ANDERSON LOCALIZATION FOR TIME PERIODIC RANDOM SCHÖDINGER OPERATORS

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Abstract. We prove that at large disorder, Anderson localization in $\mathbf{Z}^d$ is stable under localized time-periodic perturbations by proving that the associated quasi-energy operator has pure point spectrum. The formulation of this problem is motivated by questions of Anderson localization for non-linear Schrödinger equations.

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I. Introduction

Anderson localization for time *independent* random Schrödinger operators at large disorder has been well known since the seminal work of Fröhlich-Spencer [FS]. It is a topic with an extensive literature [GMP, FMSS, vDK, AM, AFHS, AESS], to name a few.

Time-independent random Schrödinger operator is an operator of the form

$$H_0 = \Delta + \gamma V, \tag{1.1}$$

on $L^2(\mathbf{R}^d)$ or $\ell^2(\mathbf{Z}^d)$, where Δ is the continuum or discrete Laplacian, γ is a positive parameter and V is a random potential. We specialize to discrete random Schrödinger operator. H_0 is then defined as the operator:

$$H_0 = \Delta + \gamma V, \text{ on } \ell^2(\mathbf{Z}^d), \tag{1.2}$$

where the matrix element Δ_{ij}, for i, $j \in \mathbf{Z}^d$ verify

$$\begin{aligned} \Delta_{ij} &= 1 \quad |i-j|_{\ell^1} = 1 \\ &= 0 \quad \text{otherwise}; \end{aligned} \tag{1.3}$$

γ is a positive parameter, the potential function V is a diagonal matrix: $V = \operatorname{diag}(v_j)$, $j \in \mathbf{Z}^d$, where $\{v_j\}$ is a family of independently identically distributed (iid) real random variables with distribution g. From now on, we write $|\,|$ for the ℓ^1 norm: $|\,|_{\ell^1}$ on $\mathbf{Z}^d$. We denote ℓ^2 norms by $\|\,\|$. The probability space Ω is taken to be $\mathbf{R}^{\mathbf{Z}^d}$ and the measure P is $\prod_{j\in\mathbf{Z}^d} g(dv_j)$.

Let $\sigma(H_0)$ denote the spectrum of H. The spectrum can be decomposed into $\sigma_{\text{pp}}(H)$, $\sigma_{\text{ac}}(H)$ and $\sigma_{\text{sc}}(H)$, where $\sigma_{\text{pp}}(H)$, the"pure point" spectrum, denotes the *closure* of the set $S(H) = \{\lambda|\lambda \text{ is an eigenvalue of} H\}$, $\sigma_{\text{ac}}(H)$ the absolutely continuous spectrum and $\sigma_{\text{sc}}(H)$ the singular continous spectrum. We have the well established fact that $\sigma(H)$ and its decompositions σ_{pp}, σ_{ac} and σ_{sc} are almost surely constant sets in $\mathbf{R}$, (see e.g., [CFKS,PF]).

Remark. The definition of σ_{pp} is different from the usual definition, e.g., from that in [RS]. This is in order that we have the stability property regarding spectral decomposition as mentioned above.

As is well known, $\sigma(\Delta) = [-2d, 2d]$. Let supp g be the support of g, we know further (see e.g., [CFKS,PF]) that

$$\sigma(H) = [-2d, 2d] + \gamma \operatorname{supp} g \quad a.s. \tag{1.4}$$

The basic result proven in the references mentioned earlier is that under certain regularity conditions on g, for $\gamma >> 1$, and in any dimension d, the spectrum of H_0 is almost surely pure point with exponentially localized eigenfunctions. This is called *Anderson localization*, after the physicist P. W. Andreson [An]. Physically this corresponds to a lack of conductivity due to the localization of electrons. Anderson was the first one to explain this phenomenon on theoretical physics ground.

The study of electron conduction is a many body problem. One needs to take into account the interactions among electrons. This is a hard problem. The operator H_0 defined in (1.2) corresponds to the so called 1-body approximation, where the interaction is approximated by the potential V. The equation governing the system is

$$i\frac{\partial}{\partial t}\psi = (\Delta + \gamma V)\psi \tag{1.5}$$

on $\mathbf{Z}^d \times [0, \infty)$.

This is the usual Schrödinger equation with a random potential. Since V is independent of t, the study of (1.5) could be reduced to the study of spectral properties of H_0. Hence the importance of spectral results on H_0 mentioned earlier.

In this paper, we consider (1.5) perturbed by a bounded, localized (in space), time-periodic potential. We study the equation:

$$i\frac{\partial}{\partial t}\psi = (\Delta + \gamma V + \lambda\mathcal{W})\psi \tag{1.6}$$

on $\mathbf{Z}^d \times [0, \infty)$, where V is as in (1.2), $\{v_j\}$ is a family of (time-independent) i.i.d. random variables; $\mathcal{W} = \mathcal{W}(t, j)$, which we further assume to be of the form:

$$\mathcal{W}(t, j) = \cos 2\pi(\omega t + \theta)W(j)\ (\omega > 0). \tag{1.7}$$

The motivation for studying (1.6) comes from questions of Anderson localization for non-linear Schrödinger equations (see e.g., [DS, FSW]), which in turn is a approximation to the many body problem of electron conduction mentioned earlier.

