into an equivalent homogeneous $\Sigma\Pi\Sigma\Pi^{[t]}$ circuit with top fan-in[1] at most $\exp(O(\frac{n}{t}\log N))$.

In particular, in order to separate VNP from VP, it would suffice to prove ($n^{\omega(\sqrt{n})}$) lower bounds for just homogeneous $\Sigma\Pi\Sigma\Pi^{[\sqrt{n}]}$ circuits.

[1]For $\Sigma\Pi\Sigma\Pi^{[t]}$ circuits where $t = \sqrt{n}$, observe that an upper bound of $\exp(O(\sqrt{n}\log N))$ on the top fan-in the circuit implies the same upper bound on size, since each product gate at the second layer computes a polynomial with at most $\exp(O(\sqrt{n}\log N))$ monomials. However for other values of t, the top fan-in bound will be the more relevant parameter for depth reduction.

In another very exciting work in this direction, Gupta, Kamath, Kayal and Saptharishi [GKKS13b] proved that strong enough ($n^{\omega(\sqrt{n})}$) lower bounds for just *depth 3 circuits* suffice to show superpolynomial lower bounds for circuits of arbitrary depth (over fields of characteristic zero). Indeed this depth reduction result showed that there exist $n^{O(\sqrt{n})}$ circuits of depth 3 that compute the determinant of an $n \times n$ matrix over characteristic zero, a fact that was not believed to be true until this work. One caveat is that in this depth reduction we lose the property of homogeneity that was true for the reduction to depth 4 circuits. At least from the point of view of proving lower bounds, the loss in homogeneity when using this reduction to depth 3 circuits might be severe, since we know only weak lower bounds for non-homogeneous depth 3 circuits over general fields [SW01].

B. Lower bounds for $\Sigma\Pi\Sigma\Pi^{[t]}$ *circuits*

In a recent breakthrough result Gupta, Kamath, Kayal and Saptharishi [GKKS13a], made the first major progress on the problem of obtaining lower bounds for depth 4 arithmetic circuits. They proved $2^{\Omega(\sqrt{n})}$ lower bounds for an explicit polynomial of degree n in $n^{O(1)}$ variables computed by a homogeneous $\Sigma\Pi\Sigma\Pi^{[\sqrt{n}]}$ circuit. These lower bounds worked for both the Permanent as well as the Determinant. The lower bounds of [GKKS13a] were later improved to $2^{\Omega(\sqrt{n}\log n)}$ in a follow-up work of Kayal, Saha, Saptharishi [KSS13]. This was shown not for the Permanent or the Determinant, but for another explicit family of polynomials in VNP. These results were all the more remarkable in the light of the results of Koiran [Koi12] and Tavenas [Tav13] who had showed that $2^{\omega(\sqrt{n}\log n)}$ lower bounds for the class of homogeneous $\Sigma\Pi\Sigma\Pi^{[\sqrt{n}]}$ circuits would suffice to separate VP from VNP. Thus, any asymptotic improvement in the exponent, in either the upper bound on depth reduction or the lower bound of [KSS13] would separate VNP from VP. We formally state the result below.

Theorem 3.2 ([GKKS13a], [KSS13]): For every n, there is an explicit family of polynomials in VNP in $N = \theta(n^2)$ variables and with degree $\theta(n)$ such that any homogeneous $\Sigma\Pi\Sigma\Pi^{[t]}$ circuit computing it must have top fan-in at least $\exp(\Omega(\frac{n}{t}\log N))$.

Both papers [GKKS13a], [KSS13] used the notion of the dimension of *shifted partial derivatives* as a complexity measure, an augmentation of the Nisan-Wigderson complexity measure of dimension of partial derivatives in order to prove the lower bounds. We define this complexity measure below.

Shifted Partial Derivatives: This was a complexity measure introduced in [Kay12] and used in [GKKS13a], [KSS13] as well as in several later works. For a field $\mathbb{F}$, an n variate polynomial $P \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ and a positive integer k, we denote by $\partial^{=k}P$, the set of all partial derivatives of order equal to k of P. For a polynomial P and a monomial m, we denote by $\partial_m P$ the partial derivative of P with respect to m.

