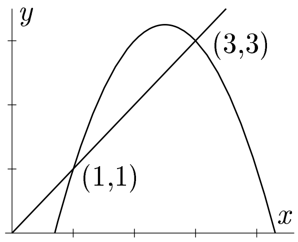
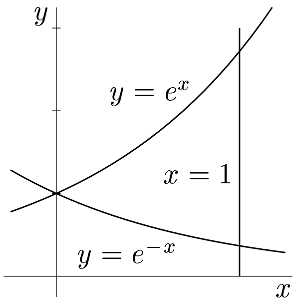
1(a) The parabola $y = -x^2 + 5x - 3$ crosses the line $y = x$ at those points x where $-x^2+5x-3 = x$, or $x^2-4x+3 = 0$. Thus, factoring, $(x-1)(x-3) = 0$, or $x = 1$ and $x = 3$. Thus, the area is: $\int_1^3 ((-x^2 + 5x - 3) - (x))\,dx =$ $\int_1^3 (-x^2 + 4x - 3)\,dx = (-x^3/3 + 2x^2 - 3x)\Big|_1^3 = 4/3.$



1(b) The volume obtained by rotation of R around the y-axis is:
$$2\pi\int_1^3 (x(-x^2 + 5x - 3) - x(x))\,dx = 2\pi\int_1^3 (-x^3 + 4x^2 - 3x)\,dx = 2\pi\left(-x^4/4 + 4x^3/3 - 3x^2/2\right)\Big|_1^3 = 16\pi/3.$$

2. The upper function is $y = e^x$ and the lower function is $y = e^{-x}$. The curves cross at $x = 0$. A sketch of the region appears at the right.



2(a) The integral for the area is: $\int_0^1 (e^x - e^{-x})\,dx = (e^x + e^{-x})\Big|_0^1 =$ $e + e^{-1} - 2$.

2(b) The volume obtained by rotation of R around the x-axis is:
$\pi\int_0^1 ((e^x)^2 - (e^{-x})^2)\,dx = \pi\int_0^1 (e^{2x} - e^{-2x})\,dx = (\pi/2)(e^{2x} + e^{-2x})\Big|_0^1 =$ $(\pi/2)(e^2 + e^{-2} - 2)$.

2(c) The volume obtained by rotation of R around the y-axis is: $2\pi\int_0^1 x(e^x - e^{-x})\,dx$. Using integration by parts, $\int xe^x\,dx = xe^x - e^x + C$ and $\int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C$. Thus, the volume is: $2\pi(xe^x - e^x + xe^{-x} + e^{-x})\Big|_0^1 = 2\pi(2e^{-1}) = 4\pi e^{-1}$.

3(a) Using $\cos^2(x) = (1 + \cos(2x))/2$, it follows that $\cos^4(x) = (1 + 2\,\cos(2x) + \cos^2(2x))/4$. Substituting $\cos^2(2x) = (1 + \cos(4x))/2$, then $\cos^4(x) = (3 + 4\,\cos(2x) + \cos(4x))/8$. Thus, $\int_0^{\pi/8}\cos^4(x)\,dx =$ $\frac{1}{8}\int_0^{\pi/8} (3 + 4\,\cos(2x) + \cos(4x))\,dx = \frac{1}{8}(3x + 2\,\sin(2x) + \frac{1}{4}\sin(4x))\Big|_0^{\pi/8} = \frac{3\pi + 8\sqrt{2} + 2}{64}$.

3(b) Using integration by parts, with $u = \tan^{-1}(x)$, $dv = dx$, $du = dx/(1 + x^2)$, $v = x$, and also that $\int \frac{x\,dx}{1 + x^2} = \frac{1}{2}\ln(1 + x^2) + C$, it follows: $\int_0^1 \tan^{-1}(x)\,dx = (x\tan^{-1}(x) - \frac{1}{2}\ln(1 + x^2))\Big|_0^1 = \frac{\pi - 2\cdot\ln(2)}{4}$.

3(c) Replacing $\sin^2(x)$ by $1 - \cos^2(x)$, and using the substitution $u = \cos(x)$, $du = -\sin(x)\,dx$ we obtain: $\int_0^{\pi/2}\sin^3(x)\,dx = \int_0^{\pi/2}(1 - \cos^2(x))\sin(x)\,dx = \int_1^0 -(1 - u^2)\,du$, where $u = \cos(\pi/2) = 0$, when $x = \pi/2$, and $u = \cos(0) = 1$, when $x = 0$. The last integeral is: $-(u - u^3/3)\Big|_1^0 = 1 - 1/3 = 2/3$.