To proceed further, we assume

(H1) g has bounded support

(H2) g is absolutely continuous with a bounded density $\tilde{g}$:

$$g(dv) = \tilde{g}(v)dv,\ \|\tilde{g}\|_\infty < \infty$$

(H3)

$$|W(j)| \leq Ce^{-b(\log\gamma)|j|}\ (C > 0, b > 0)$$

Remark. To prove Theorem 1.1 below, we only need W to decay exponentially away from the origin. In (H3), the γ dependence of the rate of decay of W is chosen to coincide with the γ dependence of the rate of decay of eigenfunctions of H_0, as our main motivation for studying (1.6) comes from non-linear Schrödinger. We assume (H1) for convenience. In the case g is unbounded, we believe that supplemented with Lifshitz tail arguments (see e.g., [PF]), the proof presented in sect. II and III would go through.

We further note that due to the presence of the parameters γ and λ, without loss, (H1,3) could be replaced by

(H1') $\operatorname{supp} g \subset [-1, 1]$

(H3') $|W(j)| \leq e^{-b(\log\gamma)|j|}\ (b > 0)$

Assume (H2), the precise spectral property of H_0 mentioned earlier is the following (see e.g., [vDK]):

Localization Theorem. *Let I be an interval in $\mathbf{R}$. There exists $m > 0$, such that for sufficiently large γ, with probability 1:*

- *$\sigma(H_0) \cap I$ is pure point,*
- *the eigenfunctions ψ_E corresponding to eigenvalues E in I satisfy*

$$\liminf_{\|i\|\to\infty} -\frac{\log|\psi_E(i)|}{\|i\|} \geq m \log \gamma.$$

Define

$$H(\theta) = H_0 + \lambda \mathcal{W}(\theta), \tag{1.8}$$

the operator that first appeared in the RHS of (1.6). We consider $H(\theta)$ as a family of Hamiltonians depending parametrically on the initial point $\theta \in \mathcal{S}^1$, a unit circle. We define T_t to be the shift operator:

$$T_t\theta = \omega t + \theta, \tag{1.9}$$

and $\theta(t) = T_t\theta \in \mathcal{S}^1$.

Let $U(t, s; \theta)$ $(t,\ s \in \mathbf{R})$ be the corresponding propagator: if at time s, a solution to (1.6) is $\psi(s)$, then

$$\psi(t) = U(t, s; \theta)\psi(s) \tag{1.10}$$

is the solution at time t. For (1.6), using well known arguments, see e.g., [Ho1, Ya], we know that $U(t, s; \theta)$ is unitary and strongly continuous in t and s. Moreover, it satisfies

$$U(t + a, s + a; \theta) = U(t, s; T_a\theta),\ (a \in \mathbf{R}). \tag{1.11}$$

As in the usual construction (see e.g., [Ho1, Ya]), we consider the enlarged space

$$\mathcal{K} = \ell^2(\mathbf{Z}^d) \otimes L^2(\mathcal{S}^1), \tag{1.12}$$

and the one-parameter family of operators $\tilde{U}(t)$ $(t \in \mathbf{R})$ acting on $\Psi \in \mathcal{K}$ by

$$\begin{aligned}[\tilde{U}(t)\Psi](\theta) &= U(0, -t; \theta)[\mathcal{T}_{-t}\Psi](\theta) \\ &= \mathcal{T}_{-t}U(t, 0; \theta)\Psi(\theta)\end{aligned} \tag{1.13}$$

with

$$[\mathcal{T}_{-t}\Psi](\theta) = \Psi(T_{-t}\theta). \tag{1.14}$$

It can be shown that the $\tilde{U}(t)$ here is a strongly continuous family of unitary operators, see e.g., [Ya]. By Stone's theorem, it can therefore be represented as

$$\tilde{U}(t) = e^{-iKt}, \tag{1.15}$$

where

$$K = \frac{\omega}{i}\frac{\partial}{\partial\theta} + \Delta + \gamma V + \lambda \cos 2\pi\theta W \tag{1.16}$$

on $\ell^2(\mathbf{Z}^d) \times L^2(\mathcal{S}^1)$ is the quasi-energy operator. When $t = T = 1/\omega$, the period of the system, $U(T, 0; 0)$ is the Floquet operator. Formally, the generalized eigenvalues and eigenfunctions of $U(T, 0; 0)$ and K are related by (see e.g., [JL])

$$\begin{aligned} K\psi &= \lambda\psi \\ U(T, 0; 0)\phi &= e^{-i\lambda T}\phi \\ \psi(\theta) &= e^{i\lambda\theta}U(\theta, 0; 0)\phi(0)\end{aligned} \tag{1.17}$$

Our main result is

Theorem 1.1. *Assume g satisfies (H1',2) and W satisfies (H3'). Let I be an interval in $\mathbf{R}$. There exists $a > 0$, such that for sufficiently large γ, with probability 1:*

- *$\sigma(K) \cap I$ is pure point,*
- *the eigenfunctions ψ_E corresponding to eigenvalues E in I satisfy*

$$\liminf_{\|i\|\to\infty} -\frac{\log|\psi_E(i)|}{\|i\|} \geq a\log\gamma. \tag{1.18}$$

for all $\theta \in \mathcal{S}^1$.