Definition 3.3 ([GKKS13a]): For an n variate polynomial $P \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ and integers $k, \ell \geq 0$, the space of ℓ shifted k^{th} order partial derivatives of P is defined as

$$\langle \partial^{=k} P \rangle_{\leq \ell} \stackrel{def}{=} \mathbb{F} - span\{\prod_{i\in[n]} {x_i}^{j_i} \cdot g : \sum_{i\in[n]} j_i \leq \ell, g \in \partial^{=k}P\}$$

The [KSS13] result was shown to hold a family of polynomials in VNP called the Nisan-Wigderson polynomials (named after Nisan-Wigderson designs). We do not know if this result holds also the Permanent. Since this class of polynomials (and mild variants of it) has subsequently been useful for several of the lower bounds proofs, we define them below.

Nisan-Wigderson Polynomials: For a prime power n, let $\mathbb{F}_n$ be a field of size n. For the set of n^2 variables $\{x_{i,j} : i, j \in [n]\}$ and $d \in [n]$, we define the degree n homogeneous polynomial $NW_{d,n}$ as

$$NW_{d,n} = \sum_{\substack{f(z)\in\mathbb{F}_n[z] \\ deg(f)<d}} \prod_{i\in[n]} x_{i,f(i)}$$

C. Lower bounds for VP *and tightness of depth reduction results*

The depth reduction results combined with the lower bounds for homogeneous $\Sigma\Pi\Sigma\Pi^{[t]}$ circuits is indeed a remarkable collection of results. As it stands, in order to separate VP from VNP, any small asymptotic improvement in the exponent on either the lower bound front or on the depth reduction front would be sufficient. In fact for any class of circuits $\mathcal{C}$ for which we can improve the depth reduction parameters of Theorem 3.1, we would get superpolynomial lower bounds for that class using Theorem 3.2.

Unfortunately, it seems that in general, we cannot hope for a better depth reduction. In a recent work, Fournier, Limaye, Malod and Srinivasan [FLMS13] gave an example of an explicit polynomial in VP (of degree n and in $N = n^{O(1)}$ variables) such that any homogeneous $\Sigma\Pi\Sigma\Pi^{[t]}$ circuit computing it must have top fan-in at least $\exp(\Omega(\frac{n}{t}\log N))$. This immediately implies that the depth reduction parameters in the result of Tavenas [Tav13] are *tight* for circuits.

Theorem 3.4 ([FLMS13]): For every n, there is an explicit family of polynomials in VP in $N = \mathsf{poly}(n)$ variables and with degree n such that any homogeneous $\Sigma\Pi\Sigma\Pi^{[t]}$ circuit computing it must have top fan-in at least $\exp(\Omega(\frac{n}{t}\log N))$.

The above result, along with the fact that the hard polynomial used by Kayal et al [KSS13] has a shifted partial derivative span only a polynomial factor away from the maximum possible value suggests that the technique of improving depth reduction and then using shifted partial derivatives may not be strong enough to separate VNP from VP[2]. In a recent result, Chillara and Mukhopadhyay [CM13] gave a clean unified way of way of lower bounding the shifted partial derivative complexities of the polynomials considered by [KSS13], [FLMS13].

D. Lower bounds for homogeneous formulas

Even though it is not possible to separate VNP from VP by just improving the depth reduction of Koiran and Tavenas, it was conceivable that it could lead to superpolynomial lower bounds for other interesting classes of circuits, for instance homogeneous arithmetic formulas, or even general arithmetic formulas. This hope was further strengthened when Kayal et al [KSS13] used these precise ideas to prove superpolynomial lower bounds for a restricted class of formulas which they called *regular* formulas (defined below).

Definition 3.5: A formula computing a degree d polynomial in n variables is said to be regular, if it satisfies the following conditions:

1) It has alternating layers of sum and product gates.
2) All gates in a single layer have the same fan-in.
3) The formal degree of the formula is at most some constant multiple of the degree of the polynomial being computed.

Kayal et al [KSS13] proved their result by showing that one can reduce any polynomial size regular formula to a $\Sigma\Pi\Sigma\Pi^{[t]}$ circuit (for a carefully chosen choice of t) of size asymptotically better in the exponent than the $\exp(\frac{n}{t}\log N)$ bound (which as we just discussed is known to be tight for circuits). This improvement in depth reduction immediately leads to superpolynomial lower bounds for regular formulas by using Theorem 3.2.

[2]The reason this statement is not completely formal is that we still do not know know if the upper bounds on the shifted partial derivative measure for $\Sigma\Pi\Sigma\Pi^{[t]}$ circuits is tight for all choices of derivatives and shifts, though the results of [FLMS13] and this paper show that they are indeed tight for many of the choices.

