SEMINARS ON RENORMALISATION THEORY

VOLUME I: Lectures on Analytic Renormalisation*

by

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FORWARD

In the summer of 1972 Seminars on renormalization theory were held at the University of Maryland under the auspices of the Center for Theoretical Physics. The aim was to study rigorous renormalization theory and, in this respect, the topics of analytic renormalization and additive renormalization in momentum space were selected out. The present volume consists of the lectures of E. R. Speer on analytic renormalization. Volume 2 will consist of lectures on additive renormalization and normal product methods by John Lowenstein.

I wish to thank John Lowenstein and Gene Speer for their cooperation, my colleagues, especially Dan Fivel, for their support and Sheila Rodriguez for her help in the organization.

P. K. Mitter

Analytic Renormalization

I. Introduction

If we study (in perturbation theory) a field theory governed by a Lagrangian $L(x) = L_o(x) + gL_I(x)$, where L_o is the free Lagrangian of the fields involved and gL_I is the interaction Lagrangian, then the S-matrix is given by

$$S = T(e^{ig\int d^4(x) L_I(x)}). \qquad (1)$$

T is the operation of time-ordering, and (1) is to be understood as a formal power series in the coupling constant g. If, for example, $L(x) = - :\phi^4(x):$, where ϕ is a hermitian scalar field, then

$$S = 1 + (-ig) \int :\phi^4(x):dx + \frac{(-ig)^2}{2} \iint T[:\phi^4(x_1): :\phi^4(x_2):]dx_1 dx_2 + \cdots + \frac{(-ig)^n}{n!} \int dx_1 \cdots \int dx_n T[:\phi^4(x_1):\cdots] + \cdots \qquad (2)$$

The time ordered products in (2), however, are not well defined. To see this, we may expand them using Wick's theorem, so that the third term becomes, for example,

$$T[:\phi^4(x_1)::\phi^4(x_2):] = :\phi^4(x_1)\phi^4(x_2): + 16:\phi^3(x_1)\phi^3(x_2): \times <0|T[\phi(x_1)\phi(x_2)]|0> + 72:\phi^2(x_1)\phi^2(x_2):<0|T[\phi(x_1)\phi(x_2)]|0>^2 + 96:\phi(x_1)\phi(x_2): <0|T[\phi(x_1)\phi(x_2)]|0>^3 + 24<0|T[\phi(x_1)\phi(x_2)]|0>^4.$$

Unfortunately, the propagator

$$\Delta_F(x_1-x_2) = \langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle$$

is actually a distribution with singularities on the light cone $(x_1-x_2)^2=0$, and its powers (higher than the first) are not well defined.

In higher orders of the preturbation expansion we deal not only with powers of the propagator, but also with other products of propagators. To each term in the Wick expansion of

$$T[:\phi^4(x_1): :\phi^4(x_2): \cdots :\phi^4(x_n):]$$

there is associated a Feynman graph $G(\mathcal{V},\mathcal{L})$. Here $\mathcal{V} = \{V_1, \cdots V_n\}$ is the set of vertices of G; $\mathcal{L} = \{1, \cdots L\}$ the set of lines. In an arbitrary fashion we call one end of each line ℓ the initial vertex V_{i_ℓ}, the other the final vertex V_{f_ℓ}; the incidence matrix

$$e_i^\ell = \begin{cases} 1 & , \text{ if } i = f_\ell \\ -1 & , \text{ if } i = i_\ell \\ 0 & , \text{ otherwise} \end{cases}$$

then completely describes the graph. The coefficient of the corresponding term in the Wick expansion is formally

$$\prod_{\ell\in\mathcal{L}} \Delta_F(x_{f_\ell}-x_{i_\ell}) = \prod_{\ell\in\mathcal{L}} \Delta_F(e_i^\ell\, x_i). \qquad (3)$$

(Here we are using the summation convention.)

The problem of renormalization is to appropriately define (3) as a tempered distribution $\mathcal{T}(x_1, \cdots x_n) = \mathcal{T}(\underline{x})$. A theorem of Bogoliubov

[1] then assures us that the term in the Wick expansion,

$$T(\underline{x}):\phi^{\alpha_1}(x_1)\cdots\phi^{\alpha_n}(x_n): \qquad (4)$$

is well defined as an operator valued distribution. (There does remain the problem of integrating (4) over all $\underline{x}$, as required by (2). We will return to this later.)

In these lectures we will study in detail one method of defining (3), called analytic renormalization [2,3]. We will then show the relation of this method to the usual subtractive schemes for renormalization, and to the axiomatic treatment given by Hepp [4]. For simplicity we will deal primarily with spinless particles, but will occasionally point out the modifications introduced by spin.

II. Analytic Renormalization

A. Analytic Regularization.

The propagator in momentum space is

$$\tilde{\Delta}_F(p) \equiv (2\pi)^{-2} \int e^{-ip\cdot x} \Delta_F(x)\, dx$$

$$= -i\ (m^2-p^2-i0)^{-1},$$

where m is the mass of the scalar field. We define a generalized propagator

$$\tilde{\Delta}_F(\lambda,p) = -i\ (m^2- p^2 - io)^{-\lambda}, \tag{2.1}$$

with λ a new complex variable, so that $\tilde{\Delta}_F(p) = \tilde{\Delta}_F(1,p)$. We will see that, if Re λ is sufficiently large, we may multiply the generalized propagators at will. The resulting analytic function of λ may be analytically continued to the physical point $\lambda = 1$, where it has a singularity. Removal of this singularity is analytic renormalization.

The distribution (2.1) is well defined according to

Theorem 1: Let Q^1, Q^2 be real quadratic forms on $\mathbb{R}^k$, with Q^2 positive semidefinite. Write $Q = Q^1 + iQ^2$. Let c be a complex variable with Re $c \geq a > 0$, Im $c \leq 0$ (for some a). For Re $\nu > 0$ define the function $f(Q,c,\nu)$ on $\mathbb{R}^n$ by

$$f(Q,c,\nu)\ (y) = (c - \sum_{i,j} y_i Q_{ij} y_j)^\nu,$$

where, for Im $z \leq 0$, $z^\nu = \exp\{\nu[\ln|z| + i \arg z]\}$, with $-\pi \leq \arg z < 0$. Then $f(Q,c,\nu)$ may be analytically continued as a tempered distribution to all values of ν; the resulting distribution is holomorphic in ν and continuous in Q and c.

Proof: Note first that, for Im $c < 0$ or for Q^2 strictly positive definite, $(c-y_i Q_{ij} y_i)$ cannot vanish, so that $f(Q,c,\nu)$ is defined and analytic for all ν. For general c,Q, f is differentiable (in y) for Re $\nu > 1$ and satisfies

$$f(Q,c,\nu) = \frac{1}{c}\left[1 - \frac{1}{2(\nu+1)} \sum_i y_i \frac{d}{dy_i}\right] f(Q,c,\nu+1). \qquad (2.2)$$

If f is smeared with a test function the derivatives in (2.2) can be transferred to the test function by integration by parts (2.2) then gives an extension of the region of definition of f to Re $\nu > -1$, or, by iteration, to all ν. The distribution defined in this manner has apparent poles at $\nu = -1, -2, \cdots$, due to the factor $\frac{1}{\nu+1}$ in (2.2). But $f(Q,c,\nu)$ is continuous in Q, c for Re $\nu > 0$ and hence, using (2.2), for all $\nu \neq -1,-2, \cdots$. Since the poles are not present for Im $c < 0$ we may choose a contour C circling $\nu = -k$ and write, for ν' inside the contour,

$$f(Q,c+i\varepsilon,\nu') = \oint_C \frac{f(Q,c+i\varepsilon,\nu)}{\nu-\nu'} d\nu; \qquad (2.3)$$

The $\varepsilon \to 0$ limit of (2.3) shows that the poles are not present for any values of Q,c; q.e.d.

$\tilde{\Delta}_F(\lambda,p)$ in (2.1) is obtained from Theorem 1 by taking $k = 4$, $c = m^2$, $Q_{\mu\nu} = g_{\mu\nu}$. To assure convergence of certain integrals it will be convenient to consider

$$\tilde{\Delta}_{F,\eta}(\lambda,p) = -i(m^2 - i\eta - p\,Qp)^{-\lambda},$$

where $Q_{\mu\nu} = Q_{\mu\nu}(\eta) = g_{\mu\nu} + i\eta\,\delta_{\mu\nu}$. By the continuity shown in Theorem 1,

$$\tilde{\Delta}_F(\underline{\lambda},p) = \lim_{\eta\to 0+} \Delta_{F,\eta}(\lambda,p).$$

To investigate multiplication of propagators we Fourier transform back to configuration space. From [5,3] we have

$$\Delta_{F,\eta} = F^{-1}\tilde{\Delta}_{F,\eta} = \int \frac{d^4p}{(2\pi)} e^{ip\cdot x}\,\tilde{\Delta}_{F,\eta}(p)$$

$$= \frac{(-i)2^{1-\lambda}b^{2-\lambda}}{\Gamma(\lambda)\sqrt{|Q|}}\left(\sqrt{-xQ^{-1}x}\right)^{\lambda-2} K_{\lambda-2}\left(b\sqrt{-xQ^{-1}x}\right).$$

Here $K_{\lambda-2}$ is the "Bessel function of imaginary argument", $b = m^2 - i\eta$, with Re $b > 0$, and $0 \leq \arg\sqrt{-xQ^{-1}x} \leq \frac{\pi}{2}$; $\sqrt{|Q|}$ is defined so that $\lim_{\eta\to 0}\sqrt{|Q|} = -i$.

Thus

$$\Delta_F = \lim_{\eta\to 0}\Delta_{F,\eta} = \frac{2^{1-\lambda}\, m^{2-\lambda}}{\Gamma(\lambda)}\left(\sqrt{-x^2+i0}\right)^{\lambda-2} K_{\lambda-2}\left(m\sqrt{-x^2+i0}\right).$$

<u>Lemma 1</u>: For Re $\lambda > m+2$, Δ_F and $\Delta_{F,\eta}$ are in C^m (m-times continously differentiable functions).

<u>Proof</u>: $K_\nu(z)$ has a power series of the form

$$K_\nu(z) = z^{-\nu}\left(\sum_0^\infty A_i z^{2i}\right) + z^{\nu}\left(\sum_0^\infty B_i z^{2i}\right)$$

and therefore,

$$\Delta_F = \frac{2^{1-\lambda}}{\Gamma(\lambda)}\{(m^2)^{2-\lambda}\,\Sigma A_i(m^2x^2)^i + (-x^2+i0)^{\lambda-2}\,\Sigma B_i(m^2x^2)^i\}.$$

For Re$(\lambda-2) > m$ this may be continuously differentiated m times. The argument for $\Delta_{F,\eta}$ is the same.

<u>Definition 1:</u> For Re $\lambda > 2$ the generalized Feynman amplitude for the graph $G(V,L)$ is

$$T(\underline{\lambda},\underline{x}) = \prod_L \Delta_F(\lambda_\ell, e_i^\ell x_i) = \lim_{\eta\to 0} \prod_L \Delta_{F,\eta}(\lambda_\ell, e_i^\ell x_i).$$

It is important to note that we introduce a separate λ_ℓ parameter for each line. We also remark that, if we were dealing with particles having spin, the propagator becomes

$$Z(\frac{d}{dx})\ \Delta_F(x),$$

with Z a polynomial. By Lemma 1, the generalized propagators

$$Z_\ell(\frac{d}{dx})\ \Delta_F(\lambda_\ell, x)$$

may also be multiplied freely if Re $\lambda_\ell > 2 + \deg Z_\ell$.

We now prceed to calculate T as an integral over Feynman parameters. At first we will keep $\eta > 0$; this will make all integrals involved absolutely convergent.

Using the integral representation [1]

$$\tilde{\Delta}^{\lambda}_{F,\eta} = \frac{-i}{(m^2-i\eta-pQp)^\lambda} = \frac{-ie^{\lambda\pi i}}{\Gamma(\lambda)} \int_0^\infty d\alpha\ \alpha^{\lambda-1}\ e^{i\alpha(pQp-m^2+i\eta)}, \qquad (2.4)$$

we may calculate the Fourier transform $\Delta_{F,\eta}$ from the formula

$$\int_{\mathbb{R}^m} dy \exp i\ [y\ \mathcal{R}y + y\cdot B] = \frac{\pi^{m/2}}{[\det(-i\mathcal{R})]^{1/2}} \exp\left(-\frac{i}{4} B\ \mathcal{R}^{-1}\ B\right) \qquad (2.5)$$

[In (2.5) $\mathcal{R}$ is a quadratic form with positive definite imaginary part. (2.5) is first proved for $\mathcal{R} = i\ \mathcal{P}$, $\mathcal{P} > 0$, by diagonalization of $\mathcal{P}$; in this case the square root of the determinant is positive. For general R the formula is established, and the correct sign of $[\det(-iR)]^{1/2}$ determined, by analytic continuation in R.] Now

$$\Delta_{F,\eta}(\lambda,x) = \frac{1}{(2\pi)^2}\int e^{ip\cdot x}\,\tilde{\Delta}_{F,\eta}(\lambda,p). \tag{2.6}$$

If we insert (2.4) into (2.6), the order of integration may be interchanged (for $\eta > 0$).