4(a) Use the substitution, $u = 1 + e^x$, $du = e^x dx$, and $e^x = u - 1$. Then, $\int \frac{e^{2x}\,dx}{\sqrt{1 + e^x}} = \int \frac{e^x e^x\,dx}{\sqrt{1 + e^x}} =$ $\int \frac{(u-1)du}{\sqrt{u}} = \int (u^{1/2} - u^{-1/2})du = (2/3)u^{3/2} - 2u^{1/2} + C = (2/3)(1 + e^x)^{3/2} - 2(1 + e^x)^{1/2} + C.$

4(b) Using $\tan^2(x) = -1 + \sec^2(x)$, $\int\tan^2(x)\,dx = \int(-1 + \sec^2(x))\,dx = -x + \tan x + C$.

4(c) First complete the square: $x^2 - 4x + 3 = (x - 2)^2 - 1 = u^2 - 1$ with $u = x - 2$. Make the substitution $u = x - 2$, $du = dx$. Then, $\int \frac{dx}{(-3 + 4x - x^2)^{3/2}} = \int \frac{du}{(1 - u^2)^{3/2}}$. In this integral, let $u = \sin\theta$, $du = \cos\theta\,d\theta$, $(1 - u^2)^{1/2} = \cos\theta$. The integral becomes; $\int \frac{\cos\theta\,d\theta}{\cos^3\theta} = \int \sec^2\theta\,d\theta = \tan\theta + C = \frac{\sin\theta}{\cos\theta} + C =$

$$\frac{u}{(1-u^2)^{1/2}} = \frac{x-2}{(-3+4x-x^2)^{1/2}} + C.$$

4(d) A single integration by parts with $u = \ln x$, $dv = x^3 dx$, $du = dx/x$, $v = x^4/4$ results in:
$\displaystyle\int x^3 \ln(x)\,dx = \frac{1}{4}x^4 \ln x - \int \frac{x^4}{4}\cdot\frac{dx}{x} = \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3\,dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C.$

4(e) Using long division of polynomials, $x^5 = x(x^4-1) + x$. Thus, $\dfrac{x^5}{x^4-1} = x + \dfrac{x}{x^4-1}$. Using partial fractions, $\dfrac{x}{x^4-1} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+1}$. Multiplying out and solving for the constants A, B, C, D, $\dfrac{x}{x^4-1} = \dfrac{1}{4}\left(\dfrac{1}{x-1} + \dfrac{1}{x+1} - \dfrac{2x}{x^2+1}\right)$. Then, $\displaystyle\int \frac{x^5\,dx}{x^4-1} = \int \left(x + \frac{1}{4}\left(\frac{1}{x-1} + \frac{1}{x+1} - \frac{2x}{x^2+1}\right)\right)dx =$
$\dfrac{x^2}{2} + \dfrac{1}{4}\ln|x-1| + \dfrac{1}{4}\ln|x+1| - \dfrac{1}{4}\ln(x^2+1) + C = \dfrac{x^2}{2} + \dfrac{1}{4}\ln\left|\dfrac{x^2-1}{x^2+1}\right| + C.$

4(f) Use the substitution $x = \tan\theta$. Then $dx = \sec^2\theta\,d\theta$, and $(1+x^2)^{1/2} = \sec\theta$. Thus, $\displaystyle\int \frac{dx}{(1+x^2)^{1/2}} =$
$\displaystyle\int \frac{\sec^2\theta\,d\theta}{\sec\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|x + \sqrt{1+x^2}\right| + C.$

5. Separating the variables, $\int dy/(\cos^2(y)) = \int dx$, or $x = \int \sec^2(y)\,dy$, or $x = \tan(y) + C$. Since $y = \pi/4$, when $x = 0$, $0 = 1 + C$, or $C = -1$. Thus, $x = \tan(y) - 1$, or $\tan(y) = x+1$, or $y = \tan^{-1}(x+1)$.