Removing the restriction on regularity and proving superpolynomial lower bounds for general formulas or even general homogeneous formulas would be a major step forward - it would be perhaps the most natural class of arithmetic circuits for which we would be able to prove lower bounds. The authors of the two papers [KSS13], [FLMS13] left open the question whether formulas (or even homogeneous formulas) can have better depth reduction than circuits (such as is true for regular formulas). If true, this would imply superpolynomial lower bounds for (homogeneous) formulas.

Unfortunately, Kumar and Saraf [KS13a] showed that this was false. The same exponential ($n^{\Omega(\sqrt{n})}$) lower bounds were also shown to hold for very simple polynomial sized formulas of just depth 4 (if one requires them to be computed by homogeneous $\Sigma\Pi\Sigma\Pi^{[\sqrt{n}]}$ circuits).

Theorem 3.6 ([KS13a]): For every n, there is an explicit family of polynomials computed by $\mathsf{poly}(n)$ sized depth 4 formulas in $N = \mathsf{poly}(n)$ variables and with degree n, such that any homogeneous $\Sigma\Pi\Sigma\Pi^{[t]}$ circuit computing it must have top fan-in at least $\exp(\Omega(\frac{n}{t}\log N))$.

Thus on the one hand these results give us extremely strong lower bounds for an interesting class of depth 4 homogeneous circuits. On the other hand, since these lower bounds also hold for polynomials in VP and for homogeneous formulas [FLMS13], [KS13a], it follows that the depth reduction results of Koiran [Koi12] and Tavenas [Tav13] to the class of homogeneous $\Sigma\Pi\Sigma\Pi^{[\sqrt{n}]}$ circuits are tight and cannot be improved even for homogeneous formulas.

E. Lower bounds for general homogeneous $\Sigma\Pi\Sigma\Pi$ circuits

Although these results discussed earlier represent a lot of exciting progress on the problem of proving lower bounds for homogeneous $\Sigma\Pi\Sigma\Pi^{[\sqrt{n}]}$ circuits, and these results seemed possibly to be on the brink of proving lower bounds for general arithmetic circuits, they still seem to give almost no nontrivial results for general homogeneous depth 4 circuits with no bound on bottom fanin (homogeneous $\Sigma\Pi\Sigma\Pi$ circuits). In addition, it was shown in [KS13a] that general homogeneous $\Sigma\Pi\Sigma\Pi$ circuits are exponentially more powerful than homogeneous $\Sigma\Pi\Sigma\Pi^{[\sqrt{n}]}$ circuits[3].

Recently, the first super-polynomial lower bounds for general homogeneous depth 4 ($\Sigma\Pi\Sigma\Pi$) circuits were proved independently by Kumar and Saraf [KS13b] who

[3]It was demonstrated that even very simple homogeneous $\Sigma\Pi\Sigma\Pi$ circuits of polynomial size might need $n^{\Omega(\sqrt{n})}$ sized homogeneous $\Sigma\Pi\Sigma\Pi^{[\sqrt{n}]}$ circuits to compute the same polynomial.

showed a lower bound of $n^{\Omega(\log \log n)}$ for a polynomial in VNP and Limaye, Saha and Srinivasan [LSS14], who showed a lower bound of $n^{\Omega(\log n)}$ for a polynomial in VP. Subsequently, Kayal, Limaye, Saha and Srinivasan greatly improved these lower bounds to obtain exponential ($2^{\Omega(\sqrt{n}\log n)}$) lower bounds for a polynomial in VNP (over fields of characteristic zero). Notice that this result also extends the results of [GKKS13a] and [KSS13] who proved similar exponential lower bounds for the more restricted class of homogeneous $\Sigma\Pi\Sigma\Pi^{[\sqrt{n}]}$ circuits. The result by [KLSS14] shows the same lower bound without the restriction of bottom fanin. Again, any asymptotic improvement of this lower bound in the exponent would separate VP from VNP.

Theorem 3.7 ([KLSS14]): Let $\mathbb{F}$ be any field of characteristic zero. There exists an explicit family of polynomials (over $\mathbb{F}$) of degree n and in $N = n^{O(1)}$ variables in VNP, such that any homogeneous $\Sigma\Pi\Sigma\Pi$ circuit computing it has size at least $n^{\Omega(\sqrt{n})}$.