[To check this we verify absolute convergence of the double integral:

$$\int d^4p \int d\alpha\, \alpha^{\lambda-1}\left|e^{i\alpha(pQp-m^2+i\eta)}\right|$$

$$= \int d^4p \int d\alpha\, \alpha^{\lambda-1}\, e^{-\alpha\,\eta[||p||^2+1]}$$

$$= \frac{\Gamma(\lambda)}{\eta^\lambda}\int \frac{d^4p}{(||p^2||+1)^\lambda} < \infty\,.]$$

Thus,

$$\Delta_{F,\eta} = \frac{-i\,e^{\frac{\lambda\pi i}{2}}}{(2\pi)^2\Gamma(\lambda)}\int_0^\infty d\alpha\,\alpha^{\lambda-1}\int d^4p\; e^{i[\alpha(pQp-m^2+i\eta+p\cdot x]}.$$

Applying (2.5) with $R = i\alpha Q$ gives

$$\Delta_{F,\eta} = \frac{i\,e^{\lambda\pi i/2}}{4\Gamma(\lambda)\sqrt{|Q|}}\int_0^\infty \alpha^{\lambda-3}\,d\alpha\,\exp[-\tfrac{i}{4}\,x\,\frac{Q^{-1}}{\alpha}\,x - i(m^2-i\eta)],$$

since the analytic continuation procedure gives $[\det(-iR)]^{1/2} = -\alpha^2\sqrt{|Q|}$, with $\sqrt{|Q|}$ defined as above.

Therefore

$$\mathcal{T}_\eta(\underline{\lambda},\underline{x}) = \prod_L \left[\frac{ie^{\lambda\pi i/2}}{4\Gamma(\lambda)\sqrt{|Q|}}\right] \int_0^\infty \cdots \int_0^\infty \Pi\, \alpha_\ell^{\lambda_\ell - 3}\, d\alpha_\ell$$

$$\exp i \left[-\frac{1}{4} \sum_{i,j,\ell} x_i e_i^\ell \frac{Q^{-1}}{\alpha_\ell} e_j^\ell x_j - (m^2 - i\eta)\Sigma\alpha_\ell\right]. \qquad (2.7)$$

Writing $A_{ij} = \sum_\ell \frac{e_{\ell i} e_{\ell j}}{\alpha_\ell}$, the exponential in (2.7) becomes

$$\exp i\left[-\frac{1}{4} x(A \otimes Q^{-1})x - (m^2 - i\eta)\Sigma\alpha_\ell\right]. \qquad (2.8)$$

Now note that $\sum_i e_i^\ell = e_{f_\ell}^\ell + e_{i_\ell}^\ell = 0$, hence $\sum_i A_{ij} = 0$. From this, for any $k\varepsilon\{1,\cdots n\}$.

$$\otimes(A \times Q^{-1})x = \sum_{i,j\neq k} (x_i - x_k)(A_{ij}\, Q^{-1})(x_j - x_k).$$

Let A' be the matrix A with k'th row and column deleted, and let $\xi_i = x_i - x_k$ $(i \neq k)$, so that (2.8) becomes

$$\exp i\left[-\frac{1}{4} \xi(A' \otimes Q^{-1})\xi - (m^2 - i\eta)\Sigma\alpha_\ell\right]. \qquad (2.9)$$

We now take the Fourier transform of (2.7):

$$\tilde{\mathcal{T}}_\eta(\underline{\lambda},\underline{p}) = \frac{1}{(2\pi)^{2n}} \int d\xi_i\, dx_k\, \mathcal{T}_\eta(\underline{\lambda},\xi)\, e^{-i\left[\sum_{i\neq k} p_i\xi_i + x_k \sum_1^n p_i\right]}. \qquad (2.10)$$

The x_k integration yields the δ-function expressing conservation of momentum. If we insert (2.7) and (2.9) into (2.10), the double integral over α and ξ is absolutely convergent whenever the graph G is connected.

[Proof: If the integrand in (2.7) is replaced by its absolute value, the integral retains a product structure: $\prod_L F(|| e_i^\ell x_i||^2)$, with F(t) exponentially decreasing for large t. By connectivity, the product is exponentially decreasing as a function of ξ, hence the ξ integration is convergent.] Thus

$$\tilde{T}_\eta(\underline{\lambda},\underline{p}) = \prod_L \left(\frac{i\, e^{\lambda\pi i/2}}{4\sqrt{|Q|}\ \Gamma(\lambda\ )}\right) (2\pi)^2\delta(\Sigma pi) \int\cdots\int \Pi\alpha_\ell^{\lambda_\ell -3}$$

$$\int \frac{d\xi}{(2\pi)^{2(n-1)}} \exp i[-\tfrac{1}{4}\,\xi(A' \otimes Q^{-1})\xi-\xi\cdot p-(m^2-i\eta)\Sigma\alpha_\ell]. \qquad (2.11)$$

The ξ integral is done by another application of (2.5), taking $R = -\frac{1}{4} A' \otimes Q^{-1}$. Then

$$[\det(-iR)]^{1/2} = \frac{(\det A')^2\ (-\sqrt{|Q|})^{n-1}}{4^{2(n-1)}},$$

and $R^{-1} = 4A'^{-1} \otimes Q$. Since $\det A' = A\binom{k}{k}$, and

$$(A'^{-1})_{ij} = \frac{A'\binom{i}{j}}{|A'|} = \frac{A\binom{k\ i}{k\ j}}{A\binom{k}{k}},$$

we finally have

$$\tilde{T}_\eta(\underline{\lambda}, p) = \prod_L \left(\frac{i\, e^{\lambda\pi i/2}}{4\sqrt{|Q|}\ \Gamma(\lambda_\ell)}\right) (4\sqrt{|Q|})^{n-1}\, (2\pi)^2\, \delta\,(\Sigma p_i) \qquad (2.12)$$

$$\int\cdots\int \frac{\Pi\alpha_\ell^{\lambda_\ell -1} d\alpha_\ell}{d(\alpha)^2} \exp i \left[\sum_{i,j\neq k} \frac{P_i D_{ij}^k Q P_j}{d} - (m^2 - i\eta)\Sigma\alpha_\ell\right]$$

Here

$$d(\alpha) = (\prod_L \alpha_\ell)\, A\binom{k}{k}$$

$$D^k_{ij}(\alpha) = (\prod_L \alpha_\ell)\, A\binom{k\ i}{k\ j} \qquad (2.13)$$

(2.12) is our desired form of the generalized Feynman amplitude. As a preliminary to its analytic continuation we now turn to the evaluation of the Symanzik polynomials $d(\alpha)$, $D^k_{ij}(\alpha)$. We remark that it may be converted to the usual Feynman-parametric form by the change of variables $\alpha_\ell = t\,\beta_\ell$, $\Sigma\,\beta_\ell = 1$ followed by an explicit evaluation of the t integral. In this form the $\eta \to 0$ limit is easily taken using Theorem 1.

B. Symanzik Functions.

We wish to relate the functions $d(\alpha)$, $D^k_{ij}(\alpha)$ to the structure of the graph. We will first work out several examples. Note that

$$A_{ij} = \sum_\ell \frac{e_i^\ell\, e_j^\ell}{\alpha_\ell} = \begin{cases} -\displaystyle\sum_{\substack{\ell \text{ joining} \\ i,j}} \frac{1}{\alpha_\ell} & \text{if } i \neq j \\ -\displaystyle\sum_{\substack{\ell \text{ incident} \\ \text{on } i}} \frac{1}{\alpha_\ell} & \text{if } i = j \end{cases}$$

Example 1: V_1 

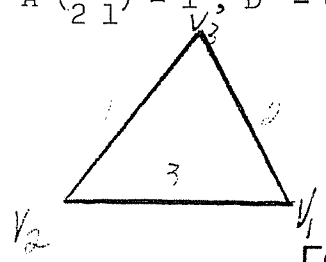
V_2. Take k = 2

$$A = \begin{bmatrix} \alpha_1^{-1} + \alpha_2^{-1} + \alpha_3^{-1} & -(\alpha_1^{-1} + \alpha_2^{-1} + \alpha_3^{-1}) \\ -(\alpha_1^{-1} + \alpha_2^{-1} + \alpha_3^{-1}) & (\alpha_1^{-1} + \alpha_2^{-1} + \alpha_3^{-1}) \end{bmatrix}$$

$$A\binom{2}{2} = \alpha_1^{-1} + \alpha_2^{-1} + \alpha_3^{-1} \; ; \; d(\alpha) = \alpha_2\alpha_3 + \alpha_1\alpha_3 + \alpha_1\alpha_2$$

$$A\binom{2\;1}{2\;1} = 1 \; ; \; D^2 = \alpha_1\alpha_2\alpha_3$$

<u>Example 2</u>:



Take $k = 3$:

$$A = \begin{bmatrix} \alpha_2^{-1} + \alpha_3^{-1} & -\alpha_3^{-1} & -\alpha_2^{-1} \\ -\alpha_3^{-1} & \alpha_1^{-1} + \alpha_3^{-1} & -\alpha_1^{-1} \\ -\alpha_2^{-1} & -\alpha_1^{-1} & \alpha_1^{-1} + \alpha_2^{-1} \end{bmatrix}$$

$$d(\alpha) = \alpha_1\alpha_2\alpha_3 \; A\binom{3}{3} = \alpha_1\alpha_2\alpha_3 \left[(\alpha_2^{-1} + \alpha_3^{-1})(\alpha_1^{-1} + \alpha_3^{-1}) - \alpha_3^{-2} \right]$$

$$= \alpha_1 + \alpha_2 + \alpha_3$$

$$D^3_{11} = \alpha_2\alpha_3 + \alpha_1\alpha_2 \; ; \quad D^3_{12} = \alpha_1\alpha_2 \; ; \quad D^3_{22} = \alpha_1\alpha_2 + \alpha_1\alpha_3$$

We will see that, in general, $d(\alpha)$ is independent of the choice of k.

To derive general formulas we introduce some notation from graph theory. A <u>tree</u> is a set of $n-1$ lines of G which form no loops; a <u>2-tree</u> is a similar set of $n-2$ lines. Note that all vertices of G are connected by a tree, while a 2-tree divides the vertices into 2 disjoint sets, each of which is connected by the 2-tree.

We now study minors of the $n \times L$ incidence matrix e_i^{ℓ}. If S is a set of $n-k$ lines at the graph, and $V_{i_1} \ldots V_{i_k}$ are k vertices, we will let $e(i_1, \ldots i_k; S)$

denote the (n-k) square determinant

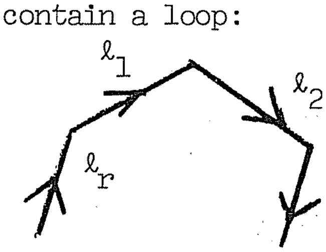
$$e(i_1, \ldots i_k; S) = \left| \{ e^{\ell}_{i} \mid \ell \in S,\ i \neq i, \ldots i_k \} \right|$$

Lemma 2: If T, T_2 are sets of n-1 and n-2 lines, respectively, then

$$e(i;\, T) = \begin{cases} \pm 1, \text{ if } T \text{ is a tree,} \\ 0, \text{ if } T \text{ is not a tree;} \end{cases}$$

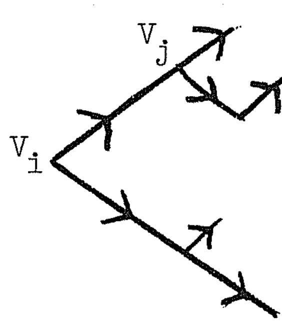
$$e(i,j;\, T_2) = \begin{cases} \pm 1, \text{ if } T_2 \text{ is a 2-tree and } i,j \text{ are} \\ \text{not connected in } T_2; \\ 0, \text{ otherwise.} \end{cases}$$

Proof: If T is not a tree it must contain a loop:



We may (by multiplying columns of the determinant by (-1), if necessary) arrange the orientation of the lines as in the figure. In that case, $\sum_{s=1}^{r} e_i^{\ell_s} = 0$, and this linear relation on the columns of the determinant implies that it must vanish.

If T is a tree we may, as above, assume that the lines of T are oriented away from i in T:



The complete expansion of the determinant is $e(i;\, T) = \sum_{\pi}(\pm) \prod_{L} e^{\ell}_{\pi(\ell)}$, where π is a 1-1, onto map $\pi : T \to \{1, \ldots \hat{i}, \ldots n\}$. To contribute, a term must have $\pi(\ell) = i_\ell$ or $\pi(\ell) = f_\ell$, for each $\ell \in T$. If ℓ is a line with $i_\ell = i$, then $\pi(\ell)$ must equal f_ℓ. But then if ℓ' is a line with $i_{\ell'} = V_{f_\ell}$, $\pi(\ell')$ must equal $f_{\ell'}$. Continuing, we see that there is only one non-zero term, that with $\pi(\ell) = f_\ell$, for all ℓ. This proves (a), the proof of (b) is similar.

