x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	2π
$\sin x$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0	-1	0

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	2π
$\cos x$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0	-1	0	1

$\sin(2x) = 2\sin x \cos x$
$\cos(2x) = \cos^2 x - \sin^2 x$

$f(x)$	$\int f(x)\,dx$
$x^r, r \neq -1$	$x^{r+1}/(r+1) + C$
x^{-1}	$\ln\lvert x\rvert + C$
e^x	$e^x + C$
$a^x, a \neq 1$	$a^x/(\ln a) + C$

$f(x)$	$\int f(x)\,dx$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x + C$

$f(x)$	$\int f(x)\,dx$
$1/\sqrt{a^2 - x^2}, a \neq 0$	$\sin^{-1}(x/a) + C$
$1/(a^2 + x^2), a \neq 0$	$(1/a)\tan^{-1}(x/a) + C$
$\tan x$	$-\ln\lvert\cos x\rvert + C$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert + C$

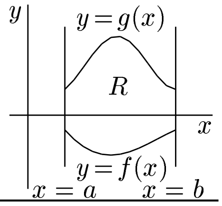
Areas and Volumes of Solids of Rotation, see figure at the right

R is the region bounded by $y = g(x)$, $y = f(x)$, $x = a$, $x = b$, with $g(x) \geq f(x)$ for x in $[a, b]$.

Area of R: $\int_a^b (g(x) - f(x))\,dx$

Volume of solid found by rotating R around the x-axis: $\int_a^b \pi((g(x))^2 - (f(x))^2)\,dx$, $(f(x) \geq 0)$

Volume of solid found by rotating R around the y-axis: $\int_a^b 2\pi\, x(g(x) - f(x))\,dx$, $(a \geq 0)$



Integration by parts: $\int u\,dv = uv - \int v\,du$. Choose dv to be easy to integrate and so that $\int v\,du$ is simpler than $\int u\,dv$.
Use for integrals of: $x^m e^{ax}$, $x^m \sin ax$, $x^m \cos ax$, $x^m \ln x$, $x^m \tan^{-1} x$, $x^m \sin^{-1} x$, $e^{ax} \sin bx$, $e^{ax} \cos bx$, $\sec^m x$, m odd.

$\int \sin^m x \cos^n x\,dx$ Reduce to a sum of integrals of the type: $\int \sin^j x(\cos x\,dx)$ and $\int \cos^k x(\sin x\,dx)$
m odd: Group $\sin x$ with dx. Replace $\sin^{m-1} x$ using $\sin^2 x = 1 - \cos^2 x$. Let $u = \cos x$, $du = -\sin x\,dx$. Expand.
n odd: Group $\cos x$ with dx. Replace $\cos^{n-1} x$ using $\cos^2 x = 1 - \sin^2 x$. Let $u = \sin x$, $du = \cos x\,dx$. Expand.
m, n both even: Use $\sin^2 x = (1 - \cos(2x))/2$, $\cos^2 x = (1 + \cos(2x))/2$. Possibly repeat, or use an earlier case.
EX: $\int \sin^5 x \cos^4 x\,dx = \int \sin^4 x \cos^4 x(\sin x\,dx) = \int (1 - \cos^2 x)^2 \cos^4 x(\sin x\,dx) = \int (1 - u^2)^2 u^4 (-du) = \int (-u^4 + 2u^6 - u^8)du = -(1/5)\cos^5 x + (2/7)\cos^7 x - (1/9)\cos^9 x + C$. EX: $\int \sin^2 x \cos^2 x\,dx = \int (1/4)(1 - \cos(2x))(1 + \cos(2x))dx = (1/4)\int (1 - \cos^2(2x))dx = (1/4)\int (1 - (1/2)(1 + \cos(4x)))dx = (1/8)\int (1 - \cos(4x))dx = (1/8)x - (1/32)\sin(4x) + C$.
$\int \sec^m x \tan^n x\,dx$ Reduce to a sum of integrals of the type: $\int \sec^j x(\sec x \tan x\,dx)$ and $\int \tan^k x(\sec^2 x\,dx)$
m even: Group $\sec^2 x$ with dx. Replace $\sec^{m-2} x$ using $\sec^2 x = 1 + \tan^2 x$. Let $u = \tan x$, $du = \sec^2 x\,dx$. Expand.
n odd: Group $\sec x \tan x$ with dx. Replace $\tan^{n-1} x$ using $\tan^2 x = \sec^2 x - 1$. Let $u = \sec x$, $du = \sec x \tan x\,dx$.
m odd, n even: Use $\tan^2 x = \sec^2 x - 1$ to express as a sum of integrals of $\sec^m x$, m odd. Use integration by parts.