This class of results represents an important step forward, since homogeneous depth 4 circuits seem a much more natural class of circuits than homogeneous depth 4 circuits with bounded bottom fanin. Given the new lower bounds for the more natural class of depth 4 homogeneous circuits (with no restriction on bottom fanin), and especially the exponential lower bounds of [KLSS14], the most obvious question that arises is the following: If one relaxes away the requirement of bounded bottom fanin, i.e. all one requires is to reduce to the class of general depth 4 homogeneous circuits, can one improve upon the upper bounds obtained by Koiran and Tavenas? If we could do this over the reals/complex numbers, then given the [KLSS14] result, this would also suffice in separating VP from VNP!

Very recently, Kumar and Saraf [KS14] built upon and extended the results of [KLSS14] to hold over all fields. They achieved this via giving a new and more combinatorial proof of the result of [KLSS14] that is not dependent on the underlying field. The combinatorial nature of the proof also allowed for much more flexibility and thus enabled them to prove the same lower bounds even for a polynomial in VP.

Theorem 3.8 ([KS14]): Let $\mathbb{F}$ be any field. There exists an explicit family of polynomials (over $\mathbb{F}$) of degree n and in $N = n^{O(1)}$ variables in VP, such that any homogeneous $\Sigma\Pi\Sigma\Pi$ circuit computing it has size at least $n^{\Omega(\sqrt{n})}$.

As an immediate corollary of the result above, we get that the depth reduction results of Koiran [Koi12] and Tavenas [Tav13] are tight even when one wants to depth reduce to the class of general homogeneous depth 4 circuits (without the constraint on bounded bottom fanin).

It is interesting to note that all the results above proving lower bounds for general homogeneous depth 4 circuits do not use the complexity of shifted partial derivatives as is, but use variations of them. Thus it feels that this method is still far from fully understood, and it might lead to other new and interesting lower bounds in the near future.

IV. Discussion and future directions

- The results on depth reduction and lower bounds suggest a natural approach toward proving lower bounds for arithmetic circuits. In order to prove superpolynomial lower bounds for a class of circuits $\mathcal{C}$, it would suffice to show that any polynomial in $\mathcal{C}$ can be computed by a $n^{o(\sqrt{n})}$ sized homogeneous depth 4 circuit. It has not been ruled out that such a statement might be true even for homogeneous formulas or even general formulas.
- Another very interesting direction of work is to prove strengthened lower bounds for circuits of larger depth. At the moment we do not even know quadratic lower bounds for homogeneous depth 5 circuits.
- Strong lower bounds for general (not necessarily homogeneous) circuits of depth 3 over general fields would also be extremely interesting. Even more so given the depth reduction results of Gupta et al [GKKS13b] to nonhomogeneous circuits of just depth 3.
- Although lower bounds for arithmetic circuits are intimately connected to the problem of derandomizing polynomial identity testing, none of these new techniques have as yet been useful for derandomizing polynomial identity testing. We still are only able to derandomize PIT for very restricted classes of depth 3 circuits. It would be extremely interesting to know if these techniques shed any light on PIT for bounded depth circuits.
- The results of [KS13b], [LSS14], [KLSS14], [KS14] all use variants of the method of shifted partial derivates to obtain the lower bounds. All the variants are able to give nontrivial results that are not believed to be provable using shifted partial derivatives alone. This suggests that we do not really fully understand the potential of these methods, and perhaps they can be used to give even much stronger lower bounds for even richer classes of circuits.

- It seems extremely worthwhile to develop and fully understand these methods. - to understand how general a class of lower bounds they can prove as well as to understand if there any any limitations to these methods. For instance it would be very interesting to understand if these methods have the potential of separating VP from VNP, or if there is some inherent underlying reason that suggests we might need different techniques.

REFERENCES

[AV08] M. Agrawal and V. Vinay. Arithmetic circuits: A chasm at depth four. In *Proceedings of the 49th Annual FOCS*, pages 67–75, 2008.

[BS83] Walter Baur and Volker Strassen. The complexity of partial derivatives. *Theoretical Computer Science*, 22:317–330, 1983.

[CM13] Suryajith Chillara and Partha Mukhopadhyay. Depth-4 lower bounds, determinantal complexity: A unified approach. *CoRR*, abs/1308.1640v3, 2013.