Lemma 3: (a) $d(\alpha) = \sum_{T} \prod_{\ell \notin T} \alpha_\ell$,

(b) $D^k_{ij}(\alpha) = \sum_{T_2} \prod_{\ell \notin T_2} \alpha_\ell$,

the sums running respectively over all trees T of G and all 2-trees T_2 of G which do not connect k with either i or j.

Proof: Set $e'^{\ell}_{i} = \alpha^{-1}_{\ell} e^{\ell}_{i}$, so that $A = e'e^t$ (here e^t = transpose of e). Applying the Binet-Cauchy theorem for the determinant of a product matrix gives

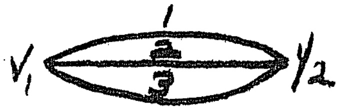
$$A\binom{k}{k} = \sum_{\substack{T \subset L, \\ T \text{ containing } n-1 \text{ lines}}} e'(i,\, T)\, e(i,\, T).$$

By Lemma 2 only terms in which T is a tree contribute, and since $e'(i,\, T) = \prod_{T} \alpha^{-1}_{\ell}\, e(i,\, T)$,

$$A\binom{k}{k} = \sum_{\text{trees } T} \prod_{\ell \in T} \alpha^{-1}_{\ell} \tag{2.14}$$

Multiplying (2.14) by $\prod_{L} \alpha_\ell$ gives (a); (b) is proved similarly.

We recompute the Symanzik functions for the graphs discussed above.

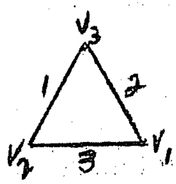
Example 3: 

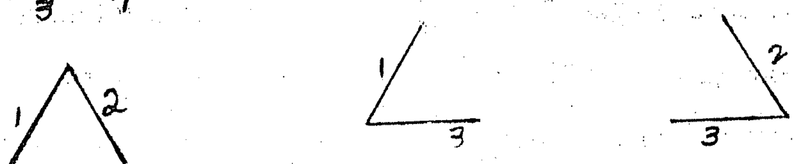
Trees T: 

$\prod_{\ell \notin T} \alpha_\ell$: $\alpha_2\alpha_3 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = d(a)$

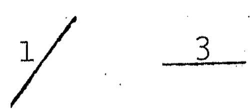
The only 2-tree is the empty set, so

$$D_1^2(\alpha) = \prod_{\ell \notin \phi} \alpha_\ell = \alpha_1\alpha_2\alpha_3$$

Example 4: 

Trees: 

$\prod_{\ell \notin T} \alpha_\ell$ $\quad \alpha_3 + \alpha_2 + \alpha_1 = d(\alpha)$

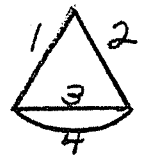
For D_{11}^3 : 2-trees: 

$\prod_{\ell \notin T_2} \alpha_\ell$: $\quad \alpha_2\alpha_3 + \alpha_1\alpha_2 = D_{11}^3(\alpha)$

For D_{12}^3 : 

(only 2-tree)

$$\alpha_1\alpha_2 = D_{12}^3(\alpha)$$

Example 5: 

Trees:

$$d(\alpha) = \alpha_3\alpha_4 + \alpha_2\alpha_4 + \alpha_2\alpha_3 + \alpha_1\alpha_4 + \alpha_1\alpha_3$$

C. Analytic Continuation.

We now wish to show that the integral in (2.12),

$$\int_0^\infty \cdots \int_0^\infty \frac{\Pi \alpha_\ell{}^{\lambda_\ell - 1}\, d\alpha_\ell}{d(\alpha)^2} \exp i \left[\sum_{i,j \neq k} \frac{p_i D^k_{ij} p_j}{d} - (m^2 - i\eta) \Sigma \alpha_\ell \right] \qquad (2.15)$$

which we know to converge for Re $\lambda_\ell > 2$, may be analytically continued to a function meromorphic in $\mathbb{C}^L$, the space of the $\underline{\lambda}$ variables. Divergence difficulties arise because the function $d(\alpha)$ vanishes when certain α's vanish. To study this behavior of $d(\alpha)$ we will introduce scaling transformations in the α variables.

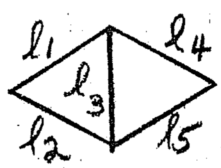
First note that $d(\alpha) = 0$ if $\alpha_\ell = 0$ for all lines ℓ in some loop. [This follows from Lemma 3, since no tree T can contain the loop and hence $\prod_{\ell \notin T} \alpha_\ell = 0$, for any T.] If $\ell_1 \ldots \ell_r$ are the lines of the loop, we make the variable change $\alpha_{\ell_j} = t\, \beta_{\ell_j}$, with the normalization $\beta_{\ell_k} = 1$, for some k. Then $d(\alpha) = t\, d'$, where $d' \neq 0$ unless the α's vanish around some other loop as well. We have isolated the zero of d in the factor t.

Suppose now that the α's vanish for lines which form two loops. If these loops have no line in common, we may scale the variables for the two loops separately;

$$\alpha_{\ell_i} = t\,\beta_{\ell_i}, \qquad \alpha_{\ell'_i} = s\beta_{\ell_i} \;;$$

$$\beta_{\ell_k} = 1 \;; \qquad \beta_{\ell'_j} = 1 \quad .$$

Then $d(\alpha) = s\,t\,d'$, with $d' \neq 0$ (unless the α's vanish for another loop). However, if the two loops overlap, we must first scale everything by one variable, then one loop by another:



$$\alpha_{\ell_i} = t\,\beta_{\ell_i} \qquad \beta_{\ell_i} = s\,\gamma_{\ell_i} \quad (i = 1,2,3)$$

$$\beta_{\ell_4} \text{ or } \beta_{\ell_5} = 1 \;; \qquad \gamma_{\ell_j} = 1, \text{ some } j.$$

Then $d(\alpha) = t^2 s\,d'$, with $d' \neq 0$ as above; t appears to the power two since it is a scaling variable for two loops.

Two further remarks are necessary. First, it is never necessary to include in the scaling a line ℓ which is not part of a loop on which the α's vanish, even if $\alpha_\ell = 0$. Secondly, since in the integration region $\{\alpha_\ell \geq 0\}$ the α_ℓ's which vanish can form loops in many ways, we make a given scaling transformation only in a region of α-space in which a partial ordering of the variables limits the loops which can be formed.

As an example consider the graph of Example 1, with $d(\alpha) = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$. d vanishes whenever any two α's vanish, and vanishes to second order when all α's are zero. In the region $0 \le \alpha_1 \le \alpha_2 \le \alpha_3$, we may transform to new variables

$$\alpha_3 = t \quad , \qquad 0 \le t \quad ,$$

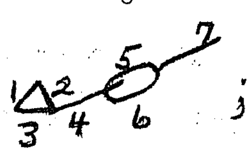
$$\alpha_2 = t\, s \quad , \qquad 0 \le s \le 1,$$

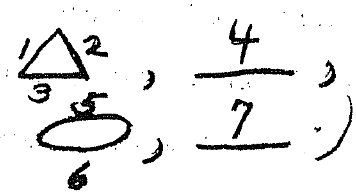
$$\alpha_1 = t\, s\, \beta, \qquad 0 \le \beta \le 1,$$

$$d\,(\alpha) = t^2\, s\,(1 + \beta + s\beta).$$

Thus the vanishing of d is isolated in the factors t^2,s, associated respectively with $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\alpha_1 = \alpha_2 = 0$. The remaining factor $1 + \beta + s\beta$ is strictly positive in the integration region.

We will now formalize these considerations. A Feynman graph is 2-connected if it cannot be disconnected by removing any vertex. Every graph G_0 is the union of its maximal 2-connected subgraphs and of single lines; these are called the pieces of G_0. (Example:

G_0 : 

; pieces of G_0 : 

)

We will in the future assume that our basic Feynman graph G is 2-connected. This represents no loss of generality, since the amplitude for a general graph is the product of the amplitudes of its pieces. Two subgraphs of G are disjoint if they have no common line; subgraphs are nonoverlapping if they are disjoint or if one is a subgraph of the other.

Definition 2: A singularity family (s-family) is a maximal family $\mathcal{E}$ of nonoverlapping subgraphs of G, each 2-connected or consisting of a single line, and satisfying

(A) no union of two or more disjoint elements of $\mathcal{E}$ is 2-connected. (We remark that this definition differs slightly from that of [3], corresponding to the definition of labeled s-family given there.)

We need some facts about any s-family $\mathcal{E}$.

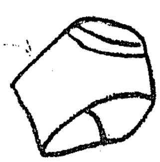
(1) G belongs to $\mathcal{E}$, by maximality, since if it did not, its addition to $\mathcal{E}$ would violate none of the conditions of Definition 2.

(2) For each $H \varepsilon \mathcal{E}$, there is precisely one line of H, called $\sigma(H)$, which lies in no subgraph of H in $\mathcal{E}$.

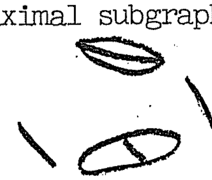
Proof: There must be at least one such line ℓ since otherwise the union of the maximal subgraphs of H in $\mathcal{E}$ would equal H, and hence be 2-connected, contradicting (A). If we remove ℓ from H, all pieces of the graph which remain must be in $\mathcal{E}$ by maximality (as in (1)). Thus ℓ is unique.

Example 6:

H: $\ell = \sigma(H)$



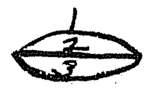
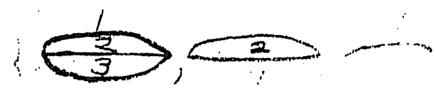
Maximal subgraphs of H in $\mathcal{E}$:



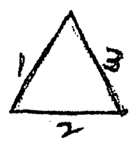
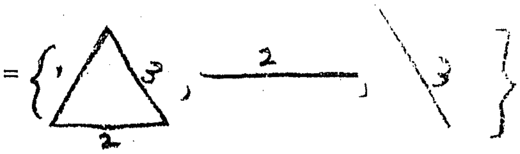
(3) For each $\mathcal{E}$ we define the domain $\mathcal{D}(\mathcal{E})$ in α-space by $\mathcal{D}(\mathcal{E}) = \left\{\alpha_\ell \geq 0 \,\middle|\, \alpha_{\sigma(H)} \geq \alpha_\ell \text{ if } \ell \varepsilon H\right\}$. Then $\bigcup \mathcal{D}(\mathcal{E}) = \left\{\alpha_\ell \geq 0\right\}$, and if $\underline{\alpha} \varepsilon \mathcal{D}(\varepsilon) \cap \mathcal{D}(\varepsilon')$, then $\alpha_\ell = \alpha_{\ell'}$ for some $\ell \neq \ell'$. Thus we can write

$$\int_{\alpha_\ell \geq 0}\int \cdots \int d\underline{\alpha}\, f(\alpha) = \sum_{\mathcal{E}} \int_{\mathcal{D}(\mathcal{E})} \cdots \int d\underline{\alpha}\, f(\alpha) \qquad (2.16)$$

for any $f(\alpha)$.

Example 7: If G = , a typical s-family is E = 

with $\mathcal{D}(E) = \{\alpha_3 \geq \alpha_2 \geq \alpha_1\}$. The decomposition (2.16) corresponds to separate integration over such region in which the order of the α_ℓ is completely specified.

If G = , a typical s-family is E = 

with $\mathcal{D}(E) = \{\alpha_1 \geq \alpha_2, \alpha_1 \geq \alpha_3\}$. Here (2.16) is a separation into 3 integrals according to which α_ℓ is largest.

Proof: If $\underline{\alpha}$ is a point for which no two coordinates agree, so that (say) $\alpha_{\ell_L} > \alpha_{\ell_{L-1}} > \ldots > \alpha_{\ell_1}$, then $\underline{\alpha}$ belongs to a unique domain $\mathcal{D}(E)$. For certainly we must have $\ell_L = \sigma(G)$. The pieces of the graph obtained by removing ℓ_L from G are then also in E (see Example 6). Let H be such a piece. Among the lines of H is one, say ℓ_k, which is maximal in the ordering given above. Then $\ell_k = \sigma(H)$, and the pieces of $H - \{\ell_k\}$ must be in E. Continuing, we generate a unique s-family E with $\underline{\alpha} \in \mathcal{D}(E)$.

(4) In $\mathcal{D}(E)$ we may introduce scaling variables t_H, $H \in E$, by

$$\alpha_\ell = \prod_{\ell \in H} t_H \quad ; \tag{2.17}$$

with $t_G \geq 0$, $1 \geq t_H \geq 0$ for other H. The Jacobian $\frac{\partial(\underline{\alpha})}{\partial(\underline{t})}$ of this transformation is

$$\frac{\partial(\underline{\alpha})}{\partial(\underline{t})} = \prod_{H \in E} t_H^{L(H) - 1} \tag{2.18}$$

(Here $L(H)$ is the number of lines in H. We will similarly write $n(H)$ for the number of vertices in H, so that $L = L(G)$, $n = n(G)$, etc.)

