Homework-2 (Graph Theory)

Due date: March 22
Collaboration is encouraged. However the writeup should be your own.

1. Given a graph $G$, show that one can order its vertices so that the greedy algorithm uses exactly $k = \chi(G)$ colors to vertex color $G$.

2. For each $k \geq 3$, find a bipartite graph with vertices $x_1, x_2, \ldots, x_n$ for which the greedy algorithm with that vertex ordering uses $k$ colors. Show that this can be done for $n = 2k - 2$. Show that it cannot be done for $n = 2k - 3$.

3. Show that a planar map can be 2-face-colored if and only if every vertex of the underlying planar graph has even degree.

4. Let $G$ be a bipartite graph of maximum degree $\Delta$. Show that the chromatic index (i.e. the minimum number of colors needed to properly color the edges) of $G$ is $\Delta$. (Hint: embed $G$ into a $D$-regular bipartite graph.)

5. Let $K_n$ denote the complete graph on $n$ vertices.
   
   (a) Show that when $n$ is odd, the chromatic index of $K_n$ is $n$.
   
   (b) Show that when $n$ is even, the chromatic index of $K_n$ is $n - 1$. (Hint: delete one vertex and use the previous part to color the edges. Then add back the deleted vertex and color the new edges.)

6. Let $v_1, v_2, \ldots, v_{3n}$ be vectors in Euclidean plane such that the Euclidean distance between every pair of distinct points is at most 1. Prove that at most $3n^2$ of the distances can exceed $1/\sqrt{2}$.

7. The upper density $ud(G)$ of an infinite graph $G$ is define to be the supremum of the densities of its large finite subgraphs, i.e.

   $$ud(G) = \lim_{n \to \infty} \sup \left\{ \frac{e(H)}{\binom{|H|}{2}} | H \subset G, n \leq |H| < \infty \right\}.$$ 

Show that for every $G$,

   $$ud(G) \in \{ 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1 - \frac{1}{r}, \ldots, 1 \}.$$