position. Show that the equation of Hamilton's principle is of the form

$$\delta \int_{t_1}^{t_2} \left(\frac{1}{2} m\dot{x}^2 + mgx - \frac{1}{2} kx^2 \right) dt = 0,$$

and obtain the Euler equation.

58. A particle of mass m is falling vertically under the action of gravity, and its motion is resisted by a force numerically equal to a constant c times its velocity $\dot{x}$. Show that the equation of Hamilton's principle takes the form

$$\delta \int_{t_1}^{t_2} \left(\frac{1}{2} m\dot{x}^2 + mgx \right) dt - \int_{t_1}^{t_2} c\dot{x}\,\delta x\, dt = 0.$$

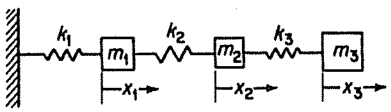

FIGURE 2.11

59. Three masses are connected in series to a fixed support, by linear springs. Assuming that only the spring forces are present, and using the notation of Figure 2.11, show that the Lagrangian function of the system is

$$L = \tfrac{1}{2}[m_1\dot{x}_1^2 + m_2\dot{x}_2^2 + m_3\dot{x}_3^2 - k_1x_1^2 - k_2(x_2 - x_1)^2 - k_3(x_3 - x_2)^2] + \text{const.},$$

where the x_i represent displacements from equilibrium. [Notice that if the x_i are given increments δx_i the total work done *by* the springs is given by

$$\delta\Phi = -\delta V = [k_2(x_2 - x_1) - k_1x_1]\,\delta x_1 + [k_3(x_3 - x_2) - k_2(x_2 - x_1)]\,\delta x_2 + [-k_3(x_3 - x_2)]\,\delta x_3.]$$

Section 2.11.

60. Obtain the Lagrangian equations relevant to the mechanical system of Problem 59.

61. A mass $4m$ is attached to a string which passes over a smooth pulley. The other end of the string is attached to a smooth pulley of mass m, over which passes a second string attached to masses m and $2m$. If the system starts from rest, determine the motion of the mass $4m$, using the coordinates q_1 and q_2 indicated in Figure 2.12.

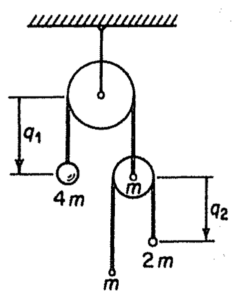

FIGURE 2.12

62. Obtain the Lagrangian equations for a triple pendulum consisting of three weights of equal mass m, connected in series to a fixed support by inextensible strings of equal length a, taking as the coordinates the angles θ_1, θ_2, and θ_3 made with the vertical by the three strings. Show also that, for small deviations from equilibrium, and small velocities, the Lagrangian function takes the approximate form

$$L = \frac{ma^2}{2}(3\dot{\theta}_1^2 + 2\dot{\theta}_2^2 + \dot{\theta}_3^2 + 4\dot{\theta}_1\dot{\theta}_2 + 2\dot{\theta}_2\dot{\theta}_3 + 2\dot{\theta}_1\dot{\theta}_3) - \frac{mga}{2}(3\theta_1^2 + 2\theta_2^2 + \theta_3^2) + \text{const.}$$

63. Two particles of equal mass m are connected by an inextensible string which passes through a hole in a smooth horizontal table, the first particle resting on the table, and the second particle being suspended vertically. Initially, the first particle is caused to describe a circular path about the hole, with an angular velocity $\omega = \sqrt{g/a}$, where a is the radius of the path, so that the suspended mass is held at equilibrium. At the instant $t = 0$, the suspended mass is pulled downward a short distance and is released, while the first mass continues to rotate.

(a) If x represents the distance of the second mass below its equilibrium position at time t, and θ represents angular position of the first particle at time t, show that the Lagrangian function is given by

$$L = m[\dot{x}^2 + \tfrac{1}{2}(a - x)^2\dot{\theta}^2 + gx] + \text{const.},$$

and obtain the equations of motion.

(b) Show that the first integral of the θ equation is of the form

$$(a - x)^2\dot{\theta} = a\sqrt{ag},$$

and that the result of eliminating $\dot{\theta}$ between this equation and the x equation becomes

$$2\ddot{x} + \left[\frac{1}{(1 - x/a)^3} - 1\right] g = 0.$$

(c) In the case when the displacement of the suspended mass from equilibrium is small, show that the suspended mass performs small vertical oscillations of period $2\pi\sqrt{2a/3g}$.

Section 2.12.

64. (a) In terms of Lagrange's function $L(q_1, \cdots, q_n; \dot{q}_1, \cdots, \dot{q}_n)$, such that $L = T - V$, show that the equations of motion become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \qquad (i = 1, 2, \cdots, n).$$

(b) Show that the generalized momentum p_i corresponding to the ith coordinate q_i is given by

$$p_i \equiv \frac{\partial T}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i}.$$