6(a) Use the substitution $u = \tan^{-1}x$, $du = \dfrac{dx}{1+x^2}$. Change the limits of integration using $\tan^{-1}0 = 0$ and as $x \to \infty$, $\tan^{-1}x \to \pi/2$. Then, $\displaystyle\int_0^\infty \frac{\tan^{-1}x}{1+x^2}\,dx = \int_0^{\pi/2} u\,du = \left.\frac{u^2}{2}\right|_0^{\pi/2} = \frac{\pi^2}{8}.$

6(b) Using partial fractions: $\dfrac{1}{(x+1)(x+2)} = \dfrac{1}{x+1} - \dfrac{1}{x+2}$. It follows that $\displaystyle\int_0^\infty \frac{dx}{(x+1)(x+2)} =$
$\left.(\ln|x+1| - \ln|x+2|)\right|_0^\infty = \left.\ln\left|\dfrac{x+1}{x+2}\right|\right|_0^\infty = 0 - \ln(1/2) = \ln 2$, where we used $\dfrac{x+1}{x+2} \to 1$, as $x \to \infty$, and therefore $\ln\left|\dfrac{x+1}{x+2}\right| \to 0$. Thus, $\displaystyle\int_0^\infty \frac{dx}{(x+1)(x+2)} = \ln 2.$

6(c) With the substitution $u = 1 - x^2, du = -2x\,dx$, or $x\,dx = -du/2$, $\displaystyle\int_0^1 \frac{x\,dx}{(1-x^2)^{3/2}} = \int_1^0 \frac{-du}{2u^{3/2}} =$
$\displaystyle\int_0^1 \frac{du}{2u^{3/2}} = \left.\frac{-1}{u^{1/2}}\right|_0^1 = \infty$, since as $u \to 0^+$, $1/u^{1/2} \to \infty$, and the integral diverges.

7. In the first two replace n by x and use L'Hospital's Rule.

7(a) 0. Here, $\displaystyle\lim_{x\to\infty} \frac{x^2}{a^x} = \lim_{x\to\infty} \frac{2x}{\ln(a)\cdot a^x} = \lim_{x\to\infty} \frac{2}{\ln(a)^2\cdot a^x} = 0$, since as $a > 1$, $\displaystyle\lim_{x\to\infty} \frac{1}{a^x} = 0$.

7(b) e^a. To find $\displaystyle\lim_{x\to\infty} (1+a/x)^x$, set $y(x) = (1+a/x)^x$. Then find $L = \displaystyle\lim_{x\to\infty} \ln(y(x)) =$
$\displaystyle\lim_{x\to\infty} \ln\left[(1+a/x)^x\right] = \lim_{x\to\infty} x\ln(1+a/x) = \infty\cdot 0$, an indeterminate form. We rewrite the expression so that it has the form $0/0$, to which L'Hospital's Rule applies. Then the limit is: $L = \displaystyle\lim_{x\to\infty} \frac{\ln(1+a/x)}{1/x} =$
$\displaystyle\lim_{x\to\infty} \left(\left(\frac{-a/x^2}{1+a/x}\right) \Big/ (-1/x^2)\right) = \lim_{x\to\infty} \frac{a}{1+a/x} = a$. It follows that $\displaystyle\lim_{x\to\infty} y(x) = \lim_{x\to\infty} e^{\ln(y(x))} = e^L = e^a$.

7(c) $\ln a$. Here, $\displaystyle\lim_{n\to\infty} \frac{\ln(a^n+n^2)}{n} = \lim_{n\to\infty} \frac{\ln(a^n(1+n^2/a^n))}{n} = \lim_{n\to\infty} \frac{\ln(a^n) + \ln(1+n^2/a^n)}{n} =$
$\displaystyle\lim_{n\to\infty} \frac{n\,\ln(a)}{n} + \lim_{n\to\infty} \frac{\ln(1+n^2/a^n)}{n} = \ln a$, since by 7(a), as $n \to \infty$, $n^2/a^n \to 0$, and so $\ln(1+n^2/a^n) \to$

$\ln 1 = 0$.

7(d) $a/2$. $\lim_{n\to\infty}(\sqrt{n^2+an}-n) = \lim_{n\to\infty}(\sqrt{n^2+an}-n)\cdot\frac{(\sqrt{n^2+an}+n)}{(\sqrt{n^2+an}+n)} = \lim_{n\to\infty}\frac{(n^2+an)-n^2}{\sqrt{n^2+an}+n} =$ $\lim_{n\to\infty}\frac{an}{\sqrt{n^2(1+a/n)}+n} = \lim_{n\to\infty}\frac{an}{n\sqrt{1+a/n}+n} = \lim_{n\to\infty}\frac{a}{\sqrt{1+a/n}+1} = a/2.$

8(a) Divergent. Use the limit comparison test with the series: $\sum_{n=1}^{\infty}\frac{1}{n}$, since $\lim_{n\to\infty}\left(\frac{n}{n^2+1}\right)\Big/\frac{1}{n} =$ $\lim_{n\to\infty}\frac{n^2}{n^2+1} = 1$, and since the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

8(b) Divergent. By L'Hospital, $\lim_{n\to\infty}\frac{\ln n}{\ln(n^2+1)} = \frac{1}{2}$, and so the n-th term of the series does not go to 0.

