Workshop 12
November 21, 2001

1. Consider the function defined by $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$.

(a) Write out the first five terms of the series (up to $n = 4$; remember that $0! = 1$).

(b) Determine the interval of convergence; this is the domain of f.

(c) Verify *in two ways* that

$$f'(x) = -2xf(x) \ : \qquad (*)$$

(i) Compute both sides of (*) in terms of the five terms you wrote out in (a);

(ii) Verify (*) using the original sigma notation. This is a bit tricky (which is why it is a good idea to do (i) first); you will have to make a change of summation index.

(d) Explain why $y = f(x)$ is a solution of the initial value problem

$$y' = -2xy, \quad y(0) = 1.$$

(e) Solve this initial value problem and thereby obtain a formula for $f(x)$ in terms of "elementary functions"—those found on your calculator.

2. Keep the notation of the previous problem, and let $s_N(x) = \sum_{n=0}^{N} (-1)^n x^{2n}/n!$, e.g., $s_3(x) = 1 - x^2 + x^4/2 - x^6/6$.

(a) Using the formula you discovered for $f(x)$, use your calculator to graph f and the partial sums s_0, s_2, s_4, and s_6 at the same time, in a window where $0 \le x \le 1.2$. Copy your graph to your workshop, identifying the different curves. Does it appear that s_N becomes a better approximation to f as N increases?

(b) Use the alternating series error formula to obtain an upper bound for the error in the approximation $f(x) \approx s_6(x)$, $0 \le x \le 1.2$. Your answer should be a single number that applies to all x in the range $0 \le x \le 1.2$. Explain why this number is consistent with the graph from (a).

3. (a) Use the formula for the sum of a geometric series ,

$$\frac{a}{1-r} = a + ar + ar^2 + ar^3 + \cdots \qquad \text{(valid for } |r| < 1\text{)},$$

to express each of the following functions as a power series $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. Give a formula for the coefficient a_n in each case.

$$f(x) = \frac{x}{1-x} \qquad\qquad h(x) = \frac{2}{3x^4 + 16}$$

(b) For each series in part (a), determine the interval of x values in which it converges. Describe what happens at the end points of the convergence interval.

(c) Use your answer to a) to express $\int_0^1 \frac{2\,dx}{3x^4 + 16}$ as the sum of an infinite series.