Proof: The function σ (see (2)) gives a 1-1 correspondence between lines of G and graphs in E, so we have the correct number of t variables. $\alpha_{\sigma(H)}$ depends on $t_{H'}$ only if H' contains H; thus the matrix $\left\{\frac{\partial \alpha_{\sigma(H)}}{\partial t_{H'}}\right\}$ is triangular and its determinant $\frac{\partial(\underline{\alpha})}{\partial(\underline{t})}$ is the product of the diagonal entries. Since

$\frac{\partial \alpha_{\sigma(H)}}{\partial t_H} = \prod_{H' \supsetneqq H} t_{H'}$,

$$\frac{\partial(\underline{\alpha})}{\partial(\underline{t})} = \prod_{H \varepsilon E} \prod_{\substack{H' \varepsilon E \\ H' \supsetneqq H}} t_{H'} = \prod_{H'} \left[t_{H'}\right]^{L(H') - 1} ,$$

because H' has $L(H') - 1$ proper subgraphs, by (2).

(5). Under the change of variables (2.17),

$$d(\alpha) = \prod_{H \varepsilon E} t_H^{L(H) - n(H) + 1} E(\underline{t}) \tag{2.19}$$

$$D^k_{ij} = t_G \prod_{H \varepsilon E} t_H^{L(H) - n(H) + 1} F^k_{ij} (\underline{t}) , \tag{2.20}$$

with E, F^k_{ij} polynomials and $E(\underline{t}) \neq 0$ for $t_H \geq 0$.

Remark: This is the key result: all the zeros of $d(\alpha)$ in the region $\mathcal{D}(E)$ are isolated in the factors t_H in (2.19). Note also that only factors t_H for H 2-connected actually occur in (2.19), (2.20), since $L(H) - n(H) + 1 = 0$ for H a single line.

Proof: From Lemma 3,

$$d(\alpha) = \sum_T \prod_{\ell \notin T} \alpha_\ell ,$$

the sum running over all trees T of G. Now a tree can intersect any subgraph H in at most n(H) - 1 lines, since otherwise it would contain a loop in H. The complement of T must therefore intersect H in at least L(H) - n(H) + 1 lines, so that $\prod_{\ell \notin T} \alpha_\ell$ contains at least L(H) - n(H) + 1 factors t_H. This proves (2.19). (2.20) follows similarly, the extra factor of t_G expresses the fact that D^k_{ij} is homogeneous of degree L - n + 2 in the α's.

To show that E does not vanish we need

Lemma 4: The set T_0 of lines which are themselves graphs in $\mathcal{E}$ forms a tree in G.

Example: In the two cases in Example 7., $\{1\}$ and $\{2,3\}$, respectively, are trees.

Proof: T_0 cannot contain a loop, since the union of lines in the loop would be a 2-connected subgraph, contradicting (A) of Def. 2. It suffices then to prove that T_0 connects all vertices. If not, there exist disjoint subsets $\mathcal{U}$, $\mathcal{U}' \subset \{V_1, \ldots V_n\}$ such that no line ℓ, joining a vertex in $\mathcal{U}$ with a vertex in $\mathcal{U}'$, is in T_0. Therefore each such line must be a $\sigma(H_i)$, where $H_1, \ldots H_k \in \mathcal{E}$ are 2-connected. Suppose H_j, $1 \leq j \leq k$, is minimal among these graphs. Because H_j is 2-connected it must contain at least two lines joining $\mathcal{U}$ with $\mathcal{U}'$, say $\sigma(H_i)$ and $\sigma(H_j)$. But by the coherence of $\mathcal{E}$ we must have $H_j \subsetneqq H_i$, so $\sigma(H_i)$ cannot lie in H_j. This contradiction proves the lemma.

Now d(α) contains a term $\prod_{\ell \notin T_0} \alpha_\ell$. For each H in $\mathcal{E}$, the graphs of $\mathcal{E}$ which are subgraphs of H form an s-family in H; applying Lemma 4 to this s-family implies that T_0 intersects H in a tree, i.e., in n(H) - 1 lines. The above term is then precisely $\prod_H t_H^{L(H) - n(H) + 1}$, so

$$d(\alpha) = \prod_H t_H^{L(H) - n(H) + 1} (1 + \ldots),$$

We now return to the integral (2.15), writing it as a sum of integrals over the domains $\mathcal{D}(E)$, then making the substitutions (2.17). Using (2.18), (2.19), and (2.20), (2.15) becomes

$$\sum_E \int_0^\infty dt_G \int_0^1 \cdots \int_0^1 \prod_{H \neq G} dt_H \prod_{H \varepsilon E} t_H^{\Lambda(H) - \frac{\mu(H)}{2} - 1} E(\underline{t})^{-2} \tag{2.21}$$

$$\times \exp i\, t_G \left[\sum_{i,j} \frac{F_{ij}^k(\underline{t})}{E(\underline{t})} p_i Q p_j - (m^2 - i\eta)(\Sigma \beta_\ell(\underline{t})) \right]$$

Here $\Lambda(H) = \sum_{\ell \varepsilon L(H)} (\lambda_\ell - 1)$ (with $L(H)$ the set of lines of H), $\mu(H) = 2L(H) - 4(n(H) - 1)$ is the superficial divergence of H, and $\beta_\ell(\underline{t}) = t_G^{-1} \alpha_\ell(\underline{t})$. (2.21) now indicates clearly the divergences of the generalized Feynman integral: the t_H integration is convergent at the physical point ($\lambda_\ell = 1$, for all ℓ) only if the graph H has negative superficial divergence.

We may now analytically continue $T(\underline{\lambda}, \underline{p})$ to all values of $\underline{\lambda}$. Consider first the t_H ($H \neq G$) integrations in (2.21). If f(t) is any infinitely differentiable function for $0 \leq t \leq 1$, the integral

$$I(\nu) = \int_0^1 t^{\nu - 1} f(t)\, dt,$$

convergent for Re $\nu > 0$, may be analytically continued by writing

$$f(t) = \sum_{i=0}^{k} \frac{f^{(i)}(0)t^{i}}{k!} + g_k(t),$$

where $g_k(t)$ has a zero of order $(k + 1)$ at $t = 0$ [5]. Thus

$$I(\nu) = \sum_{i=0}^{k} \frac{f^{(i)}(0)}{i!\ (\nu + i)} + \int_0^1 t^{\nu - 1} g_k(t), \tag{2.22}$$

and (2.22) provides an analytic continuation of $I(\nu)$ valid for Re $\nu > -(k + 1)$. Stated differently: on $[0,1]$ we may write the distribution $t^{\nu - 1}$ in the form

$$t^{\nu - 1} = \sum_{0}^{k} \frac{\delta^{(i)}(t)}{i!(\nu + i)} + \left[\text{term analytic for Re } \nu > -(k + 1)\right] \tag{2.23}$$

so that $t^{\nu - 1}$ is meromorphic in ν with simple poles at $\nu = -i$ $(i \geq 0)$ of residue $\delta^{(i)}/i!$. The factors $t_H^{\Lambda(H) - 1/2\,\mu(H) - 1}$ in (2.21) are to be interpreted as distributions in this sense.

The t_G integral in (2.21) may be done explicitly, using the formula

$$\int_0^\infty t^{\nu - 1} e^{-itx}\, dt = \frac{e^{-\frac{\nu \Pi i}{2}}\, \Gamma(\nu)}{(x)^{\nu}}$$

valid for Re $\nu > 0$ and Im $\mathbf{x} < 0$ (see (2.4)). From (2.13) and (2.21),

$$\mathcal{T}_\eta(\underline{\lambda}, \underline{p}) = \prod_{L}\left(\frac{i}{4\sqrt{|Q|}\ \Gamma(\lambda_\ell)}\right)(4\sqrt{|Q|})^{n - 1}\ (2\Pi)^2\ \delta(\Sigma p_i)\ \Gamma(\Lambda(G) - \frac{\mu(G)}{2})$$

$$\sum_{E} \int_0^1 \cdots \int_0^1 \prod_{H \neq G} t_H^{\Lambda(H) - \frac{1}{2}\mu(H) - 1}\, dt_H\ E(t)^{-2} \left[(m^2 - i\eta)(\Sigma\beta_\ell) \right. \tag{2.24}$$

$$- \sum_{i,j} \frac{F^k_{ij} \; p_i Q p_i}{E} \Bigg]^{\frac{\mu(G)}{2} - \Lambda(G)}$$

In the form (2.24) we may let $\eta \to 0$, according to Theorem 1, since $(\Sigma \beta_\ell) \geq 1$. (note $\beta_{\sigma(H)} = 1$). In this limit $\sqrt{|Q|} \to -i$, so

$$\mathcal{T}(\underline{\lambda},\underline{p}) = \prod_L \left(\frac{-1}{4\Gamma(\lambda_\ell)}\right) (-4i)^{n-1} (2\pi)^2 \, \delta(\Sigma p_i) \, \Gamma(\Lambda(G) - \frac{\mu(G)}{2})$$

$$\sum_E \int_0^1 \cdots \int_0^1 \prod_{H \neq G} t_H^{\Lambda(H) - \frac{1}{2}\mu(H) - 1} dt_H \, E(t)^{-2} \left[m^2(\Sigma\beta_\ell) - \Sigma \frac{p_i F_{ij} p_j}{E} \right]^{\frac{\mu(G)}{2}} \qquad (2.25)$$

We emphasize that the powers of t_H are defined in the sense of (2.23). We have thus proved

Theorem 3: The generalized Feynman amplitude $\mathcal{T}(\underline{\lambda},\underline{p})$ is a meromorphic function on $\mathbb{C}^L$, with possible simple poles on the varieties

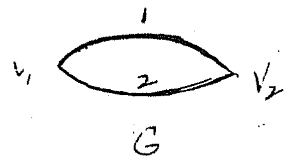
$$\Lambda(H) - \frac{1}{2}\mu(H) = 0,-1,-2,\ldots .$$

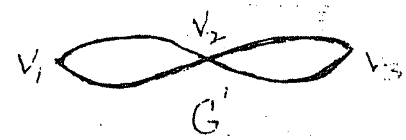
If G contains divergent subgraphs, $\mathcal{T}$ will have a complicated singularity at the point $\lambda_\ell = 1$, all ℓ. We now discuss methods of removing this singularity in order to define a finite, renormalized amplitude.

D. Renormalization

The most obvious procedure for removing the singularity is to set all variables λ_ℓ equal to a single complex variable z. The resulting function of z has a pole (of order at most $L - n + 1$) at $z = 1$, so that one could take the constant term in the Laurent series at $z = 1$ as a renormalized amplitude. We give an example to show why this procedure is unacceptable.

Example 8: Let G, G' be the graphs





The amplitude $T_G(\lambda_1,\lambda_2)(x_1,x_2)$ depends only on $\xi = x_2 - x_1$, so that

$$T_G(z,z)(x_1,x_2) = f(z,\xi) = \frac{a}{z-1} + b(\xi) + c(\xi)(z-1) + \ldots \qquad (2.26)$$

Here a is a non-zero constant, and b, c are non-trivial functions of ξ. In this case $b(\xi)$ is a correctly renormalized amplitude for G.

Exercise: Verify these properties of a, b, and c. Use the remark following Theorem 5.

The amplitude for G' is the product of the amplitudes for its parts. Thus if $\eta = x_3 - x_2$,

$$T_{G'}(z,z,z,z)(x_1,x_2,x_3) = f(z,\xi)\ f(z,\eta)\ . \qquad (2.27)$$

Expanding each factor in (2.27) according to (2.26) gives constant term

$$b(\xi)\ b(\eta) + a[c(\xi) + c(\eta)] \qquad (2.28)$$

However, (2.28) is not a correctly renormalized amplitude for G', which must have the form

$$b(\xi)\ b(\eta) + (\text{constant})$$

(see Section III). The proposed rule fails.

This difficulty is not associated with the reducibility of G'. The problem is that, under the proposed rule, poles associated with one part of the diagram cause unwanted higher z-derivatives associated with other parts; the problem will arise whenever a proper subgraph is superficially divergent.

We now give an acceptable method of removing the singularity. It is convenient to axiomatize this process to emphasize the essential features.

Definition 3: Let $j(\underline{\lambda})$ be the function

$$j(\underline{\lambda}) = \prod \Lambda(H) ,$$

the product taken over all nonempty subgraphs of G. Let A be the space of all functions f, meromorphic in a neighborhood of $\underline{\lambda}^o = (1,1,\ldots 1)$, such that $j(\underline{\lambda})\, f(\underline{\lambda})$ is analytic at $\underline{\lambda}^o$, i.e., f has at most simple poles on the varieties $\Lambda(H) = 0$. An evaluator is a map $W : A \to \mathbb{C}$ satisfying:

(W1) Linearity. W is linear.

(W2) Continuity. If $\{f_n\} \in A$ are such that $\{jf_n\}$ are analytic in a fixed neighborhood of $\underline{\lambda}^o$ and $jf_n \to 0$ uniformly in that neighborhood, then $Wf_n \to 0$.

(W3) Extension. If $f \in A$ is analytic at $\underline{\lambda}^o$, then $Wf = f(\underline{\lambda}^o)$

(W4) Symmetry. If π is a permutation of $\{1,\ldots L\}$, and f_π is defined by $f_\pi(\lambda_1 \ldots \lambda_L) = f(\lambda_{\pi(1)},\ldots \lambda_{\pi(L)})$, then $Wf_\pi = Wf$.