8(c) Absolutely convergent, since the series converges by the ratio test: $\frac{|a_{n+1}|}{|a_n|} = \frac{((n+1)!)^3}{(3n+3)!}\cdot\frac{(3n)!}{(n!)^3} = \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} \to \frac{1}{27}$, as $n\to\infty$.

8(d) Divergent. Use the integral test, substituting $u = \ln x$, $du = \frac{dx}{x}$: $\int_2^{\infty}\frac{dx}{x\sqrt{\ln x}} = 2\sqrt{\ln x}\,\Big|_2^{\infty} = \infty$.

8(e) Absolutely convergent, since the series converges by the ratio test. Here, $\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2} = \frac{1}{2}\left(\frac{n+1}{n}\right)^2 \to \frac{1}{2}$, as $n\to\infty$.

8(f) Conditionally convergent. If $a_n = n/(n^2+1)$, then $a_n > 0$, $\{a_n\}$ is decreasing, and $\lim_{n\to\infty} a_n = 0$. By the alternating series test, the series $\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+1}$ converges. By problem 8(a), the series of absolute values diverges, and thus the series is conditionally convergent.

9. Use $R(t) = R_0e^{kt}$, for constants R_0 and k and with t measured in years. Then, $R(0) = R_0$ and $R(2) = R_0/2$, since after 2 years half of the original substance is left. Thus, $R_0e^{2k} = R_0/2$, or $e^{2k} = 1/2$. Thus, $k = -\ln 2/2$ and $R(t) = R_0e^{-\ln 2\cdot t/2}$. If $R(t) = .1R_0$, then $.1R_0 = R_0e^{-\ln 2\cdot t/2}$, or $.1 = e^{-\ln 2\cdot t/2}$. Taking ln( ), $\ln(1/10) = -\ln 2\cdot t/2$, or $-\ln 10 = -\ln 2\cdot t/2$, or $t = 2\ln 10/\ln 2$.

10. The radius of convergence is $3^{1/3}$ and the interval of convergence is $[-3^{1/3}, 3^{1/3}]$. Here, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{3n+3}}{(n+1)^3 3^{n+1}}\cdot\frac{n^3 3^n}{|x|^{3n}} = \left(\frac{n}{n+1}\right)^3\cdot\frac{|x|^3}{3} \to \frac{|x|^3}{3}$, as $n\to\infty$. It follows that the series converges absolutely if $\frac{|x|^3}{3} < 1$, or $|x|^3 < 3$, or $|x| < 3^{1/3}$. When $x = \pm 3^{1/3}$, the series is respectively, $\sum_{n=1}^{\infty}\frac{1}{n^3}$ and $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$, and these series converge absolutely.

11(a) From the geometric series $\frac{1}{1-u} = \sum_{n=0}^{\infty}u^n$, if $|u| < 1$. Setting $u = -t$, $\frac{1}{1+t} = \sum_{n=0}^{\infty}(-1)^n t^n$, if $|t| < 1$. Thus, $\ln(1+x) = \int_0^x\frac{dt}{1+t} = \int_0^x\sum_{n=0}^{\infty}(-1)^n t^n\,dt = \sum_{n=0}^{\infty}(-1)^n\int_0^x t^n\,dt = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}$, if $|x| < 1$.

11(b) The next term of the series occurs at $n = 11$ and is, in absolute value, $1/(12\cdot 2^{12}) = 1/(2^{14}3)$. This is an upper bound for the absolute value of the error.

12(a) Since, if $|x| < 1$, $\tan^{-1}(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$, if $|t| < 1$, $\tan^{-1}(t^2) = \sum_{n=0}^{\infty}\frac{(-1)^n t^{4n+2}}{2n+1}$.

It follows that: $F(x) = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+2}}{2n+1}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^x t^{4n+2}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)(4n+3)}$.

12(b) Replacing x by $1/2$, the integral is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(4n+3)2^{4n+3}}$.

