

1. An urn contains  $r = r_1 + r_2$  packages of candy; of these,  $r_1$  contain one piece and  $r_2$  contain three pieces.  
 (a) If a package were to be drawn from the urn at random, what is the probability it would contain one piece of candy? Three pieces? If two packages were drawn (without replacement), what are the probabilities of obtaining altogether two, four, or six pieces of candy?

For (b) and (c) below, we suppose that a sample of  $n$  packages ( $n \leq r$ ) is drawn from the urn, one at a time, without replacement.

(b) Let  $X_i$  be the number of pieces of candy in the  $i^{\text{th}}$  package drawn. Find  $E[X_i]$ ,  $\text{Var}(X_i)$ , and  $\text{Cov}(X_i, X_j)$  for  $i \neq j$ . Hint: use your results from (a).

(c) Let  $X$  be the total number of pieces of candy obtained. Find  $E[X]$  and  $\text{Var}(X)$ .

2. Let  $X$  and  $Y$  be continuous random variables with joint density

$$f(x, y) = \begin{cases} cx^2ye^{-xy}, & \text{if } 1 \leq x \leq 2 \text{ and } y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) If we know that  $X$  takes value  $3/2$ , what is the (conditional) distribution of  $Y$ ? No calculations are needed!.

(b) Find the constant  $c$ .

(c) Find  $E[X]$ ,  $E[Y]$ , and  $\text{Cov}(X, Y)$ .

(d) Find the probability density function for the random variable  $Z = Y/X$ .

3. (a) Xerxes and Yvonne play a game in which each, independently, chooses a real number between 0 and 2; their choices are called  $X$  and  $Y$ , respectively. Xerxes wins if his choice is at least twice Yvonne's. Suppose that both choose "at random", so that their choices are uniformly distributed on the interval  $[0, 2]$ . What is the probability that Xerxes wins?

(b) Suppose that each time the game is played Yvonne contributes \$2.00 and Xerxes \$1.00 to a pot, with the winner of the game collecting the entire \$3.00, and that they play the game 100 times in this fashion. Use the central limit theorem to estimate the probability that Yvonne comes out at least \$35.00 ahead.

4. Alice and Bob play a game with a coin which shows heads with probability  $p$ . Alice flips the coin three times, and then Bob does also. The one with the most heads (out of his or her three flips) wins.

(a) What is the probability that the two players have a total of at least five heads?

(b) What is the probability that a tie occurs?

(c) What is the probability that Bob has already won the game after his first flip?

5. Three urns are numbered 1 through 3; urn  $k$  contains  $k$  balls numbered 1 through  $k$ . We select an urn at random, draw a ball from it, note the number of the ball, replace the ball, and then draw again from the same urn. If it is known that the first ball drawn has number 1, find (a) the probability mass function of the number of the selected urn; (b) the probability mass function of the number of the second ball drawn.

6. Let  $X$  and  $Y$  be independent continuous random variables with densities

$$f_X(x) = \begin{cases} x/2, & \text{if } 0 \leq x \leq 2, \\ 0, & \text{otherwise;} \end{cases} \quad f_Y(y) = \begin{cases} 3e^{-3y}, & \text{if } y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find the density function  $f_Z(z)$  for  $Z = X + Y$  (a) by computing  $F_Z(z)$ ; (b) using (3.2) of Chapter 6.

7. Let  $X_1, X_2, \dots, X_{25}$  be independent random variables, each of which is uniformly distributed on the interval  $[0, 2]$ , and let  $X = \sum_{i=1}^{25} X_i$ .

(a) Find the mean and variance of  $X$ .

(b) Use the central limit theorem to estimate the probability that  $|X - 24| \leq 3$ .

(c) Use Chebyshev's inequality to find a number  $a$  such that you are absolutely sure that  $P\{|X - E[X]| \geq a\} \leq 0.1$ .

8. A single die is rolled; let us denote the number shown by  $X$ . Then a (possibly) biased coin, with probability  $X/6$  of showing heads, is flipped; the number of heads (0 or 1) obtained is denoted  $Y$ . (For example, if a 3 is rolled then a fair coin is flipped.)

(a) Find the joint probability mass function, and the marginal probability mass functions, of  $X$  and  $Y$ .

(b) Find  $E[X]$ ,  $E[Y]$ ,  $\text{Var}(X)$ ,  $\text{Var}(Y)$ , and  $\text{Cov}(X, Y)$ .

(c) Find the probability mass function of the random variable  $Z = X + Y$ .

9. A modified form of poker is played with an ordinary deck of 52 playing cards, but hands contain six cards. Calculate the probability that a randomly dealt hand contains: (i) four cards of the same rank and two from different ranks; (ii) three pairs, each of a different rank; (iii) two triples, each containing three cards of the same rank; (iv) only two ranks; (v) six cards in sequence; (vi) six cards, all of different ranks; (vii) a triple, a pair, and a single card, all of different ranks.

10. A hat contains cards numbered 1 to  $n$ ; one card is drawn at random and its number is denoted  $X$ . Calculate the moment generating function  $M_X(t)$  of  $X$ . (Hint: review the formula for the sum of a finite geometric series.) Now use the moment generating function to find the mean and variance of  $X$ . Check your answers by the usual direct calculations.

11. (a) State Markov's inequality as a theorem, defining all symbols used and including all necessary hypotheses.

(b) State Chebyshev's inequality as a theorem, defining all symbols used and including all necessary hypotheses.

12. A drug testing program, working with a group of patients suffering from a certain disease, treats 20% of them with drug A, 30% with drug B, and 50% with drug C. Drug A has a cure rate of 50%, drug B of 60%, and drug C of 90%. If a certain patient recovers, what is the probability he was treated with drug A? You may leave your answer as an unsimplified arithmetic expression.

13. A large urn contains  $N$  balls of each of 20 different colors (that is, a total of  $20N$  balls). 10 balls are selected at random; we let  $X$  be the total number of different colors obtained, and write  $X = \sum_{i=1}^{20} X_i$ , with  $X_i$  a Bernoulli random variable indicating whether or not the  $i^{\text{th}}$  color is obtained.

(a) Suppose that the selection is without replacement. Find  $E[X]$  and  $\text{Var}(X)$ . (Hint: compute  $E[X_i]$  by computing  $P\{X_i = 0\}$  using equally likely outcomes; do  $E[X_i X_j]$  similarly.)

(b) Find by elementary reasoning the correct values in (a) when  $N = 1$ , and verify that the answers you obtained there are correct in that case.

(c) Suppose that the selection is with replacement. Find  $E[X]$  and  $\text{Var}(X)$ . Your answer should not depend on  $N$ .

(d) Show that your answers in (c) are the  $N \rightarrow \infty$  limit of your answers in (a). Explain why this should be true.

14. Each morning, Fred makes a random choice of one of three routes to take to work. After  $n$  trips, ( $n > 0$ ), what is the probability that he has traveled each route at least once? Hint: use inclusion-exclusion to calculate the probability of the complementary event.

15. Rose and Bob play a game in which each in turn draws a ball from the urn containing one red and three blue balls, and then replaces it; Rose draws first.

(a) Suppose that Rose wins if she draws a red ball before Bob draws a blue ball, and Bob wins if he draws a blue ball before Rose draws a red ball. What is the probability that Rose wins? Hint: condition on the results of the first two draws.

(b) If the rules are modified so that Rose wins if she draws a red ball before Bob draws a blue ball *twice*, what then is the probability that Rose wins?

16. Discuss *one* of the following classical problems in probability, describing the problem (or model, or situation) and giving some details about the solution: gambler's ruin, the problem of the points, coupon collecting, the matching problem.