(W5) Reality. If $f \in A$, and $f^* \in A$ is defined by $f^*(\underline{\lambda}) = f(\overline{\underline{\lambda}})$, then $Wf^* = \overline{Wf}$.

(W6) Factorization. If f_1, f_2 depend on disjoint sets of λ's (e.g., $f_1 = f_1(\lambda_1,\ldots \lambda_k)$, $f_2 = f_2(\lambda_{k+1},\ldots \lambda_L)$), then $W\, f_1 f_2 = (Wf_1)(Wf_2)$.

Since $T(\underline{\lambda})$ is clearly in A, we may define an analytically renormalized amplitude by WT. [Technically, T is a distribution in $\underline{p}$ (or $\underline{x}$), and WT is the distribution $(WT)(\psi) = W[T(\psi)]$, where ψ is a test functions. This causes no confusion in what follows; for a somewhat fuller discussion see [3].) It should be noticed that the rule discussed in Example 8 fails to satisfy (W6).

The standard example of an evaluator is to take the constant term in an iterated Laurent series, then symmetrize over the orders of iteration. This process may be given as a contour integral. Let $R_1 < R_2 < . . . < R_L$ satisfy

$$R_i > \sum_{j=1}^{i-1} R_j \quad , \tag{2.29}$$

and let C_j be the contour $|z - 1| = R_j$, oriented counterclockwise. Then for $f \in A$,

$$\mathcal{W}f = \frac{1}{(2\pi i)^L L!} \sum_{\pi} \int_{C_{\pi(1)}} d\lambda_1 .. \int_{C_{\pi(L)}} d\lambda_L \; \frac{f(\underline{\lambda})}{(\lambda_1 - 1) \; ... \; (\lambda_L - 1)} \quad , \tag{2.30}$$

the sum running over all permutations π of $\{1, ... L\}$. The integral in (2.30) is well defined, because by (2.29) f is not singular on the contour. Moreover, (2.30) is independent of the specific choice of $\{R_i\}$. (2.30) is easily seen to satisfy (W1) - (W6).

III. The Renormalization Properties

We have now given a natural definition of a "renormalized" amplitude for a Feynman graph, using analytic continuation and discarding the resulting singularities. The surprising thing is that this process has a relation to physics. We will show that the analytically renormalized amplitude satisfies properties of causality, unitarity, etc; in addition, we show that it may be obtained by the "infinite subtractions" characteristic of the usual renormalization theory.

A. Axioms of Renormalization.

In [4], Hepp gives a beautiful axiomatic characterization of renormalization, which we now discuss. Suppose that, corresponding to every graph $G(V,L)$ in a theory, we have defined a tempered distribution written symbolically as

$$\mathcal{F}[\prod_{L} \Delta_F(x_{f_\ell} - x_{i_\ell})] \tag{3.1}$$

Because anti-time ordering (in which the propagator is replaced by its complex conjugate) is necessary in the discussion of unitarity and causality, we suppose also that we have defined a tempered distribution

$$\mathcal{F}[\prod_{L} \bar{\Delta}_F(x_{f_\ell} - x_{i_\ell})] \quad , \tag{3.2}$$

corresponding to the graph in which each line is assigned an anti-propagator $\bar{\Delta}_F$.

[For a complete discussion of the equivalence of different renormalization schemes it is necessary to define renormalized amplitudes for "generalized graphs". We will return to this point in Section IV.]

The distributions (3.1) and (3.2) are renormalized amplitudes for $G(V,L)$ if the following three axioms hold. First note that $\prod_L \Delta_F$ is itself well defined on test functions in the space $S^o = S^o(\mathbb{R}^{4n})$; i.e. those which vanish in a neighborhood of the set $\{\underline{x} \mid x_i = x_j \text{ for some } i \neq j\}$. Then we require:

(F1) $\underset{L}{F}(\Pi\Delta_F)$ is a Lorentz invariant extension of $\Pi\Delta_F$ from S^o to S, and

$$\underset{L}{F}(\Pi\Delta_F) = \overline{\underset{L}{F}(\Pi\bar{\Delta}_F)} \quad .$$

We now impose unitarity. If (3.1) is to be the renormalized amplitude, then the S-matrix, say in a $:\phi^k:$ theory, will be given by a formal power series in g:

$$S = \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} \prod_{i=1}^{n} \binom{k}{\alpha_i} \int \sum_G \underset{L}{F}(\Pi\Delta_F) : \phi^{\alpha_1}(x_1) \ldots \phi^{\alpha_n}(x_n) : dx_1 \ldots dx_n; \quad (3.3)$$

the sum running over all ways of contracting pairs of the fields $:\phi^{k-\alpha_1}(x_1):$, $\ldots :\phi^{k-\alpha_n}(x_n):$. The adjoint S^+ will be given by a similar sum, with $(-ig)^n$ replaced by $(ig)^n$, and with amplitude $F(\Pi\bar{\Delta}_F)$. If we now write down the unitarity equation $S^+S = 1$, reduce the resulting products of normal products by Wick's theorem, and regroup terms according to the total power of g, we find a series identical to (3.3) except that $F(\Pi\Delta_F)$ is replaced by

$$\sum_{\substack{U' \cup U'' = V \\ U' \cap U'' = \phi}} (-1)^{|U'|} F(\Pi' \bar{\Delta}_F) F(\Pi'' \Delta_F) \prod_{\text{conn}} \Delta_+(x_a - x_b) \qquad (3.4)$$

In (3.4), Π' runs over those lines of $G(V, L)$ connecting two vertices of U' Π'' over lines connecting two vertices of U''; $\prod_{\text{conn}}$ over lines connecting $V_a \in U'$ to $V_b \in U''$; $|U'|$ is the number of elements in U', and we have written

$$\Delta_+(x_a - x_b) = \langle 0 | \phi(x_a) \phi(x_b) | 0 \rangle \quad . \qquad (3.5)$$

The unitarity axiom is:

(F2) (3.4) must vanish for each graph $G(V, L)$ containing one or more vertices.

Finally, we impose causality by requiring the retarded functions to have appropriate support properties. When the retarded functions are expressed in terms of the time-ordered functions, this becomes

(F3) For each graph $G(V,L)$ and each $V_j \varepsilon V$, the distribution

$$\sum_{\substack{U' \cup U'' = V \\ U' \cap U'' = \phi \\ j \varepsilon U'}} (-1)^{|U'|} F(\Pi' \bar{\Delta}_F) F(\Pi'' \Delta_F) \prod_{\text{conn}} \Delta_+(x_a - x_b) \tag{3.6}$$

vanishes unless $x_j - x_i$ is in $\bar{V}_+$ (the forward light cone) for each $V_i \varepsilon V$.

It is relatively easy to verify that analytic renormalization satisfies these axioms. For (F1), the Lorentz invariance is immediate. The property of extending $\Pi\Delta_F$ from S^o is an immediate consequence of the counterterm interpretation given in Section B, so we defer its discussion until that time. Finally, we introduce an anti-propagator, depending on λ, by

$$\tilde{\Delta}^*(\lambda,p) = i(m^2 - p^2 + i0)^{-\lambda}$$

$$= \overline{\Delta(\bar{\lambda},p)} \quad ,$$

and a corresponding amplitude $T^*(\underline{\lambda},p)$ defined for large Re λ_ℓ by

$$T^*(\underline{\lambda},\underline{x}) = \prod_L \Delta^*(\lambda_\ell, x_{f_\ell} - x_{i_\ell}) \quad .$$

T^* may be analytically continued to all $\underline{\lambda}$ and has the same singularity structure as T, so we define

$$F(\prod_L \bar{\Delta}_F) = WT^* \ .$$

(F1) then requires that

$$\overline{WT^*} = WT \quad ,$$

and this follows from (W5), the reality property of the evaluator.

We will prove (F2) and (F3) by defining a function $\Delta_+(\lambda)$ which is related to $\Delta_F(\lambda)$ much as Δ_+ is to Δ_F. For large Re λ_ℓ we will verify that

$$\sum_{\mathcal{U}',\mathcal{U}''} (-1)^{|\mathcal{U}'|} \Pi' \overset{*}{\Delta}_F(\lambda_\ell)\ \Pi''\Delta_F(\lambda_\ell) \prod_{\text{conn}} \Delta_+(\lambda_\ell) = 0 \qquad (3.7)$$

(compare 3.4), and that

$$\sum_{\substack{\mathcal{U}',\mathcal{U}'' \\ j\varepsilon\mathcal{U}'}} (-1)^{|\mathcal{U}'|} \Pi' \overset{*}{\Delta}_F(\lambda_\ell)\ \Pi''\Delta_F(\lambda_\ell) \prod_{\text{conn}} \Delta_+(\lambda_\ell) \qquad (3.8)$$

has support in$\{x_j - x_i \varepsilon \bar{V}_+$, all i$\}$. Analytically continuing (3.7) and (3.8) to the physical point $\underline{\lambda}^o$, and applying the evaluator, will establish (E2) and (E3).

The distributions $(m^2 - p^2)_+^{-\lambda}$, $(m^2 - p^2)_-^{-\lambda}$ [5] are defined for Re $\lambda < 1$ by

$$(m^2 - p^2)_{(\pm)}^{-\lambda} = \begin{cases} |m^2 - p^2|^{-\lambda} & \text{if } m^2 (\gtrless) p^2 \\ 0 & \text{otherwise} \end{cases}$$

and by analytic continuation for other λ. (This analytic continuation may be accomplished as in Theorem 1; however, the poles at $\lambda = 1,2,\ldots$ are actually present in this case.) These distributions agree with $(m^2 - p^2 \pm i0)^{-\lambda}$, in appropriate regions, up to a phase, so that

$$(m^2 - p^2 \pm i0)^{-\lambda} = (m^2 - p^2)_+^{-\lambda} + e^{\mp i\pi\lambda}(m^2 - p^2)_-^{-\lambda} .$$

We now argue in analogy to the relation

$$\Delta_F + \bar{\Delta}_F = \Delta_+ + \Delta_- \quad ,$$

where supp $\tilde{\Delta}_{(\pm)} \subset \{p^o (\gtrless) 0\}$. Thus

$$\tilde{\Delta}_F(\lambda) + \tilde{\Delta}_F^*(\lambda) = -i[e^{i\pi\lambda} - e^{-i\pi\lambda}](m^2 - p^2)_-^{-\lambda}$$

$$= 2 \sin \pi\lambda \, (m^2 - p^2)_-^{-\lambda} , \qquad (3.9)$$

and since $(m^2 - p^2)_-^{-\lambda}$ has support in the region $p^2 > m^2$,

$$\tilde{\Delta}_\pm(\lambda) = \theta(\pm p^o) 2 \sin \pi\lambda \, (m^2 - p^2)_-^{-\lambda} \qquad (3.10)$$

is a well defined tempered distribution.

Lemma 5: $\Delta_\pm(\lambda)$ is in C^m for Re $\lambda > m + 2$, and

$$\begin{aligned} \Delta_F(\lambda,x) &= \theta(x^o) \, \Delta_+(\lambda,x) + \theta(-x^o) \, \Delta_-(\lambda,x) \\ \Delta_F^*(\lambda,x) &= \theta(x^o) \, \Delta_-(\lambda,x) + \theta(-x^o) \, \Delta_+(\lambda,x) \end{aligned} \qquad (3.11)$$

hold (as identities between measurable functions).

Remark: The equations (3.9), (3.10), and (3.11) show that the relations among $\Delta_F(\lambda), \Delta_F^*(\lambda)$, and $\Delta_\pm(\lambda)$ completely parallel those among Δ_F, $\bar{\Delta}_F$, and $\Delta_\pm$. Although it is implicit in (3.9), (3.10) that $\Delta_\pm(1) = \Delta_\pm$, this may also be seen directly: $(m^2 - p^2)_-^{-\lambda}$ has a pole at $\lambda = 1$ with residue $\delta(p^2 - m^2)$, so that

$$\begin{aligned} \tilde{\Delta}_\pm(1) &= \lim_{\lambda\to 1} \theta(\pm p^o) \, 2 \sin \pi\lambda (m^2 - p^2)_-^{-\lambda} \\ &= -2\pi \, \delta(p^2 - m^2) \, \theta \, (\pm p^o) \end{aligned}$$

as expected.

Proof of Lemma 5: The Fourier transform of (3.10) may be calculated to give

$$\Delta_\pm(\lambda,x) = \frac{2^{1-\lambda}m^{2-\lambda}}{\Gamma(\lambda)} \left(\sqrt{-x^2 + i\sigma(x^0)0}\,\right)^{\lambda-2}$$

$$\times K_{\lambda-2}\left(m\sqrt{-x^2 + i\sigma(x^0)0}\,\right)$$

[where $\sigma(x^0)$ = sign of x^0]. The argument of Lemma 1 shows that this is C^m for Re $\lambda > m + 2$, and direct comparison with the Fourier transform of Δ_F yields (3.11).

<u>Lemma 6:</u> With $\Delta_\pm$ defined by (3.10), and Re $\lambda_\ell > 2$, (3.7) holds, and (3.8) has the proper support properties.