Integrals involving $\sqrt{a^2 - x^2}$. Set $x = a \sin\theta$. Then, $dx = a\cos\theta\,d\theta$, $\sqrt{a^2 - x^2} = a\cos\theta$.
Integrals involving $\sqrt{a^2 + x^2}$. Set $x = a\tan\theta$. Then, $dx = a\sec^2\theta\,d\theta$, $\sqrt{a^2 + x^2} = a\sec\theta$.
EX: $\int x^5 (1 - x^2)^{3/2}\,dx = \int \sin^5\theta \cos^3\theta(\cos\theta\,d\theta) = \int \sin^5\theta \cos^4\theta\,d\theta = -(1/5)\cos^5\theta + (2/7)\cos^7\theta - (1/9)\cos^9\theta + C = -(1/5)(1 - x^2)^{5/2} + (2/7)(1 - x^2)^{7/2} - (1/9)(1 - x^2)^{9/2} + C$

Integrals of rational functions: $\int (f(x)/g(x))\,dx$, with $f(x), g(x)$ polynomials.
(1) If $\deg f(x) \geq \deg g(x)$, divide $g(x)$ into $f(x)$ using long division of polynomials to get quotient $q(x)$ and remainder $r(x)$ with $\deg r(x) < \deg g(x)$. Then $\int (f(x)/g(x))\,dx = \int q(x)\,dx + \int (r(x)/g(x))\,dx$. Use next method on last integral.
(2) If $\deg f(x) < \deg g(x)$, use the method of partial fractions. First factor $g(x)$ into a product of linear factors $x + A$ and quadratic factors $x^2 + Bx + C$ with no real roots, or $g(x) = \underbrace{(x + A)^m \cdots\cdots}_{\text{linear factors}}\underbrace{(x^2 + Bx + C)^m \cdots\cdots}_{\text{quadratic factors}}$.
Then, $f(x)/g(x)$ is a sum of terms:

$$\underbrace{\frac{D_1}{(x + A)} + \frac{D_2}{(x + A)^2} + \cdots + \frac{D_m}{(x + A)^m}}_{m \text{ terms for each linear factor } (x+A)^m} \quad \text{and} \quad \underbrace{\frac{E_1 x + F_1}{(x^2 + Bx + C)} + \frac{E_2 x + F_2}{(x^2 + Bx + C)^2} + \cdots + \frac{E_n x + F_n}{(x^2 + Bx + C)^n}}_{n \text{ terms for each quadratic factor } (x^2+Bx+C)^n}$$

To find the constants D_i, E_j, F_k, multiply both sides by $g(x)$. Then, multiply the terms out and equate the coefficients of the same powers of x on each side of the equation. Finally, solve the resulting system of linear equations.
EX: $\frac{x^2+x+1}{x^2(x-1)(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+E}{x^2+1}$. First multiply both sides by $x^2(x - 1)(x^2 + 1)$ to obtain: $x^2 + x + 1 = Ax(x - 1)(x^2 + 1) + B(x - 1)(x^2 + 1) + Cx^2(x^2 + 1) + Dx^3(x - 1) + Ex^2(x - 1)$, or $x^2 + x + 1 = (A + C + D)x^4 + (-A + B - D + E)x^3 + (A - B + C - E)x^2 + (-A + B)x + (-B)$. Thus, $A + C + D = 0$, $-A + B - D + E = 0$, $A - B + C - E = 1$, $-A + B = 1$, $-B = 1$. Solving, $A = -2$, $B = -1$, $C = 3/2$, $D = 1/2$, $E = -1/2$.
If $g(x)$ is a product of distinct linear factors, there is a short cut. Multiply both sides by $g(x)$ as before.
EX: $\frac{x^2-2}{(x-2)x(x+1)} = \frac{A}{x-2} + \frac{B}{x} + \frac{C}{x+1}$. Then: $x^2 - 2 = Ax(x + 1) + B(x - 2)(x + 1) + C(x - 2)x$. Let x equal each root of $g(x)$: $x = 2, 0, -1$. Then $x = 2$ gives $2 = 6A$; $x = 0$, $-2 = -2B$; $x = -1$, $-1 = 3C$. Thus. $A = 1/3$, $B = 1$, $C = -1/3$.