Remark. Theorem 1.1 is deduced from localization properties of H_0. For related KAM type of method used to study perturbations of dense pure point spectrum, see e.g., [Ho2, TW].

Using (1.13-1.15), Theoorem 1.1 and that any $\phi \in \ell^2(\mathbf{Z}^d)$ can be embedded in $\mathcal{K} = \ell^2(\mathbf{Z}^d) \otimes L^2(\mathcal{S}^1)$ as $\psi = \phi \otimes 1$, we obtain

Corollary 1.2. *Assume g satisfies (H1',2) and W satisfies (H3'). For sufficiently large γ, for all $\phi \in \ell^2(\mathbf{Z}^d)$, all $\epsilon > 0$, there exists $R > 0$, such that*

$$\sup_t \sum_{|j|>R} |(U(t,0;\theta)\phi)(j)|^2 < \epsilon \quad a.s. \tag{1.19}$$

Corollary 1.2 implies that Anderson localization is *stable* under bounded, localized time periodic perturbations. It is a type of quantum stability result. For other results of related interests, see e.g., [Be, Co, Sa].

Finally, we sketch the main ideas to prove Theorem 1.1. As in all other proofs of localization, we use the established mechanism. We follow most closely [vDK]. The operator $K_0 = K(\lambda = 0)$ plays an important role, as in Fourier space:

$$K_0 = 2\pi n\omega + \Delta + \gamma V, \; n = 0, \pm 1, \pm 2... \tag{1.20}$$

We need two ingredients, both probabilistic in nature:

(i) A wegner estimate on regularity of eigenvalue spacing;

(ii) An initial localization estimate on finite volume Green's function.

In sect. II, we prove (i), which can be reduced to an estimate on the number of eigenvalues of K_Λ (K restrcted to a finite set $\Lambda \subset \mathbf{Z}^d$) in a given spectral interval. The bound is obtained by using the Helffer-Sjöstrand representation of $f(K_\Lambda)$ for $f \in C_0^\infty(\mathbf{R})$.

In sect. III, we prove (ii). The initial localization estimate on $(E - K_\Lambda)^{-1}$ is obtained from initial localization estimate on $(E - K_{0,\Lambda})^{-1}$ and localization properties of W in (H3'). The initial estimate on $(E - K_{0,\Lambda})^{-1}$ is provided by *uniform* localization estimates on H_0. The fact that $\sigma(\Delta)$ is bounded for the discrete Laplacian plays an essential role here. In both proofs of (i) and (ii), we use Hilbert-Schmidt properties of $(E - K_\Lambda)^{-1}$ and $(E - K_{0,\Lambda})^{-1}$.

II. Wegner Estimate for K

Recall from sect. I, the quasi-energy operator K:

$$\begin{aligned} K &= \frac{\omega}{i}\frac{\partial}{\partial\theta} + H(\theta) \\ &= \frac{\omega}{i}\frac{\partial}{\partial\theta} + \Delta + \gamma V + \lambda\cos 2\pi\theta W \end{aligned}$$

on $\ell^2(\mathbf{Z}^d)\times L^2(\mathcal{S}^1)$, as defined in (1.16).

To prove Theorem 1.1, we proceed in the usual way. We need a Wegner estimate on regularity of eigenvalue spacing and an initial estimate on the Green's function. Toward that end, let Λ be a finite subset in $\mathbf{Z}^d$. Define

$$\begin{aligned} \Delta_\Lambda(i,j) &= \Delta(i,j) \quad \text{if}\, i,\, j\in\Lambda \\ &= 0 \quad \text{otherwise;} \end{aligned}$$

and

$$K_\Lambda = \frac{\omega}{i}\frac{\partial}{\partial\theta} + \Delta_\Lambda + \gamma V + \lambda\cos 2\pi\theta W$$

on $\ell^2(\Lambda)\times L^2(\mathcal{S}^1)$.

In this section, we prove the Wegner estimate. By using the standard shift (in V) argument, see e.g., the proof of Proposition 3.1 in [W], we know that for $\epsilon << 1$

Proposition 2.1.

$$Prob(dist(E,\sigma(K_\Lambda)\leq\epsilon)\leq C\frac{N_I(E)\epsilon|\Lambda|}{\gamma} \tag{2.1}$$

where $I=(E-1,E+1)$, $N_I(E)$ is the number of eigenvalues in I.