<u>Proof:</u> Insert (3.11) and the obvious decomposition

$$\Delta_+(x) = \Theta(x^0)\,\Delta_+(x) + \Theta(-x^0)\,\Delta_+(x) \tag{3.12}$$

into (3.7), producing a sum of products in which each line is associated with a factor of the form $\Theta(\pm\, x^0)\,\Delta_\pm(x)$. It is convenient to represent such a factor by an orientation for the line, corresponding to the time-ordering imposed by the $\Theta(\pm x^0)$, and a sign corresponding to the $\Delta_\pm$ involved:

$$\Theta(x_a^0 - x_b^0)\,\Delta_\pm(x_a - x_b) \quad : \quad V_a \overset{\pm}{\longleftarrow} V_b \quad .$$

The decompositions (3.8) and (3.10) are thus

$$\begin{aligned} \Delta_F(x_a - x_b) &= a \overset{+}{\longleftarrow} b \;+\; a \overset{+}{\longrightarrow} b \quad ; \\ \bar{\Delta}_F(x_a - x_b) &= a \overset{-}{\longleftarrow} b \;+\; a \overset{-}{\longrightarrow} b \quad ; \\ \Delta_+(x_a - x_b) &= a \overset{+}{\longleftarrow} b \;+\; a \overset{-}{\longrightarrow} b \quad . \end{aligned} \tag{3.13}$$

Now consider a typical term arising from (3.9), with each line oriented and signed. We must identify from which U', U'' term in (3.7) it could have come.

Let $W' \subset V$ be the set of vertices with an outgoing positive line, W'' the vertices with an outgoing negative line, and $W = V - (W \cup W')$ the vertices with only incoming lines. By (3.13) we see that necessarily

$$U' \subset W' \cup W \quad ; \qquad\qquad (3.14)$$
$$U'' \subset W'' \cup W \quad .$$

Now if W is empty, so that every vertex has an outgoing line, the product of θ functions will imply that the term has support in some variety $x_i^o = x_j^o$ and hence vanishes as a distribution (recall that all distributions here are locally integrable functions). If W is not empty, the sum over U', U'' corresponding to (3.14) will vanish due to the factor $(-1)^{|U'|}$ in (3.7). This completes the verification of (3.7).

To check the support properties of (3.8) we again insert (3.11) and (3.12). In this instance all terms cancel, by the above mechanism, except those in which $W' = V - \{V_j\}$, $W = \{V_j\}$. But the θ-functions force this term to have support in the region $\{x_j^o \geq x_i^o$, for all i}; since (3.8) is Lorentz invariant the support must actually be in $\{x_j^o - x_i^o \in \bar{V}_+$, all i}.

We have now verified (3.7), and the support properties of (3.8), for Re λ_ℓ sufficiently large. However, these equations are easily continued to all values of $\underline{\lambda}$: the functions $\Delta_+(\lambda_\ell)$ may be multiplied, for all values of λ_ℓ, because the support properties in momentum space imply that the convolution integrals are taken over a finite region (see [1] for a discussion of the same argument for the usual Δ_+). If for U V we write T_U for the generic amplitude corresponding to all lines joining vertices of U, the analytic continuation of (3.7) is

$$\sum_{U',U''} (1)^{|U'|} T^*_{U'}(\underline{\lambda})\ T_{U''}(\underline{\lambda}) \prod_{\text{conn}} \Delta_+(\lambda_\ell) = 0 \quad , \qquad (3.15)$$

and of (3.8)

$$\sum_{\substack{U',U'' \\ j\in U'}} (-1)^{|U'|} T^{*}_{U'}(\underline{\lambda}) \quad T_{U''}(\underline{\lambda}) \prod_{\text{conn}} \Delta_{+}(\lambda_{\ell}) \ . \tag{3.16}$$

Analytic continuation does not affect the support properties of (3.16).

To complete the proof of (F2) and (F3), apply the evaluator W to (3.15) and (3.16). The factors $T_{U'}$, $T_{U''}$, and $\prod_{\text{conn}} \Delta_+$ depend on disjoint sets of λ's, and $\prod_{\text{conn}} \Delta_+$ is analytic at $\underline{\lambda}^{o}$. Thus from the factorization and extension properties of W, (3.15) becomes

$$\sum_{U',U''} (-1)^{|U'|} (WT_{U'}) \ (WT_{U''}) \prod_{\text{conn}} \Delta_{+} = 0 \quad ,$$

which is the desired unitarity property. Similarly, applying W to (3.16) shows that

$$\sum_{j\,\varepsilon U'} (-1)^{|U'|} (WT_{U'}) (WT_{U''}) \prod_{\text{conn}} \Delta_{+}$$

has the desired support properties. This completes the verification of the axioms.

B. Counterterm Interpretation

Here we relate analytic renormalization to the original idea of renormalization theory: that the divergences represent unobservable effects and may be removed by appropriate redefinition of parameters like mass, charge, etc. In a more general context the idea is as follows. Suppose one has a rule for regularizing the amplitudes (e.g., analytic regularization, Pauli-Villars regularization, etc.) with a regularization parameter r, so that

$$\Delta_F = \lim_{r \to r_o} \Delta_{F,r}$$

while

$$T_r = \Pi \, \Delta_{F,r}(x_{f_\ell} - x_{i_\ell})$$

is well defined. In a theory arising from a Lagrangian density $L = L_o + gL_I$, we say that a renormalization rule is implementable by counterterms in the Lagrangian if we can find a new density, of higher order in g and depending on r,

$$L_C(r) = \sum_{i=2}^{\infty} g^i L_{Ci}(r),$$

so that the renormalized series formed with L is the $r \to r_o$ limit of the regularized series formed with $L + L_C(r)$. Since the $r \to r_o$ limit of L_C itself does not exist, the terms in L_C are often called "infinite" counterterms. In a renormalizable theory the new Lagrangian $L + L_C$ may be interpreted as the Lagrangian of bare fields, involving the bare mass and charge.

The condition that a renormalization be implementable by counterterms forces it to have a certain subtractive structure, which we now describe (for details of the relation of this structure to the counterterms, see [1,3,6]). We will work with analytic regularization throughout, although the discussion is independent of the particular form of regularization.

Definition 4: A graph G is one particle irreducible (IPI) if it cannot be disconnected by removing a single line. A generalized vertex in a graph $G = G(V,L)$, with vertices $V = \{V_1, \dots V_n\}$, is a subset $U \subset V$; the graph $G(U)$ is the subgraph of G with vertices U and with all lines $\ell \in L$ for which $i_\ell, f_\ell \in U$. A vertex part for $U = \{V_1' \dots V_m'\}$ is a distribution $X(\underline{\lambda};U) \in S(R^{4m})$ having the form

$$\chi(\underline{\lambda};\mathcal{U}) = \begin{cases} 1, & \text{if } m = 1 \\ 0, & \text{if } m > 1 \text{ and } G(\mathcal{U}) \text{ not IPI} \\ D[\delta(x_1'-x_2')\ldots\,\delta(x_{m-1}'-x_m')], & \text{otherwise.} \end{cases}$$

Here D is a translation invariant constant (in x) coefficient differential operator, so that in p-space, if $G(\mathcal{U})$ is IPI,

$$\tilde{\chi}(\underline{\lambda};\mathcal{U}) = \delta(\sum_{i=1}^{m} p_i')P(\underline{\lambda};\, p_1'\ldots p_m')$$

with P a polynomial. Moreover, for a minimal renormalization, the degree of P (or D) is at most $\mu(G(\mathcal{U}))$, the superficial divergence of $G(\mathcal{U})$.

Suppose now that we have assigned a vertex part to each generalized vertex in the graph $G(V,L)$. For any partition Q of V into generalized vertices $\mathcal{U}_1\ldots\,\mathcal{U}_{k(Q)}$,

$$T_{Q,\chi}\,(\underline{\lambda}) = \prod_{1}^{k(Q)} \chi(\mathcal{U}_i) \prod_{\text{conn}} \Delta(\lambda_\ell)\ , \qquad (3.17)$$

where $\prod_{\text{conn}}$ is over lines connecting different generalized vertices. [(3.17) is well defined, for Re λ_ℓ sufficiently large, according to Lemma 1, since the differential operators in the vertex parts can act on the propagators. For other λ's it is defined by analytic continuation.]

Definition 6: A tempered distribution $F(\Pi\Delta_\ell)$ is an additively renormalized Feynman amplitude for $G(V,L)$ if there exists a set of vertex parts $\chi(\underline{\lambda};\mathcal{U})$ such that

$$F(\Pi\Delta_\ell) = \lim_{\underline{\lambda}\to\underline{\lambda}^0} \sum_{Q} T_{Q,\chi}(\underline{\lambda}) \quad . \qquad (3.18)$$

We remark that the sum in (3.18) is over all partitions Q. In particular, if Q is the partition $\{V_1\}, \ldots \{V_n\}$, $T_{Q,X} = T$, the generalized amplitude for $G(V,L)$. The other terms in (3.18) are subtractions to kill the singularity of T at $\underline{\lambda}^o$.

We now show that the analytically renormalized amplitude WT has the structure (3.18)[7]. A natural subtractive structure for W as an operator on A will first be developed, and then these subtractions will be related to the form in (3.18). We must first extend the notion of an evaluator.

Definition 7: An analytic evaluator is a map $V : A \to A_o$ (where $A_o \subset A$ is the subspace of functions analytic at $\underline{\lambda}^o$) satisfying

(V1) Linearity.

(V2) Continuity. If $\{f_n\} \in A$, and $f_n \to o$ in the sense of (W2), then $Vf_n \to o$ uniformly in the neighborhood of $\underline{\lambda}^o$.

(V3) Extension. $Vf = f$ if $f \in A_o$.

(V4) Symmetry. If f_π is as in (W4), then $V(f_\pi) = (Vf)_\pi$.

(V5) Reality. $(Vf)^* = V(f^*)$

(V6) Factorization. If f_1, f_2 depend on disjoint sets of λ's, then $V[f_1 f_2] = Vf_1 Vf_2$, and an additional technical assumption

(V7) If $f \in A$ is independent of λ_ℓ, so is Vf.

Conditions (V1) - (V6) should be compared with (W1) - (W6); in particular, (V2) shows that the operator V retains more information about f than the operator W. Note that if V is an analytic evaluator, then W defined by $Wf = Vf(\underline{\lambda}^o)$ is an evaluator. The specific example of an evaluator given earlier is easily extended to define an analytic evaluator, by modeling the definition on the extraction of the regular part of a Laurent series of a function of a single complex variable:

$$f(\underline{\lambda}) = \frac{1}{(2\pi i)^L L!} \sum_\pi \int_{C_{\pi(1)}} d\mu_1 \ldots \int_{C_{\pi(L)}} d\mu_L \frac{f(\mu)}{(\mu_1-\lambda_1)\ldots(\mu_L-\lambda_L)}, \qquad (3.19)$$

where in (3.19) it is assumed that $|\lambda_1 - 1|$ is less than the radius R_1 of the smallest contour.

Given an analytic evaluator V, we may define the operation of removing the singularity associated with some subset of the λ variables. Thus for H a subgraph of G, let V_H be the operator obtained by applying V to $f \in A$ while keeping the λ_ℓ, $\ell \notin H$, constant and away from singularities. $V_H f$ will not be singular when the λ_ℓ, $\ell \in H$, approach 1, but may have singularities on $\Lambda(H') = 0$, if H' is not a subset of H. By convention we define V_H, for H the empty graph, to be the identity operator on A. Note also that $V_G = V$.

We need one preliminary result.

<u>Lemma 7:</u> Let H,H' be subgraphs with $H' \subset H$. Suppose $f \in A$ is such that all singularities of f in H actually lie in H'; specifically, f is <u>not</u> singular on any $\{\Lambda(H'') = 0\}$ with $H'' \subset H$ but $H'' \not\subset H'$. Then

$$V_H f = V_{H'} f. \tag{3.20}$$

<u>Proof</u>: From the hypotheses, the function

$$g(\underline{\lambda}) = \prod_{H'' \subset H'} \Lambda(H'') \, f(\underline{\lambda}) \tag{3.21}$$

is not singular on the variety $\{\lambda_\ell = 0 | \text{ all } \ell \in H\}$, except at points where some sum of λ_ℓ's, for $\ell \notin H$, vanishes. At other points on the variety g may be expanded in a multiple Taylor series, so that g is an infinite sum of products

$$f_{H'} f_{H-H'} f_{G-H} \tag{3.22}$$

where f_K depends only on the λ_ℓ for $\ell \in K$. From (3.21), f will also be a sum of terms of the form (3.22). By the continuity and linearity of V, it suffices to verify (3.20) on each individual term; but (3.20) follows immediately on

(3.22) from the factorization and extension properties of V, q.e.d.

Definition 8: If H is any subgraph of G, define the operator $S(H)$ on A by

$$S(H) = \sum_{H' \subset H} (-1)^{|H-H'|} \; V_{H'}, \tag{3.23}$$

where the sum includes the empty graph ($H' = \emptyset$), and $|H-H'|$ = number of lines in H not also in H'.

For $f \in A$, $S(H)f$ is called the singular part of f associated with H. This terminology is justified by the first part of the next lemma.

