Workshop 10
November 9, 2001

How big is the remainder? We know three answers.
(**Warning:** Two of these formulas were incorrect on workshop 9.)

Let $s = \sum_{n=1}^{\infty} a_n$, $\quad s_N = \sum_{n=1}^{N} a_n$, $\quad$ and $\quad R_N = s - s_N = \sum_{n=N+1}^{\infty} a_n$. Then:

- If $a_n = f(n)$ and $f(x)$ satisfies the hypotheses of the intgeral test, then $|R_N| \le \int_N^{\infty} f(x)\,dx$.
- If $0 \le a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $|R_N| \le \sum_{n=N+1}^{\infty} b_n$.
- If $\sum a_n$ satisfies the hypotheses of the alternating series test, then $|R_N| \le |a_{N+1}|$.

1. (a) For each power series below, use the ratio test to determine all values of x for which the series converges absolutely, then analyze the behavior of the series at the endpoints in order to determine the interval of convergence.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{nx^n}{n^2+1}$ $\qquad$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2(x-1)^n}{2^n}$ $\qquad$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n x^n}{n^2}$

(b) Can you cook up a power series whose interval of convergence is the interval $(0, 1]$, that is, the interval defined by $0 < x \le 1$? How about $(0, \infty)$? Give an explicit series or explain why you can't.

2. Consider the two series.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n\,2^n} = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \cdots$$

(a) Explain why each series converges.

(b) It happens that the sum of each of these series equals $\ln 2$. (We shall learn why, in another week or so.) Granted this fact, suppose that you had to compute $\ln 2$ to an accuracy of 10^{-6}. Which series is better suited for this calculation? Justify your answer by figuring out how many terms of each series you must add up to get a partial sum with the desired accuracy, that is, to have $|R_N| \le 10^{-6}$. (Hint for the second series: estimate the "tail" R_N by comparison with the tail of a certain geometric series.)

3. Consider the alternating series $\displaystyle\sum_{n=1}^{\infty} (-1)^n a_n$ with $a_n = \frac{1}{n} + \frac{(-1)^n}{n}$.

(a) Write down explicitly the first 10 terms of the series.

(b) Verify that $a_n \ge 0$ and $\lim_{n\to\infty} a_n = 0$.

(c) Does the series converge? Explain.

(d) Is the alternating series test applicable? Explain.

4. For which positive integers k does $\displaystyle\sum_{n=1}^{\infty} \frac{k^n n!}{n^n}$ converge? Hint: $\displaystyle\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.