Applying Proposition 2.1 to K_Λ, we have

Lemma 2.2 (Wegner estimate for K_Λ).

$$Prob(dist(E,\sigma(K_\Lambda)\leq\epsilon)\leq C(\frac{\epsilon}{\omega})|\Lambda|^4(\frac{2d+\gamma}{\gamma}) \tag{2.2}$$

where C only depends on λ and the probability distribution g.

Proof. Let $f\in C_0^\infty(\mathbf{R};\mathbf{R}^+)$, $f(x)=1$, $x\in I$, $f(x)=0$ if $x\geq E+2$ or $x\leq E-2$. Then

$$\begin{aligned} N_I(E) &\leq \operatorname{Tr} f(K_\Lambda) \\ &= \operatorname{Tr}\{(\frac{i}{2\pi})\int\partial_{\bar z}\tilde f(z)(z-K_\Lambda)^{-1}d\bar z\wedge dz\} \end{aligned} \tag{2.3}$$

where $\tilde f\in C_0^\infty(\mathbf{C})$ is an almost analytic extension of f, i.e., $\tilde f=f$ on $\mathbf{R}$ and $\partial_{\bar z}\tilde f$ vanishes on $\mathbf{R}$ to infinite order, see e.g., [HS, D]. Let

$$\begin{aligned} \mathcal{W} &= \lambda\cos 2\pi\theta W \\ K_{0,\Lambda} &= K_\Lambda - \mathcal{W} = \frac{\omega}{i}\frac{\partial}{\partial\theta} + \Delta_\Lambda + \gamma V. \end{aligned} \tag{2.4}$$

For simplicity of notation, we write K_0 for $K_{0,\Lambda}$, K for K_Λ. We note that $\frac{\omega}{i}\frac{\partial}{\partial\theta}$ commutes with $H_\Lambda = \Delta_\Lambda + \gamma V$. Passing to the dual variable of θ by Fouries series, (and abusing the notation), we have

$$\mathcal{W} = \lambda(\frac{T_+ + T_-}{2})W \qquad (2.5)$$
$$K_0 = 2\pi n\omega + \Delta_\Lambda + \gamma V \quad n = 0, \pm 1, \pm 2...$$

where $T_\pm$ are unit shift operators on $\mathbf{Z}$:

$$(T_\pm f)(n) = f(n \pm 1). \qquad (2.6)$$

Using the resolvent equation twice, we have

$$\begin{aligned}(z-K)^{-1} &= (z-K_0)^{-1} - (z-K_0)^{-1}\mathcal{W}(z-K_0)^{-1} \\ &\quad + (z-K_0)^{-1}\mathcal{W}(z-K_0)^{-1}\mathcal{W}(z-K)^{-1}.\end{aligned} \qquad (2.7)$$

Since $(z-K_0)^{-1}$ is diagonal in n, $\mathcal{W}$ only has off-diagonal elements, the second term in the RHS of (2.7) is traceless. Substituting (2.7) into (2.3), we then obtain (2.8)

$$\begin{aligned}N_I(E) &\leq \mathrm{Tr}\{(\frac{i}{2\pi})\int \partial_{\bar z}\tilde f(z)(z-K_0)^{-1}d\bar z\wedge dz\} \\ &\quad + \mathrm{Tr}\{(\frac{i}{2\pi})\int \partial_{\bar z}\tilde f(z)(z-K_0)^{-1}\mathcal{W}(z-K_0)^{-1}\mathcal{W}(z-K)^{-1}d\bar z\wedge dz\} \\ &\overset{\mathrm{def}}{=} I_1 + I_2\end{aligned}$$

We first evaluate I_1: Since $\sigma(\Delta) \subset [-2d, 2d]$, we have

$$\sigma(\Delta + \gamma V) \subset [-2d-\gamma, 2d+\gamma]. \qquad (2.9)$$

Recall that $\mathrm{supp}\, f = (E-2, E+2)$. So we can take $\mathrm{supp}\, \Re\tilde f = (E-2, E+2)$; i.e., $\Re z \in (E-2, E+2)$.

When evaluating the trace in I_1, we only need to sum over n, such that

$$\begin{aligned}&|z - 2\pi n\omega| \leq 2d+\gamma+1 \\ &|n - \frac{z}{2\pi\omega}| \leq \frac{1}{2\pi\omega}(2d+\gamma+1) \\ &|n - \frac{E}{2\pi\omega}| \leq \frac{1}{2\pi\omega}(2d+\gamma+3).\end{aligned} \qquad (2.10)$$