Suppose the integral $\int_a^b f(x)\,dx$ is approximated by dividing $[a, b]$ into n equal segments. Let K be an upper bound for $\lvert f''(x)\rvert$ on $[a, b]$. The midpoint rule error is at most $K(b - a)^3/24n^2$, the trapezoidal error at most $K(b - a)^3/12n^2$.

$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$ $\quad$ $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$ $\quad$ $\int_1^\infty \frac{dx}{x^a}$ converges if $a > 1$, diverges if $a \leq 1$.
Compar: If $0 \leq f(x) \leq g(x)$, let $I_f = \int_a^\infty f(x)\,dx$, $I_g = \int_a^\infty g(x)\,dx$. If I_g converges, so does I_f. If I_f diverges, so does I_g.
L'Hospital: If $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$. Some limits: If $a > 0$, $\lim_{x\to\infty} x^a = \infty$, $\lim_{x\to\infty} 1/x^a = 0$. If $a > 1$, $\lim_{x\to\infty} a^x = \infty$, $\lim_{x\to\infty} 1/a^x = 0$. $\lim_{x\to\infty} \ln x = \infty$. $\lim_{x\to\pm\infty} \tan^{-1} x = \pm\pi/2$.

Arc length of $y = f(x)$ from $x = a$ to $x = b$ is $\int_a^b \sqrt{1 + (dy/dx)^2}\, dx = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$.

Differential equations–separation of variables. Factor one side of the equation so that $dy/dx = f(x)g(y)$. Then $dy/g(y) = f(x)dx$. Integrate, not forgetting the constant C of integration. Use the initial condition $(x, y) = (x_0, y_0)$ to find C. EX: If $dy/dx = x + xy$, and $y = 2$, if $x = 0$, then $dy/dx = x(1 + y)$, or $dy/(1 + y) = x\, dx$. Integrating, $\ln(1 + y) = x^2/2 + C$. Thus, $1 + y = e^{x^2/2+C}$, or $y = Ae^{x^2/2} - 1$, with $A = e^C$. Set $x = 0$, $y = 2$, $2 = A - 1$ and $y = 3e^{x^2/2} - 1$.

Exponential population growth: $P = P_0 e^{kt}$, where P_0 is the initial population and k is a constant.

$$\lim_{n\to\infty} \frac{n^a}{b^n} = 0, \text{ if } b > 1; \quad \lim_{n\to\infty} \frac{a^n}{n!} = 0; \quad f, g \text{ poly.}, \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{an^j + \text{lower terms}}{bn^k + \text{lower terms}} = \begin{cases} a/b, & \text{if } j = k; \\ 0, & \text{if } j < k; \ \pm\infty, \text{ if } j > k. \end{cases}$$

Geometric series $\sum_{n=0}^{\infty} x^n = 1/(1 - x)$, if $|x| < 1$. $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$, diverges if $p \le 1$.

Divergence Test: The series $\sum a_n$ diverges if either $\lim_{n\to\infty} a_n$ does not exist or exists and is not 0.

Integral Test: Suppose that f is a positive, continuous, decreasing function. Let $a_n = f(n)$ for each positive integer n. Then, the integral $\int_1^\infty f(x)dx$ and the series $\sum_{n=1}^{\infty} a_n$ converge or diverge together. Use for $\sum 1/n^p$, $\sum 1/(n(\ln\, n)^p)$).
EX: For $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$, find $\int_2^\infty \frac{dx}{x\sqrt{\ln(x)}} = \int_{\ln\, 2}^{\infty} \frac{du}{\sqrt{u}} = 2\sqrt{u}\Big|_{\ln\, 2}^{\infty} = \infty$, using $u = \ln\, x$, $du = \frac{dx}{x}$. The series diverges.

