Workshop 9

November 2, 2001

How big is the remainder? We know three answers.

Let $s = \sum_{n=1}^{\infty} a_n$, $s_N = \sum_{n=1}^{N} a_n$, and $R_N = s - s_N = \sum_{n=N+1}^{\infty} a_n$. Then:

- If $a_n = f(n)$ and $f(x)$ satisfies the hypotheses of the intgeral test, then $|R_N| \le \int_N^\infty f(x)\,dx$.
- If $0 \le a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $|R_N| \le \sum_{n=N+1}^{\infty} b_n$.
- If $\sum a_n$ satisfies the hypotheses of the alternating series test, then $|R_N| \le |a_{N+1}|$.

1. Consider the following three series:

$$(i)\quad \sum_{n=1}^{\infty} \frac{4-\sin n}{n^2+1}\,, \qquad (ii)\quad \sum_{n=1}^{\infty} \frac{4-\sin n}{n+1}\,, \qquad (iii)\quad \sum_{n=1}^{\infty} \frac{4-\sin n}{2^n+1}\,.$$

(a) Use the comparison test to determine whether or not each of these series converge.

(b) For each series which converges, give an approximation of its sum, together with an error estimate, as follows. First calculate the sum s_5 of the first 5 terms. Then find a number r such that you know that the error $R_5 = s - s_5$ satisfies $|R_5| \le r$, using the methods in the box above. (This is called *estimating* the error.) In one case you will have to first estimate using another series, then estimate that error using an integral.

2. For each of the following series, determine whether the three conditions of the alternating series test are satisfied. If the answer is 'YES', determine how many terms of the series need be added up to get an approximation to the exact sum of the series within an error tolerance of 10^{-20}.

$$(i)\quad \sum_{n=1}^{\infty} \frac{(-1)^n}{(n^3+4)^2}\,, \qquad (ii)\quad \sum_{n=2}^{\infty} \frac{(-1)^n n}{n+\ln n}\,, \qquad (iii)\quad \sum_{n=2}^{\infty} (-1)^n \frac{\sin^2 n}{n}\,.$$

3. This problem is based on the fact that *bounded monotonic sequences converge.*

(a) Given that $1 < x < y$, find a similar relation between $\frac{1}{x}$ and $\frac{1}{y}$. Then show that if $1 < x < y < 4$ then $1 < 4 - \frac{1}{x} < 4 - \frac{1}{y} < 4$. (Note: you must show that *each* of these three inequalities is true).

(b) Compute five terms $(a_1, \ldots, a_5)$ of the recursively defined sequence

$$a_1 = 1, \qquad a_{n+1} = 4 - \frac{1}{a_n}.$$

(c) Show that $1 < a_1 < a_2 < 4$; then, using (a), show that the sequence $\{a_n\}$ is increasing and bounded. Does it converge? Explain (see p. 700).

(d) Find the limit of the sequence in (b) (call the limit Q, and take the $n \to \infty$ limit on both sides of the recursion to obtain an equation for Q.)