Otherwise $z - K_0$ is invertible, the integrand is analytic in z and the integral is 0 for such n by using Stokes' formula. Hence

$$\begin{aligned}I_1 = (\frac{i}{2\pi})\sum_{j\in\Lambda}\sum_{|n-\frac{E}{2\pi\omega}|\leq\frac{1}{2\pi\omega}(2d+\gamma+3)} \\ \int \partial_{\bar z}\tilde f(z)(z - 2\pi n\omega - \Delta - \gamma V)^{-1}(j,j)d\bar z\wedge dz.\end{aligned} \qquad (2.11)$$

As is the standard practice, we split the $d\bar{z}\wedge dz$ integration into $|\Im z|\geq\alpha$ and $|\Im z|\leq\alpha$ for some $\alpha>0$ to be chosen conveniently. So

$$
\begin{aligned}
|I_1| \leq&(\frac{1}{2\pi})|\Lambda|\frac{2(2d+\gamma+3)+1}{2\pi\omega}\\
&\sup_{j\in\Lambda}\Big(\Big|\int_{|\Im z|\geq\alpha}\partial_{\bar{z}}\tilde{f}(z)(z-2\pi n\omega-\Delta-\gamma V)^{-1}(j,j)d\bar{z}\wedge dz\Big|\\
&\qquad+\Big|\int_{|\Im z|\leq\alpha}\partial_{\bar{z}}\tilde{f}(z)(z-2\pi n\omega-\Delta-\gamma V)^{-1}(j,j)d\bar{z}\wedge dz\Big|\Big)\\
=&\frac{|\Lambda|}{4\pi^2\omega}(2(2d+\gamma+3)+1)\big(\mathcal{O}(1)\frac{1}{\alpha}+\mathcal{O}_M(1)|\Im z|^M\frac{1}{|\Im z|}\big)
\end{aligned}
\tag{2.12}
$$

for all $M\in\mathbf{N}^+$, where we used $|\partial_{\bar{z}}\tilde{f}(z)|\leq\mathcal{O}_M(1)|\Im z|^M$ for all M and self-adjointness.

Choosing $M=\alpha=1$, we then obtain

$$
|I_1|\leq\mathcal{O}(1)\frac{|\Lambda|}{\omega}(2d+\gamma),
\tag{2.13}
$$

where $\mathcal{O}(1)$ is uniform in E, ω, d and γ.

We now estimate I_2, where the main complication comes from the term $(z-K)^{-1}$. The only control we have is via $\Im z$. We split the sum over n similar to (2.11). Anticipating ahead, we split the sum into $|n-\frac{E}{2\pi\omega}|\leq\frac{1}{2\pi\omega}(2d+\gamma+3)+1$ and its complement:

$$
\begin{aligned}
&I_2\\
=&(\frac{i}{2\pi})\sum_{j\in\Lambda}\Big(\sum_{|n-\frac{E}{2\pi\omega}|\leq\frac{1}{2\pi\omega}(2d+\gamma+3)+1}\int\partial_{\bar{z}}\tilde{f}(z)[(z-K_0)^{-1}\mathcal{W}(z-K_0)^{-1}\mathcal{W}(z-K)^{-1}](n,j,n,j)d\bar{z}\wedge dz\\
&+\sum_{|n-\frac{E}{2\pi\omega}|>\frac{1}{2\pi\omega}(2d+\gamma+3)+1}\int\partial_{\bar{z}}\tilde{f}(z)[(z-K_0)^{-1}\mathcal{W}(z-K_0)^{-1}\mathcal{W}(z-K)^{-1}](n,j,n,j)d\bar{z}\wedge dz\Big)\\
=&r_1+r_2.
\end{aligned}
\tag{2.14}
$$

r_1 can be estimated in the same way as in (2.12). We write out the kernel:

$$
\begin{aligned}
&[(z-K_0)^{-1}\mathcal{W}(z-K_0)^{-1}\mathcal{W}(z-K)^{-1}](n,j,n,j)\\
=&\sum_{n',n'',j',j''}(z-2\pi n\omega-\Delta-\gamma V)^{-1}(n,j,n,j')\mathcal{W}(n,j',n',j')\\
&\qquad(z-2\pi n'\omega-\Delta-\gamma V)^{-1}(n',j',n',j'')\mathcal{W}(n',j'',n'',j'')\\
&\qquad(z-K)^{-1}(n'',j'',n,j).
\end{aligned}
\tag{2.15}
$$

We note that from (2.5), $|n-n'|=1$, $|n''-n|\leq 2$. Taking $M=3$ instead of 1 and summing over j, j', j'', n, n', n'', we obtain

$$
|r_1|\leq\mathcal{O}(1)\frac{|\Lambda|^3}{\omega}(2d+\gamma).
\tag{2.16}
$$

Estimation of r_2 is different from that of I_1, as a priori we cannot conclude that the integrand is analytic in z for such large n. Instead, we do the following:
(2.17)

$$|r_2| \leq \mathcal{O}(1) \sum_{j,j',j''} \sum_{|n-\frac{E}{2\pi\omega}|\geq \frac{1}{2\pi\omega}(2d+\gamma+3)+1} \int |\partial_{\bar z}\tilde f(z)| \sum_{n',n''} |[(z-K_0)^{-1}\mathcal{W}(z-K_0)^{-1}\mathcal{W}](n,j,n'',j'')| \frac{1}{|\Im z|} d\bar z\wedge dz.$$