Lemma 8: (a) $S(H)f = 0$, unless f is singular on some varieties $\Lambda(H_i)$, with $H_1 \cup \ldots H_K = H$. (b) If $f = f_1 f_2$, where f_i depends only on $\lambda_\ell, \ell \in H_i$, and H_1, H_2 are disjoint, then

$$S(H)f = \begin{cases} 0 \;, \text{ unless } H \subset H_1 \cup H_2 \\ (S(H \cap H_1)f_1)(S(H \cap H_2)f_2) \;, \text{ otherwise.} \end{cases}$$

Example: If L = 5, and $f \in A$ has the form

$$f(\underline{\lambda}) = \frac{g(\underline{\lambda})}{(\lambda_1 - 1)(\lambda_2 + \lambda_3 - 2)(\lambda_1 + \lambda_2 + \lambda_4 - 3)}$$

with g analytic at λ^0, then Lemma 8(a) implies that $S(H)f = 0$ unless the set of lines of H is one of $\emptyset$, {1}, {2,3}, {1,2,3}, {1,2,4}, {1,2,3,4}.

Proof of Lemma 8: (a) If f does not have the structure indicated, there must be some line $\ell \in H$ such that, if $H'' \subset H$ and f is singular on $\Lambda(H'')$, then $\ell \notin H''$. By Lemma 7, if $H' \subset H$ but $\ell \notin H'$, then

$$V_{H'} f = V_{H' \cup \{\ell\}} f \quad ,$$

so that the H' and $H' \cup \{\ell\}$ terms in (3.23) cancel when applied to f. This verifies (a).

(b) If $H \not\subset H_1 \cup H_2$, then $S(H)f = 0$ by (a)(since the singularities of f, being contained in H_1 and H_2, could not have union H). Otherwise, the result follows from (3.23) and the factorization properties of V.

Theorem 4: The analytic evaluztor V has the decomposition

$$V = \sum_{H \subset G} S(H). \tag{3.24}$$

Proof: If one inserts (3.23) into the right hand side of (3.24) and rearranges, the coefficient of V_H vanishes unless $V_H = V_G = V$, q.e.d.

We remark that since $S(\phi) = 1$ (the identity on A) (3.24) expresses Vf as a sum of f itself together with counterterms corresponding to singularities of f associated with various subgraphs.

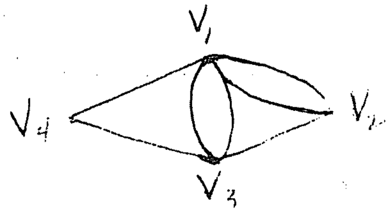
Example: If $L = \{1\}$, $S(\phi) = 1$, $S(1) = (V - 1)$. If $L = \{1,2\}$, $S(\phi) = 1$, $S(1) = V_1 - 1$, $S(2) = V_2 - 1$, $S(1,2) = V_{12} - V_1 - V_2 + 1$.

We now relate (3.24) to the desired decomposition (3.17), and begin by defining the vertex parts appearing there. Suppose that $U = \{V_1' \ldots V_m'\}$ is a generalized vertex of $G(V,L)$, and write $G' = G(V_1', \ldots V_m')$ (see Definition 4). An IPI graph H is subordinate to G' $(H \prec G')$ if H also has vertices $\{V_1', \ldots V_m'\}$. Then define

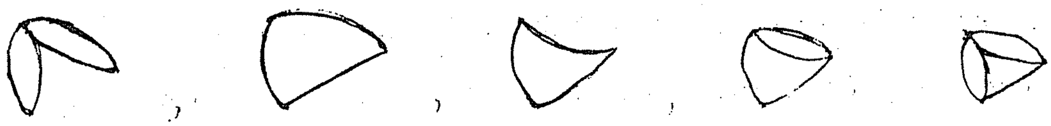
$$X(\underline{\lambda};U) = \sum_{H \prec G} S(H) T_{G'}(\underline{\lambda}) , \tag{3.25}$$

where $T_{G'}$ is the generalized amplitude for G'. Note that (3.25) vanishes unless G' is IPI, since each H in the sum is required to be IPI.

Example: If $G(V,L)$ is the graph



then some of the graphs subordinate to $G(V_1,V_2,V_3)$ are



Lemma 9: The distribution (3.25) is a vertex part for $\{V_1',\ldots V_m'\}$.

Proof: Recall the specific form of $T_{G'}(\underline{\lambda})$ given in (2.25), which we now write as

$$T_{G'}(\underline{\lambda}) = \Gamma(\Lambda(G') - \frac{\mu(G')}{2}) \sum_{E} T_E(\underline{\lambda}) \quad , \tag{3.26}$$

$$T_E(\underline{\lambda}) = \delta(\Sigma p_i')f(\underline{\lambda}) \int_0^1 \cdots \int_0^1 \prod_{\substack{H'\varepsilon E \\ H'\neq G'}} t_{H'}^{\Lambda(H') - \frac{\mu(H')}{2} - 1} dt_{H'} E(t)^{-2}$$
$$\times [m^2 \Sigma\beta_\ell - \sum_{i,j} \frac{p_i F^k_{ij} p_j}{E}]^{\frac{\mu(G')}{2} - \Lambda(G')} \tag{3.27}$$

where $f(\underline{\lambda})$ is an entire function. We will show that for each $H \prec G'$ and each E,

$$S(H)[\Gamma(\Lambda(G') - \frac{\mu(G')}{2})T_E] = \delta(\Sigma p_i')P(\lambda;p_1'\ldots p_m') \quad , \tag{3.28}$$

with P a polynomial of degree at most $\mu(G')$.

Suppose first that $H = G'$. T_E has singularities $\Lambda(H') = 0$, where H' does not contain $\sigma(G')$, therefore, if $\mu(G) < 0$ so that the gamma function is not singular of $\underline{\lambda}^0$, (3.28) will vanish, by Lemma 8 (a). If $\mu(G) \geq 0$, we use

$\Gamma(z - k) = [(-1)^k (k!)^{-1}(z - k)^{-1}$ + regular part] to write

$$\Gamma(\Lambda(G') - \frac{\mu(G')}{2})T_E = \frac{(-1)^{\mu(G')}}{(\frac{\mu(G')}{2})!\Lambda(G')} [T_E]_{\lambda_{\sigma(G')} = -\sum_{\ell \neq \sigma(G')} \lambda_\ell} + \text{remainder} \quad (3.29)$$

The remainder in (3.29) does not contain the singularity $\{\Lambda(G') = 0\}$, and hence is annihilated by $S(G')$, while the first term is already at the form (3.28), and this form will be preserved under the application of $S(G')$.

For a term $H \prec G'$, $H \neq G'$ in (3.25), we suppose that H has 2-connected pieces $H_1, \ldots H_k$. By Lemma 8 (a), (3.28) will vanish unless $H_i \varepsilon E, i = 1, \ldots k$. In the neighborhood of $\underline{\lambda}^0$ the distributions in (3.27) have the form

$$t_{H_i}^{\Lambda(H_i) - \frac{\mu(H_i)}{2} - 1} = \frac{\delta^{(\frac{\mu(H_i)}{2})}(t_{H_i})}{\Lambda(H_i)[\frac{\mu(H_i)}{2}]!} + \text{reg. part.} \quad (3.30)$$

If this decomposition (for all $H_1, \ldots H_k$) is inserted into (3.27), Lemma 8 (a) implies that every term in the resulting sum is annihilated by $S(H)$, except the one involving $\prod_{i=1}^{k} \delta^{(\frac{1}{2}\mu(H_i))}(t_{H_i})$. We will prove below that

$$F_{ij}^k(t) \Big|_{t_{H_1} = \ldots = t_{H_k} = 0} = 0 . \quad (3.31)$$

This implies that the surviving term is actually a polynomial in the p_l' of degree at most $\sum_{i=1}^{k} \mu(H_i) < \mu(G)$, multiplied by $\delta(\Sigma p_i')$, so that (3.28) is again satisfied.

To prove (3.31), recall that F^k_{ij} is related to a certain sum over 2 trees:

$$t_{G'} \prod_{E} t_{H'}^{L(H')-n(H')+1} F^k_{ij}(t) = \sum_{T_2} \prod_{\ell \varepsilon T_2} \alpha_\ell \qquad (3.32)$$

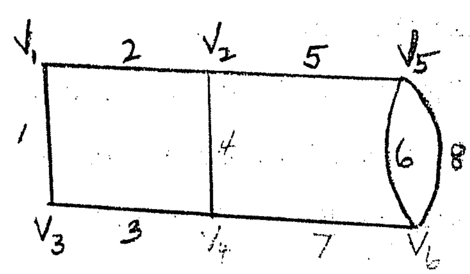
Since the H_i are the pieces of a 2-connected graph H whose vertices are $\{V_1', \ldots V_m'\}$, each 2-tree in G', in order not to connect all vertices of G', must fail to connect all vertices of some H_i. Thus T_2 can intersect that H_i in at most $n(H_i) - 2$ lines, and the corresponding term on the right hand side of (3.32) will have a factor

$$t_{H_i}^{L(H_i) - n(H_i) + 2}$$

The corresponding term in $F^k_{ij}(t)$ will have a factor t_{H_i}; this completes the proof of (3.31) and of Lemma 9.

It is convenient to extend the notion of subordinacy. Let Q be a partition of $\{V_1, \ldots V_n\}$ into k generalized vertices $\{V^i_1, \ldots V^i_{r(i)}\}$, with $r(i) \geq 2$ for $i = 1, \ldots k'$ and $r(i) = 1$ for $i = k' + 1, \ldots k$. A subgraph H is <u>subordinate</u> to $Q (H \prec Q)$ if H has connected components $H_1, \ldots H_{k'}$, and $H_i \prec G_i = G(V^i_1 \ldots V^i_{r(i)})$, for $i = 1, \ldots k'$.

<u>Example:</u> In the graph



the subgraph H containing lines 1,2,3,4,6, and 8, is subordinate to $Q = \{V_1, V_2, V_3, V_4\}, \{V_5, V_6\}$.

Lemma 10: With vertex parts X defined by (3.25),

$$T_{Q,X} = \sum_{H \prec Q} S(H) T_G$$

(see 3.17), with T_G the generalized amplitude for the graph $G = G(V,L)$.

Proof: Let $G_1, \ldots G_{k'}$ be related to Q as above; let L_{conn} be those lines in L not contained in any G_i. If Re λ_ℓ, $\ell \varepsilon L_{conn}$, is sufficiently large, T_G has a product structure (Lemma 1)

$$T_G = \prod_{i=1}^{k'} T_{G_i} \prod_{L_{conn}} \Delta(\lambda_\ell) \quad . \qquad (3.33)$$

We may apply $S(H)$ $(H \prec Q)$ to (3.33) since in defining $S(H)$ the λ_ℓ, $\ell \notin H$, are held constant, and this constant may be taken with large real part. Then using Lemma 8 (b),

$$\sum_{H \prec Q} S(H) T_G = \sum_{H \prec Q} \prod_{i=1}^{k'} (S(H_i) T_{G_i}) \prod_{L_{conn}} \Delta(\lambda_\ell)$$

$$= \prod_{i=1}^{k'} \Big(\sum_{H_i \prec G_i} S(H_i) T_{G_i} \Big) \prod_{L_{conn}} \Delta(\lambda_\ell)$$

$$= T_{Q,X} \quad .$$

We are now ready for our main theorem.

Theorem 5: Analytic renormalization is additive in the sense of Definition 6.

Proof: Suppose that the analytic renormalization is defined by an evaluator W. Then there is an analytic evaluator V with $Wf = (Vf)(\underline{\lambda}^o)$ (this has been shown by the example (3.19) for the standard W; for a general proof the reader is referred to [1]). Then

$$\mathcal{W}T_G = (\mathcal{V}T_G)(\underline{\lambda}^o)$$

$$= \lim_{\underline{\lambda} \to \underline{\lambda}^o} [\sum_H S(H)T_G](\underline{\lambda})$$

$$= \lim_{\underline{\lambda} \to \underline{\lambda}^o} \sum_Q \sum_{H \prec Q} (S(H)T_G)(\underline{\lambda})$$

$$= \lim_{\underline{\lambda} \to \underline{\lambda}^o} \sum_Q T_{Q,X} \quad ,$$

using Theorem 4 and Lemma 10. It should be noted that if a subgraph H is not subordinate to any Q, then necessarily some connected component of H is not IPI, and $S(H)T_G = 0$ by Lemma 8 (a).

Remark: (a) In section IIIA we omitted the proof that $\mathcal{W}T$ is an extension of $(\Pi\Delta_F)$ from S^o. This follows immediately from Theorem 5, however. For if Q is any partition other than $\{V_1, \ldots V_n\}$, say $Q = \{\mathcal{U}_1, \ldots \mathcal{U}_k\}$ with $\mathcal{U}_i = \{V_1^i, \ldots V_{r(i)}^i\}$ and $r(1) > 1$, then $T_{Q,X}$ is a distribution concentrated on $x_1^1 = \ldots = x_{r(1)}^1$ and vanishes on S^o. If $\psi \varepsilon S^o$, then, Theorem 5 implies

$$(\mathcal{W}T)(\psi) = \lim_{\lambda \to \lambda_o} (T\psi)$$

$$= (\Pi\Delta_F)(\psi) \quad ,$$

q.e.d.