[FLMS13] Hervé Fournier, Nutan Limaye, Guillaume Malod, and Srikanth Srinivasan. Lower bounds for depth 4 formulas computing iterated matrix multiplication. *Electronic Colloquium on Computational Complexity (ECCC)*, 20:100, 2013.

[GK98] Dima Grigoriev and Marek Karpinski. An exponential lower bound for depth 3 arithmetic circuits. In *Proceedings of the Thirtieth Annual ACM Symposium on the Theory of Computing (STOC)*, pages 577–582, 1998.

[GKKS13a] A. Gupta, P. Kamath, N. Kayal, and R. Saptharishi. Approaching the chasm at depth four. *In Proceedings of CCC*, 2013.

[GKKS13b] Ankit Gupta, Pritish Kamath, Neeraj Kayal, and Ramprasad Saptharishi. Arithmetic circuits: A chasm at depth three. *Electronic Colloquium on Computational Complexity (ECCC)*, 20:26, 2013.

[GR98] D. Grigoriev and A. Razborov. Exponential complexity lower bounds for depth 3 arithmetic circuits in algebras of functions over finite fields. In *Proceedings of the 39th Annual Symposium on Foundations of Computer Science*, FOCS '98, pages 269–278, 1998.

[Kay12] Neeraj Kayal. An exponential lower bound for the sum of powers of bounded degree polynomials. *Electronic Colloquium on Computational Complexity (ECCC)*, 19:81, 2012.

[KI04] V. Kabanets and R. Impagliazzo. Derandomizing polynomial identity tests means proving circuit lower bounds. *Computational Complexity*, 13(1-2):1–46, 2004.

[KLSS14] Neeraj Kayal, Nutan Limaye, Chandan Saha, and Srikanth Srinivasan. An exponential lower bound for homogeneous depth four arithmetic formulas. *Electronic Colloquium on Computational Complexity (ECCC)*, 2014.

[Koi12] P. Koiran. Arithmetic circuits: The chasm at depth four gets wider. *Theoretical Computer Science*, 448:56–65, 2012.

[KS13a] Mrinal Kumar and Shubhangi Saraf. The limits of depth reduction for arithmetic formulas: It's all about the top fan-in. *Electronic Colloquium on Computational Complexity (ECCC)*, 20:153, 2013.

[KS13b] Mrinal Kumar and Shubhangi Saraf. Superpolynomial lower bounds for general homogeneous depth 4 arithmetic circuits. *CoRR*, abs/1312.5978, 2013.

[KS14] Mrinal Kumar and Shubhangi Saraf. On the power of homogeneous depth 4 circuits. *Electronic Colloquium on Computational Complexity (ECCC)*, 2014.

[KSS13] Neeraj Kayal, Chandan Saha, and Ramprasad Saptharishi. A super-polynomial lower bound for regular arithmetic formulas. *Electronic Colloquium on Computational Complexity (ECCC)*, 20:91, 2013.

[LSS14] Nutan Limaye, Chandan Saha, and Srikanth Srinivasan. Super-polynomial lower bounds for depth-4 homogeneous arithmetic formulas. *To appear in STOC' 14*, 2014.

[NW95] N. Nisan and A. Wigderson. Lower bounds on arithmetic circuits via partial derivatives. In *Proceedings of the 36th Annual FOCS*, pages 16–25, 1995.

[Raz10] Ran Raz. Elusive functions and lower bounds for arithmetic circuits. *Theory of Computing*, 6(1):135–177, 2010.

[Str73] V. Strassen. Die berechnungskomplexiät von elementarsymmetrischen funktionen und von interpolationskoeffizienten. *Numer. Math*, 20:238–251, 1973.

[SW01] A. Shpilka and A. Wigderson. Depth-3 arithmetic circuits over fields of characteristic zero. *Computational Complexity*, 10(1):1–27, 2001.

[Tav13] Sébastien Tavenas. Improved bounds for reduction to depth 4 and depth 3. In *MFCS*, pages 813–824, 2013.

[Val79] L. G. Valiant. Completeness classes in algebra. In *Proceedings of the 11th Annual STOC*, STOC '79, pages 249–261, New York, NY, USA, 1979. ACM.

[VSBR83] Leslie G. Valiant, Sven Skyum, S. Berkowitz, and Charles Rackoff. Fast parallel computation of polynomials using few processors. *SIAM Journal of Computation*, 12(4):641–644, 1983.