Using the fact that $|n-n'|=1$, $|n''-n|\leq 2$, the sum over n, n', n'' is convergent. We obtain

$$|r_2| \leq \mathcal{O}(1)|\Lambda|^3. \tag{2.18}$$

Combining (2.16, 2.18) with (2.13) in (2.8), we have

$$N_I(E) \leq \mathcal{O}(1)\frac{|\Lambda|^3}{\omega}(2d+\gamma), \tag{2.19}$$

where $\mathcal{O}(1)$ is uniform in E, d, Λ, ω and γ. Substituting (2.19) into (2.1), we obtain the lemma. □

III. Initial estimate for localization and proof of Theorem 1.1

The initial estimate for localization for K is deduced from localization estimates on $H_0 = \Delta + \gamma V$:

Proposition 3.1. *There exist $a>0$, $\gamma_0>0$, $L_0>0$, such that if we let $L_{n+1}=L_n^\alpha$ $(1<\alpha<2)$, $i\in\mathbf{Z}^d$, $\Lambda_n=[-L_n,L_n]^d+i$. Then for $\gamma>\gamma_0$, with probability $\geq 1-\frac{1}{L_n^p}$ $(p>2d)$, for all $j_n\in\partial\Lambda_n$, all $E\in[-2d-\gamma,2d+\gamma]$*

$$|(E-H_{\Lambda_n})^{-1}(i,j_n)| \leq Ce^{-a\log\gamma|i-j_n|} \quad (C>0,\ a>0). \tag{3.1}$$

Proof. (3.1) is obtained by patching together the usual localization proof, see e.g., [vDK]. We will thus only mention that aspect. As in all large disorder case, $L_0=\mathcal{O}(1)$. The Wegner estimate for H_0 is:

$$\text{Prob}(\text{dist}(E,\sigma(H_\Lambda)\leq\epsilon) \leq \frac{C\epsilon|\Lambda|}{\gamma}, \tag{3.2}$$

where C only depends on the distribution g, see e.g., [vDK]. To get (3.1) for L_0, we take

$$\epsilon = \mathcal{O}(1)\sqrt{\gamma}. \tag{3.3}$$

So using (3.2), we have

$$\|(E-H_{\Lambda_{L_0}})^{-1}\| \leq \frac{\mathcal{O}(1)}{\sqrt{\gamma}} \tag{3.4}$$

with probability

$$\geq 1 - \frac{\mathcal{O}(1)}{\sqrt{\gamma}}. \tag{3.5}$$

(Recall that $L_0 = \mathcal{O}(1)$.) Further, we see that for E', such that $|E-E'| \leq \frac{1}{2}\mathcal{O}(1)\sqrt{\gamma}$,

$$\|(E - H_{\Lambda_{L_0}})^{-1}\| \leq \frac{2\mathcal{O}(1)}{\sqrt{\gamma}} \tag{3.6}$$

with the same probability as in (3.5). (This step is deterministic.)

So we have that (3.6) holds with probability $\geq 1 - \frac{\mathcal{O}(1)}{\sqrt{\gamma}}$ for an interval of length $\mathcal{O}(1)\sqrt{\gamma}/2$. This is the same $\mathcal{O}(1)$ constant as in (3.4). It is important to note that the $\mathcal{O}(1)$ constant is *independent* of the specific interval.

Using (3.6, 3.5) as our initial input in the localization mechanism and keeping the dependence on γ explicit, we obtain as in [vDK] that

$$|(E - H_{\Lambda_n})^{-1}(i, j_n)| \leq Ce^{-a\log\gamma|i-j_n|} \quad (C > 0,\ a > 0) \tag{3.7}$$

with probability $\geq 1 - \frac{1}{\sqrt{\gamma}L_n^p}$, $(p > 2d)$, for all $E \in I$, with $|I| = \mathcal{O}(1)\sqrt{\gamma}/2$. Dividing $[-2d-\gamma, 2d+\gamma]$ into $2\mathcal{O}(1)\sqrt{\gamma}$ number of intervals of size $\mathcal{O}(1)\sqrt{\gamma}/2$, we obtain the proposition. □

Let $K_{0,\Lambda}$ be defined as in (2.4). We have

Lemma 3.2. *There exist $a > 0$, $L > 0$, such that if $\gamma >> 1$ and if we let $i \in \mathbf{Z}^d$, $\Lambda = [-L, L]^d + i$, then for all $j \in \partial\Lambda$, x, $y \in [0, 1)$*

$$|(E - K_{0,\Lambda})^{-1}(i, x; j, y)| \leq C\Big(\frac{\gamma + 2d}{\omega}\Big)e^{-a\log\gamma|i-j|} \quad (C > 0,\ a > 0) \tag{3.8}$$

with probability $\geq 1 - 1/L^p$ $(p > 2d)$.