(b) It should be noted that both in checking the axioms and verifying the counterterm structure, the factorization property of $\mathcal{W}$ was crucial. This underscores the failure of the simpler method of evaluation already rejected in Section IID. It is easily seen, however, that the proofs go through if the same λ parameter is assigned to all lines joining a given pair of vertices. This procedure has been used in [8].

IV. Finite Renormalization

In this section we will give a brief discussion of the equivalence of different forms of renormalization, together with an example showing the equivalence of analytic renormalization with the usual Bogoliubov-Parasink-Hepp formulation.

To discuss the equivalence it is necessary to broaden the general discussion of renormalization given in III A. (see [4]). Suppose that, in addition to our graph $G(V,L)$, we also specify a partition Q of V into (disjoint) generalized vertices $U_1,\ldots U_k$. Suppose further that to each U_i we have assigned a generalized vertex part $\hat{X}(U_i)$, where $\hat{X}$ is a distribution with the form

$$\hat{X}(U_i) = \begin{cases} 1\,, & m = 1 \\ 0\,, & \text{if } G(U) \text{ not IPI} \\ D[\delta(x_1^i - x_2^i)\ldots\delta(x_{r(i)-1}^i - x_{r(i)}^i)] & \end{cases} \qquad (4.1)$$

when $U_i = \{V_1^i \ldots V_{r(i)}^i\}$, and D is a constant coefficient differential operator of degree at most $\mu(G(U_i))$. This form is the same as that of Def. 4 except that there is no $\underline{\lambda}$ dependence. $\hat{X}$ is sometimes called a finite vertex part, as compared to vertex parts which become singular as a regularization is removed. $(G(V,L),\ Q,\hat{X})$ is called a <u>generalized Feynman graph</u>; we require that a renormalization F assign a finite value to the corresponding amplitude:

$$F\Big(\prod_{i=1}^{k} \hat{X}(U_i) \prod_{\text{conn}} \Delta_\ell\Big) \qquad (4.2)$$

(compare 3.1). The renormalized amplitudes for the generalized graphs are then required to satisfy Lorentz invariance, unitarity, and causality axioms parallel to those discussed in III A.

Definition 9: A finite renormalization is a map assigning to every generalized graph $(G(V,L), Q,\hat{X})$ a finite vertex part $\hat{X}_o(V)$ (with structure (4.2)), with the restriction that if Q is the partition $\{V\}$, then $\hat{X}_o(V) = \hat{X}(V)$. Two renormalization operations F, F' are said to differ by a finite renormalization if for some finite renormalization $\hat{X}_o$ and every generalized graph $(G(V,L),Q,\hat{X})$,

$$F'(\Pi X(U_i)\Pi\Delta) = \sum_P F(\Pi\hat{X}_o(W_j) \prod_{\text{conn}} \Delta_\ell) \tag{4.3}$$

In (4.3) the sum is over all partitions $P = \{W_1,\ldots W_s\}$ of V which are at least as coarse as Q (for all i, there is a j with $U_i \subset W_j$).

It should be noted that one term on the right hand side of (4.3) (the P = Q term) is

$$F(\Pi X(U_i)\Pi\Delta).$$

The structure of (4.3) is similar to that of (3.18). There is a similar interpretation in terms of counterterms in the Lagrangian: (4.3) says that the renormalization F' may be implemented by adding finite counterterms (corresponding to the $\hat{X}_o$), then computing renormalized amplitudes using F and the new Lagrangian.

We quote without proof [4]

Theorem 6: Any two renormalizations differ by a finite renormalization. Moreover, given a renormalization F and finite renormalization $\hat{X}_o$, F' as defined by (4.3) is a renormalization.

Let us now take up some specific examples. First, note that analytic renormalization extends immediately to a definition of quantities such as (4.2).

For by Lemma 1,

$$T_{Q,\hat{X}} = \prod_{i=1}^{k} \hat{X}(U_i) \prod_{\text{conn}} \Delta(\lambda_\ell)$$

is well defined for Re λ_ℓ sufficiently large. $T_{Q,X}$ may then be analytically continued to the physical point $\underline{\lambda}^o$, just as in II C., and is found to lie in A, so that (4.2) is defined to be

$$WT_{Q,\hat{X}} \quad .$$

Just as in III A, this rule is shown to satisfy the generalized invariance, unitarity and causality axioms.

Another example is BPH renormalization [1,6]. This scheme is explicitly subtractive in the sense of Definition 6, the counterterms being defined recursively. We again adopt analytic regularization, although Pauli-Villars or Hepp regularization is more usually used.

Definition 10: Let $(G(V,L),Q,\hat{X})$ be a generalized Feynman graph, with $Q = \{U_1,\ldots U_k\}$. For each subset $\{U_1',\ldots U_m'\}$, define inductively quantities $\bar{R}$, Y:

$$\bar{R}(\underline{\lambda};U_1',\ldots U_m') = \sum_P \prod_{i=1}^{j} Y(\underline{\lambda};U_1^i,\ldots U_{r(i)}^i) \prod_{\text{conn}} \Delta(\lambda_\ell) \quad , \tag{4.4}$$

where the sum is over all partitions P of $\{U_1',\ldots U_m'\}$ into $\{U_1^i\ldots U_{r(i)}^i\}$, $i = 1,\ldots j$, with $j > 1$, and $\prod_{\text{conn}}$ as usual runs over lines joining vertices lying in distinct sets of this partition;

$$Y(\underline{\lambda};U_1'\ldots U_m') = \begin{cases} \hat{X}(U_1), & \text{if } m = 1 \\ 0 \text{ if } G(U_1'\ldots U_m') \text{ is not IPI} \\ -M_\mu(G(V_1',\ldots V_s'))R(\underline{\lambda};U_1',\ldots U_m') & \text{otherwise} \end{cases} \tag{4.5}$$

Here $\{V_1',\ldots V_s'\}$ are the vertices in the generalized vertices $\{\mathcal{U}_1',\ldots\mathcal{U}_m'\}$, and $G(\mathcal{U}_1' \ldots \mathcal{U}_m')$ is obtained from $G(V_1' \ldots V_s')$ (Definition 4) by collapsing each $\mathcal{U}_i'$ to a single point. M_μ is the operation of replacing $f(\underline{\lambda};p_1' \ldots p_s')$ in

$$\bar{R} = \delta(\Sigma p_1')f(\underline{\lambda};p_1' \ldots p_s')$$

by its Taylor series in p up to order μ.

Finally, the renormalized amplitude for $(G(V,L),Q,\hat{X})$ is

$$R(\underline{\lambda};\mathcal{U}_1,\ldots\mathcal{U}_k) = \bar{R}(\underline{\lambda};\mathcal{U}_1,\ldots\mathcal{U}_k) + V(\underline{\lambda},\mathcal{U}_1\ldots\mathcal{U}_k) \quad . \tag{4.6}$$

The main theorem [1,9,3] is

Theorem 7: $R(\underline{\lambda};\mathcal{U}_1\ldots\mathcal{U}_k)$ is analytic when Re $\lambda_\ell > 1 - \frac{1}{2L}$, (where $L = |L|$), and

$$R(\underline{\lambda}^o;\mathcal{U}_1,\ldots\mathcal{U}_k)$$

as a renormalized amplitude for $(G(V,L),Q,\hat{X})$.

We note that if (4.4) is inserted in (4.6), it is clear that the renormalization operation R has the subtractive structure of Definition 6.

It will now be shown explicitly that analytic and subtractive renormalization differ by a finite renormalization. For simplicity we will check this property on graphs rather than generalized graphs. Then we wish to find a finite renormalization $\hat{X}_o$ such that, for $G = G(V,L)$,

$$\mathcal{W}T_G = R'(\underline{\lambda}^o;V_1,\ldots V_n) \tag{4.7}$$

with

$$R'(\underline{\lambda}^o;V_1,\ldots V_n) = \sum_P R(\underline{\lambda}^o;\mathcal{U}_1\ldots\mathcal{U}_k) \quad , \tag{4.8}$$

where P partitions V into $U_1,\ldots U_k$, and $R(\underline{\lambda}^o,U_1\ldots U_k)$ is the BPH amplitude for $(G\ (V,L),P,\hat{X}_o)$.

Equation (4.8) is a direct transcription of (4.3). However, it is possible to define R' in a different but equivalent way. Specifically, we may modify (4.4)-(4.6) (again, we work with an ordinary graph here):

$$\bar{R}'(\underline{\lambda};V_1',\ldots V_m') = \sum_P \prod_{i=1}^{j} Y'(\underline{\lambda};V_1^i,\ldots V_{r(i)}^i) \prod_{\text{conn}} \Delta(\lambda_\ell) \tag{4.9}$$

$$Y'(\underline{\lambda};V_1'\ldots V_m') = \begin{cases} 1 & \text{if} \quad m=1 \\ 0 & \text{if } G(V_1'\ldots V_m') \text{ not IPI} \\ -M_\mu(G(V_1'\ldots V_m'))\bar{R}'(\underline{\lambda};V_1',\ldots V_m') & \\ \quad + \hat{X}_o(\{V_1',\ldots V_m'\}), & \text{otherwise,} \end{cases} \tag{4.10}$$

$$R'(\underline{\lambda},V_1\ldots V_n) = \bar{R}'(\underline{\lambda};V_1,\ldots V_n) + Y'(\underline{\lambda};V_1,\ldots V_n) \quad . \tag{4.11}$$

That is, we add in the finite renormalization as we go along, in (4.10). The equivalence of (4.9)-(4.11) with (4.8) is an easy exercise in rearranging terms [2,3]. The physical interpretation is interesting: in (4.8), as pointed out above, we are consistently using the rule R for renormalization, but have modified our interaction Lagrangian by the addition of finite terms. However, R itself may be implemented by "infinite" counterterms; (4.10) says that if we modify those infinite terms by the correct finite addition, we produce R' instead. Thus R' may be viewed as arising from the same (infinite) Lagranian in each case.

In proving (4.7) it is convenient to use (4.9)-(4.11) instead of (4.8).

Theorem 8: If the finite renormalization $\hat{X}_o$ in (4.10) is defined to be

$$\hat{X}_o(V_1',\ldots V_m') = \mathcal{W}M_\mu \mathcal{T}_{G(V_1',\ldots V_m')}(\underline{\lambda}) \qquad (4.12)$$

whenever $m > 1$ and $G(V_1' \ldots V_m')$ is IPI, (where in (4.12) $\mu = \mu(G(V_1' \ldots V_m'))$) then

$$\mathcal{W}\mathcal{T}_G = \mathcal{R}'(\underline{\lambda}^o;V_1,\ldots V_n) \quad . \qquad (4.13)$$

Proof: As a preliminary result we show that for $m > 2$,

$$\mathcal{W}\mathcal{Y}'(\underline{\lambda};V_1' \ldots V_m') = 0 \; . \qquad (4.14)$$

The proof is by induction on m. From (4.10) and (4.12), (4.14) becomes

$$\mathcal{W}M_\mu[-\sum_P \prod_{i=1}^{j} \mathcal{Y}'(\underline{\lambda};V_1^i \ldots V_{r(i)}^i) \prod_{conn} \Delta(\lambda_\ell) + \mathcal{T}_{G(V_1'\ldots V_m')}] \; . \qquad (4.15)$$

The $P = \{\{V_1'\},\ldots\{V_m'\}\}$ term in (4.15) precisely cancels the $\mathcal{T}_{G(V_1' \ldots V_m')}$ term, so (because $\mathcal{W}$ and M_μ commute) it suffices to show that

$$\mathcal{W}\left[\prod_{i=1}^{j} \mathcal{Y}'(\underline{\lambda};V_1^i \ldots V_{r(i)}^i) \prod_{conn} \Delta(\lambda_\ell)\right] = 0 \quad , \qquad (4.16)$$

where $r(i) > 1$ for some i. But the factors in (4.16) depend on disjoint λ's, so the factorization property of $\mathcal{W}$, and the induction assumption (4.14) applied to that factor for which $r(i) > 1$, implies (4.14).

The proof of the theorem is similar. From (4.11),

$$\mathcal{R}'(\underline{\lambda};V_1 \ldots V_n) = \sum_P \prod_{i=1}^{j} \mathcal{Y}'(\underline{\lambda};V_1^i \ldots V_{r(i)}^i) \prod_{conn} \Delta(\lambda_\ell) \quad , \qquad (4.17)$$

where the sum is now over <u>all</u> partitions P of $\{V_1, \ldots V_n\}$. Apply $\mathcal{W}$ to both sides of (4.17). By Theorem 7 and the extension property of $\mathcal{W}$, the left hand side becomes

$$\mathcal{R}(\underline{\lambda}^o; V_1 \ldots V_n).$$

On the right hand side, all terms in which $r(i) > 1$ for some i vanish, using (4.14) and the same argument which established (4.16). The only remaining term, with $P = \{\{V_1\}, \ldots \{V_n\}\}$, is equal to $\mathcal{T}_G$, so that on application of $\mathcal{W}$ (4.17) becomes precisely (4.13). This completes the proof.

For an argument showing the construction of a finite renormalization giving $\mathcal{R}$ from $\mathcal{W}$, the reader may consult [10].

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