Turn in starred problems Tuesday 3/28/2017.

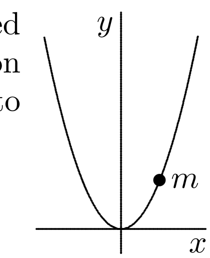
Posted with this assignment is a scan of a page from *Methods of Applied Mathematics*, by Francis Hildebrand (Dover Publications, New York, 1992). Some of the listed problems are from this sheet.

Hildebrand, pp. 206–207: 59, 60, 61*, 62, 63*.

12.A Use Hamilton's principle to derive the form of Newton's equation in spherical coordinates:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho\cos\phi.$$

12.B* A wire is bent into the shape of the parabola $y = x^2$ and supported so that the y axis is vertical. A particle of mass m slides without friction on the wire, under the influence of gravity. We will use as a coordinate to describe this system the x coordinate of the particle.



(a) Find the Lagrangian $L(x, \dot{x})$ of this system.

(b) Find the Euler-Lagrange equation obtained from Hamilton's principle.

(c) Show from your answer in (b) that the total energy $T + V$ is conserved.

12.C* Consider a system with Lagrangian $L(t, q, \dot{q}) = L(t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$, so that by Hamilton's principle the actual motion $q(t) = \big(q_1(t), \ldots, q_n(t)\big)$ of the system will be a stationary point of $I(q)$, that is, will satisfy the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0, \qquad i = 1, \ldots, n.$$

(a) Suppose that for some i, $1 \le i \le n$, the Lagrangian does not depend on q_i. Show that the quantity $\dfrac{\partial L}{\partial \dot{q}_i}\big(t, q(t), \dot{q}(t)\big)$ is a constant of the motion. (Very easy!)

(b) Suppose that the Lagrangian does not depend explicitly on t, that is, that $L(t, q, \dot{q}) = L(q, \dot{q})$. Show that the quantity

$$\sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\big(q(t), \dot{q}(t)\big) - L(q(t), \dot{q}(t)$$

is a constant of the motion.

(c) In class we wrote down Newton's equations in polar coordinates. Show that in this case (a) implies conservation of angular momentum when the potential energy depends only on r and not on θ: $W(r, \theta) \to W(r)$.

(d) Suppose that L has the form $L(q, \dot{q}) = T(q, \dot{q}) - V(q)$, where T and V are the kinetic and potential energies and T has the form

$$T(q, \dot{q}) = \frac{1}{2}\sum_{i,j=1}^{n} t_{ij}(q)\dot{q}_i\dot{q}_j$$

for some functions $t_{ij}(q)$ which satisfy $t_{ij}(q) = t_{ji}(q)$, $i, j = 1, \ldots, n$. (In particular, L does not depend explicitly on t). Show that (b) gives conservation of total energy $T + V$.