Proof. Using Fourier series, we have

$$(E - K_{0,\Lambda})^{-1}(i, x; j, y) = \sum_{n=0,\pm1,\ldots} (E - 2\pi n\omega - H_0)^{-1}(i, j, n)e^{in(x-y)}. \tag{3.9}$$

(3.10)

$$\begin{aligned}
|(E - K_{0,\Lambda})^{-1}(i, j; x - y)| &\leq \sum_{n=0,\pm1,\ldots} |(E - 2\pi n\omega - H_0)^{-1}(i, j, n)| \\
&\leq \sum_{|n-\frac{E}{2\pi\omega}|\leq\frac{1}{\pi\omega}(2d+\gamma)} |(E - 2\pi n\omega - H_0)^{-1}(i, j, n)| \\
&\quad + \sum_{|n-\frac{E}{2\pi\omega}|>\frac{1}{\pi\omega}(2d+\gamma)} |(E - 2\pi n\omega - H_0)^{-1}(i, j, n)| \\
&\leq \mathcal{O}(1)\frac{2d+\gamma}{\omega}e^{-a\log\gamma|i-j|} + \mathcal{O}(1)\frac{1}{\omega}e^{-a\log(2d+\gamma)|i-j|} \\
&\leq C\frac{2d+\gamma}{\omega}e^{-a\log\gamma|i-j|}
\end{aligned}$$

with probability $\geq 1 - 1/L^p$, where we assumed $\gamma >> 1$ and used (3.1) in estimating the first sum and standard elliptic estimate on the second sum. □

In order to prove Proposition 3.4 below, we need a slight generalization of Lemma 3.2, which we state without proof as

Corollary 3.3. *There exist $a > 0$, $L > 0$, such that if $\gamma >> 1$ and if we let $i \in \mathbf{Z}^d$, $\Lambda = [-L, L]^d + i$, then for all j, $j' \in \Lambda$, $|j - j'| \geq L/4$, y, $y' \in [0, 1)$*

$$|(E - K_{0,\Lambda})^{-1}(i, x; j, y)| \leq C(\frac{\gamma + 2d}{\omega})e^{-a \log \gamma |i-j|} \quad (C > 0,\, a > 0) \tag{3.11}$$

with probability $\geq 1 - 1/L^p$ $(p > 2d)$, the p here is not necessarily the same as in Lemma 3.2.

Using assumption (H3') and Proposition 3.1, we are now ready to prove

Proposition 3.4 (Initial estimate for K_Λ). *There exist $\tilde{a} > 0$, $L \in \mathbf{N}^+$, such that if we let $i \in \mathbf{Z}^d$, and $\Lambda = [-L, L|^d + i$, then for $\gamma >> 1$, all $j \in \partial\Lambda$, x, $y \in [0, 1)$*

$$|(E - K_\Lambda)^{-1}(i, x; j, y)| \leq C\frac{\gamma + 2d}{\omega}e^{-\tilde{a} \log \gamma |i-j|} \quad (C > 0,\, \tilde{a} > 0). \tag{3.12}$$

with probability $\geq 1 - \frac{1}{L^p}$ $(p > 2d)$

Proof. We deduce (3.12) from (3.11) by using the resolvent equation, the Wegner estimate in (2.2) and localization property of W in (H3'). Iterating the resolvent equation twice, we have (writing K for K_Λ, K_0 for $K_{0,\Lambda}$):

$$\begin{aligned} (E - K)^{-1}(i, x; j, y) = {} & (E - K_0)^{-1}(i, x; j, y) \\ & -[(E - K_0)^{-1}\mathcal{W}(E - K_0)^{-1}](i, x; j, y) \\ & +[(E - K_0)^{-1}\mathcal{W}(E - K)^{-1}\mathcal{W}(E - K_0)^{-1}](i, x; j, y) \\ & \stackrel{\text{def}}{=} I_1 + I_2 + I_3. \end{aligned} \tag{3.13}$$

We use (3.8) to estimate the first term in the RHS of (3.13). To estimate the second term, we use (3.8, 3.11). Let $\tilde{b} = \epsilon \min(a, b)$ for some $\epsilon > 0$ to be determined later. We write

$$\begin{aligned} I_2 = {} & e^{-\tilde{b} \log \gamma |i-j|} I_2 e^{\tilde{b} \log \gamma |i-j|} \\ \stackrel{\text{def}}{=} {} & e^{-\tilde{b} \log \gamma |i-j|} \tilde{I}_2. \end{aligned} \tag{3.14}$$

We only need to bound $\tilde{I}_2$.