Comparison Test: Assume $0 \le a_n \le b_n$. If $\sum b_n$ converges, so does $\sum a_n$. If $\sum a_n$ diverges, so does $\sum b_n$.
EX: Note that $\frac{1}{n^2 \ln\, n} \le \frac{1}{n^2}$, if $n \ge 3$, as $\ln\, n \ge 1$, if $n \ge 3$. Since $\sum \frac{1}{n^2}$ converges, so does $\sum \frac{1}{n^2 \ln\, n}$.

Limit Comparison: If $a_n, b_n > 0$, $\lim_{n\to\infty}(a_n/b_n) = L$, with $L \ne 0, \infty$, $\sum a_n$ and $\sum b_n$ both converge or diverge.
EX: Compare $\sum \frac{\sqrt{n}}{n+1}$ to $\sum \frac{1}{\sqrt{n}}$, since $\left(\frac{\sqrt{n}}{n+1}\right) / \left(\frac{1}{\sqrt{n}}\right) = \frac{n}{n+1} \to 1$, as $n \to \infty$. Since $\sum \frac{1}{\sqrt{n}}$ diverges, both diverge.

Alternating Series: If $\sum_{n=0}^{\infty} (-1)^n a_n$ is a series with $a_n \ge 0$, $a_{n+1} \le a_n$ for all n and $\lim_{n\to\infty} a_n = 0$, the series converges.

Absolute Convergence: The series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges. Then also $\sum a_n$ converges.
Conditional Convergence: The series $\sum a_n$ converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Ratio Test: Suppose that $\lim_{n\to\infty} |a_{n+1}/a_n| = L$. If $L < 1$, $\sum a_n$ converges. If $L > 1$, $\sum a_n$ diverges. If $L = 1$, the test fails. Often useful for series involving a^n or $n!$. For power series, use this to find the radius of convergence.
EX: For $\displaystyle\sum \frac{3^n}{n!}$, $\displaystyle\frac{|a_{n+1}|}{|a_n|} = \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \frac{3}{n+1} \to 0$, as $n \to \infty$. As $L = 0 < 1$, the series converges absolutely.

Series	Test
$\sum \frac{1}{n(ln\, n)^4}$	Integral
$\sum \frac{n^2}{(n+1)2^n}$	Ratio
$\sum \frac{(-1)^n n}{n^2+1}$	Alter. Ser.

Series	Test
$\sum \frac{(n!)^2}{(2n)!}$	Ratio
$\sum \frac{n^2+1}{n^2+n}$	Divergence
$\sum \frac{n^2+n}{n^4+1}$	Lim. Comp. to $\sum \frac{1}{n^2}$

Series	Test
$\sum \frac{1}{(\ln\, n)2^n}$	Comp: $\sum \frac{1}{(\ln\, n)2^n} \le \sum \frac{1}{2^n}$
$\sum \frac{\ln\, n}{n}$	Comp: $\sum \frac{1}{n} \le \sum \frac{\ln\, n}{n}$
$\sum \frac{\sin\, n}{n^2}$	Comp: $\sum \frac{\lvert\sin\, n\rvert}{n^2} \le \sum \frac{1}{n^2}$

Estimating Sums with the Integral Test: The error in approximating $\sum_{n=1}^{\infty} a_n$ with $\sum_{n=1}^{N} a_n$ is at most $\int_N^\infty f(x)dx$.
EX: How large should N be to insure that $\sum_{n=1}^{N} n\, e^{-n^2}$ is within $1/10^{10}$ of $\sum_{n=1}^{\infty} n\, e^{-n^2}$? We need $\int_N^\infty x\, e^{-x^2}\, dx \le 1/10^{10}$. Integrating, $e^{-N^2}/2 \le 1/10^{10}$, or $10^{10}/2 \le e^{N^2}$, or $10\, \ln\, 10 + \ln\, 2 \le N^2$, or $\sqrt{10\, \ln\, 10 + \ln\, 2} \le N$, N an integer.
With Alternating Series: The error in approximating $\sum_{n=0}^{\infty} (-1)^n a_n$ with $\sum_{n=0}^{N} (-1)^n a_n$ has absolute value at most a_{n+1}.