13(a) Here, $f(x) = x^{1/4}$, $f'(x) = (1/4)x^{-3/4}$, $f''(x) = (-3/4^2)x^{-7/4}$, $f'''(x) = (21/4^3)x^{-11/4}$, $f^{(4)}(x) = (-231/4^4)x^{-15/4}$. It follows that $f(16) = 2$, $f'(16) = 1/2^5$, $f''(16) = -3/2^{11}$, $f'''(16) = 21/2^{17}$. Thus, $T_3(x) = 2 + \frac{1}{2^5}(x-16) - \frac{3}{2^{12}}(x-16)^2 + \frac{7}{2^{18}}(x-16)^3$.

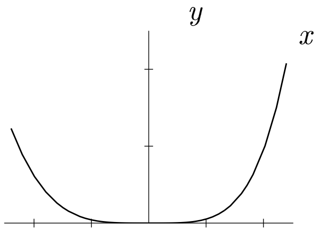
13(b) Taylor's inequality says that $|f(x) - T_3(x)| \le M|x-16|^4/4!$ with M an upper bound for $|f^{(4)}(x)|$ with x in the interval $[16, 18]$. To find an upper bound for $|f^{(4)}(x)|$, we must find x in $[16, 18]$ so that $|f^{(4)}(x)|$ is as large as possible. Since $|f^{(4)}(x)| = (231/4^4)x^{-15/4}$ is decreasing for x in $[16, 18]$, $|f^{(4)}(x)|$ is largest when x is as small as possible, i.e., when $x = 16$. Thus, $|f^{(4)}(x)| \le (231/4^4)16^{-15/4} = (231)/(4^4 \cdot 16^{15/4})$. Since $16^{1/4} = 2$ and $4^4 = 2^8$, $4^4 \cdot 16^{15/4} = 2^8 \cdot 2^{15} = 2^{23}$. Thus, $|f^{(4)}(x)| \le (231)/(2^{23})$ and $M = (231)/(2^{23})$. As x is in $[16, 18]$, $|x-16| \le 2$, and $|x-16|^4 \le 2^4$. Thus, $|f(x) - T_3(x)| \le \frac{M|x-16|^4}{4!} \le \left(\frac{231}{2^{23}}\right)\frac{(2^4)}{(2^3 \cdot 3)} = \frac{77}{2^{22}}$.

14(a) The graph appears at the right.

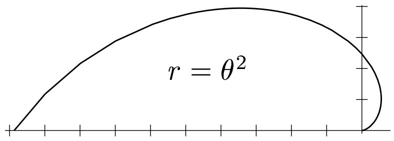
14(b) Since $\frac{dx}{dt} = -3\cos^2 t \sin t$, $\frac{dy}{dt} = 3\sin^2 t \cos t$, $\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{3\sin^2 t \cos t}{-3\cos^2 t \sin t} = -\tan t$.

14(c) At $t = \pi/6$, $x = \cos^3(\pi/6) = 3\sqrt{3}/8$, $y = \sin^3(\pi/6) = 1/8$, and $dy/dx = -\tan(\pi/6) = -1/\sqrt{3}$. Thus, the equation of the tangent line is $y - \frac{1}{8} = \frac{-1}{\sqrt{3}}\left(x - \frac{3\sqrt{3}}{8}\right)$, or $y = \frac{-1}{\sqrt{3}}x + \frac{1}{2}$.

14(d) Here, $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t = 9\sin^2 t \cos^2 t(\cos^2 t + \sin^2 t) = 9\sin^2 t \cos^2 t$. Thus, the portion of the curve in the first quadrant has arc length: $\int_0^{\pi/2} \sqrt{9\sin^2 t \cos^2 t}\,dt = \int_0^{\pi/2} 3\sin t \cos t\,dt = \frac{3}{2}\sin^2 t\Big|_0^{\pi/2} = \frac{3}{2}$. It follows that the total arc length is $4 \cdot (3/2) = 6$.



15. The graph appears at the right.



15(a) The area is: $\int_0^{\pi} r^2/2\,d\theta = \int_0^{\pi} \theta^4/2\,d\theta = \theta^5/10\Big|_0^{\pi} = \pi^5/10$.

15(b) Here, as $r = \theta^2$, $\frac{dr}{d\theta} = 2\theta$. Thus, the arc length is:

$$\int_0^{\pi} \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_0^{\pi} \sqrt{\theta^4 + 4\theta^2}\,d\theta = \int_0^{\pi} \theta\sqrt{\theta^2 + 4}\,d\theta = \frac{(\theta^2+4)^{3/2}}{3}\Big|_0^{\pi} = \frac{(\pi^2+4)^{3/2} - 8}{3}.$$