$$\begin{aligned} |\tilde{I}_2| \leq {} & e^{\tilde{b} \log \gamma |i-j|} \sum_k \int dt |[(E - K_0)^{-1}|\mathcal{W}|^{1/2}](i, x; k, t)| \\ & |[|\mathcal{W}|^{1/2}(E - K_0)^{-1}](k, t; j, y)| \\ \leq {} & e^{\tilde{b} \log \gamma |i-j|} \sum_k [\{\int dt[(E - K_0)^{-1}(i, x; k, t)]^2 |W(k)|\}^{1/2} \\ & \{\int dt[(E - K_0)^{-1}(k, t; j, y)]^2 |W(k)|\}^{1/2}] \\ \leq {} & e^{\tilde{b} \log \gamma |i-j|} \sum_k [\{\sum_{n=0,\pm 1,\pm 2\ldots} [(E - 2\pi n\omega - H_0)^{-1}(i, k)]^2 |W(k)|\}^{1/2} \\ & \{\sum_{n'=0,\pm 1,\pm 2\ldots} [(E - 2\pi n'\omega - H_0)^{-1}(k, j)]^2 |W(k)|\}^{1/2}] \\ \leq {} & \mathcal{O}(1)\frac{2d + \gamma}{\omega}|\Lambda|^q \end{aligned} \tag{3.15}$$

for some $q > 0$, with probability $\geq 1 - \frac{1}{L^p}$ $(p > 2d)$, where we estimated the sum over n similar to (3.10), and we used (3.11) and the Wegner estimate for H_0 in (3.2) with $\epsilon = |\Lambda|^{-q}$, q adjusted according to p.

We now estimate I_3. Similar to (3.14), we write

$$\begin{aligned} I_3 =& e^{-\tilde{b}\log\gamma|i-j|} I_3 e^{\tilde{b}\log\gamma|i-j|} \\ \stackrel{\text{def}}{=}& e^{-\tilde{b}\log\gamma|i-j|}\tilde{I}_3. \end{aligned} \tag{3.16}$$

$$\begin{aligned} |\tilde{I}_3| \leq& e^{\tilde{b}\log\gamma|i-j|}\sum_{k,k'}\int dt\int dt' |[(E-K_0)^{-1}|\mathcal{W}|^{1/2}](i,x;k,t)||\mathcal{W}|^{1/2}[(E-K)^{-1}|\mathcal{W}|^{1/2}](k,t;k',t')| \\ & |[|\mathcal{W}|^{1/2}(E-K_0)^{-1}](k',t';j,y)| \\ \leq& e^{\tilde{b}\log\gamma|i-j|}\sum_{k,k'}[\{\int dt\int dt'|W(k)|[(E-K)^{-1}(k,t;k',t')]^2|W(k')|\}^{1/2} \\ & \{\int dt[(E-K_0)^{-1}(i,x;k,t)]^2|W(k)|\}^{1/2} \\ & \{\int dt'[(E-K_0)^{-1}(k',t';j,y)]^2|W(k')|\}^{1/2}] \\ \leq& e^{\tilde{b}\log\gamma|i-j|}\{\sum_{k,k'}\int dt\int dt'|W(k)|[(E-K)^{-1}(k,t;k',t')]^2|W(k')|\}^{1/2} \\ & \{\sum_k\int dt[(E-K_0)^{-1}(i,x;k,t)]^2|W(k)|\}^{1/2}\{\sum_{k'}\int dt'[(E-K_0)^{-1}(k',t';j,y)]^2|W(k')|\}^{1/2} \\ \stackrel{\text{def}}{=}& S_1S_2S_3 \end{aligned} \tag{3.17}$$

S_2, S_3 can be similarly estimated as in (3.15).

$$S_1 \leq \mathcal{O}(1)\|(E-K)^{-1}\|_{\text{HS}}, \tag{3.18}$$

where $\| \|_{\text{HS}}$ denotes the Hilbert-Schmidt norm. From the resolvent equation, we have

$$(E-K)^{-1} = (E-K_0)^{-1} - (E-K_0)^{-1}\mathcal{W}(E-K)^{-1}. \tag{3.19}$$

To estimate the H-S norm, we sum over n similar to (3.10, 3.16). Using (3.2, 2.2), we obtain

$$\begin{aligned} \|(E-K)^{-1}\|_{\text{HS}} &\leq \|(E-K_0)^{-1}\|_{\text{HS}}(1+\lambda\|(E-K)^{-1}\|_{L^2}) \\ &\leq \mathcal{O}(1)|\Lambda|^s\frac{2d+\gamma}{\omega}\lambda \end{aligned} \tag{3.20}$$

for some $s > 0$, with probability $\geq 1 - 1/L^p$ (s depends on p).

Combining the estimates on I_1, I_2 and I_3 in (3.8), (3.14-3.20), adjusting q, s and L, we obtain (3.12). □

Proof of Theorem 1.1. Using Lemma 2.2 and Proposition 3.4, Theorem 1.1 follows via the standard route of localization proofs and polynomial boundedness of generalized eigenfunctions of K, see e.g., [Si]. (See also [vDK], in particular proof of Lemma 3.1.) □

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