Radius of convergence: For the power series, $\sum c_n(x - a)^n$, there is a number $R \ge 0$ so that if $|x - a| < R$, the series converges absolutely and if $|x - a| > R$, the series diverges. If $|x - a| = R$, the series may converge or diverge. Find R with the ratio test. The interval of convergence I is the set of points where the series converges. I contains the interval $(a - R, a + R)$ and may or may not contain the boundary points $a - R$ and $a + R$.
EX: For $\displaystyle\sum \frac{nx^{2n}}{3^n}$, $\displaystyle\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)|x|^{2n+2}}{3^{n+1}} \cdot \frac{3^n}{n|x|^{2n}} = \left(\frac{n+1}{n}\right) \cdot \frac{|x|^2}{3} \to \frac{|x|^2}{3}$, as $n \to \infty$. The series converges absolutely if $|x|^2/3 < 1$, or $|x|^2 < 3$, or $|x| < \sqrt{3}$. If $|x| = \sqrt{3}$, the series is $\sum n$ and diverges. Then, $R = \sqrt{3}$ and $I = (-\sqrt{3}, \sqrt{3})$.

EX: Use power series integration to find the power series of $\tan^{-1} x$ at $a = 0$. Setting $x = -t^2$ in the geometric series:
$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n} \text{ and } \tan^{-1} x = \int_0^x \frac{dt}{1+t^2} = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n}\, dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

Taylor Polynomials: The n-th Taylor poly. of f is $T_n(x) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2/2! + \cdots + f^n(a)(x-a)^n/n!$, assuming f has n derivatives at a. Also $f(x) = T_n(x) + R_n(x)$, $|R_n(x)| \le M|x-a|^{n+1}/(n+1)!$, with M an upper bound of $|f^{n+1}(t)|$ with t in the interval with endpoints a and x. EX: Find $T_2(x)$ of $f(x) = (1+x)^{1/2}$ at $a = 0$. Estimate $|R_2(x)|$ with x in $[0, .1]$. Here, $f(x) = (1+x)^{1/2}$, $f'(x) = (1/2)(1+x)^{-1/2}$, $f''(x) = (-1/4)(1+x)^{-3/2}$, $f'''(x) = (3/8)(1+x)^{-5/2}$. Then, $f(0) = 1$, $f'(0) = 1/2$, $f''(0) = -1/4$. Thus, $T_2(x) = 1 + x/2 - x^2/8$. Also, $|R_2(x)| \le M|x|^3/3!$, with M an upper bound of $|f'''(t)| = (3/8)(1+t)^{-5/2}$, with t in $[0, .1]$. Since $1 + t$ is increasing, $(3/8)(1+t)^{-5/2}$ is decreasing on $[0, .1]$ and its maximum value occurs at $t = 0$ and is $3/8$. Thus, $M = 3/8$, $|x| \le .1$, and $|R_2(x)| \le (3/8)(.1)^3/6 = (.1)^3/16$.

$e^x = \sum_{n=0}^{\infty} x^n/n!$ $\sin\, x = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)!$ $\cos\, x = \sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n)!$

Parametric curves: If $x = f(t), y = g(t)$, then the slope of the tangent line is given by $\frac{dy}{dx} = (\frac{dy}{dt})/(\frac{dx}{dt}) = \frac{g'(t)}{f'(t)}$. The arc length from $t = a$ to $t = b$ is $\int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}$. Polar Coordinates: $x = r\, \cos\, \theta$, $y = r\, \sin\, \theta$. $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$. The area bounded by $y = f(\theta)$, $\theta = a$, $\theta = b$ is $\int_a^b \frac{1}{2} r^2\, d\theta = \int_a^b \frac{1}{2}(f(\theta))^2\, d\theta$.