Dispersive Analysis of Charge Transfer Models

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Abstract

We prove L^p estimates for charge transfer Hamiltonians, including matrix and inhomogeneous generalizations; such equations appear naturally in the study of multi-soliton systems.

1 Introduction

This paper is devoted to the study of dispersive properties of the model corresponding to the time-dependent charge transfer Hamiltonian

$$H(t) = -\frac{1}{2}\triangle + \sum_{j=1}^{m} V_j(x - \vec{v}_j t)$$

with rapidly decaying smooth potentials $V_j(x)$ and a set of mutually non-parallel constant velocities $\vec{v}_j$. Our main focus is on the L^p decay estimates for the solutions of the time-dependent problem

$$\frac{1}{i}\partial_t \psi + H(t)\psi = 0$$

associated with a charge transfer Hamiltonian $H(t)$. The well-known L^p estimates for the free Schrödinger equation ($H(t) = -\frac{1}{2}\triangle$) on $\mathbb{R}^n$ are

$$\|e^{it\frac{\triangle}{2}} f\|_{L^p} \le C_p\,|t|^{-n(\frac{1}{2}-\frac{1}{p})}\|f\|_{L^{p'}}, \quad p \ge 2, \;\; \frac{1}{p} + \frac{1}{p'} = 1.$$

They imply the Strichartz estimates

$$\|e^{it\frac{\triangle}{2}} f\|_{L^q_t L^r_x} \le C_q \|f\|_{L^2}, \quad 2 \le r, q \le \infty, \;\; \frac{n}{r} + \frac{2}{q} = \frac{n}{2}, \quad n \ge 3 \qquad [GV, KT]$$

Such estimates play a fundamental role, among other things in the theory of nonlinear dispersive equations. The extension of such theories to inhomogeneous problems (either due to curvature, local potentials or coherent structure such as solitons, vortices etc. ...) then motivated the efforts

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to establish the L^p decay estimates for the general *time independent* Schrödinger operators of the type $H = -\frac{1}{2}\triangle + V(x)$. In this case the estimates should take the form

$$\|e^{-itH}P_c(H)\psi_0\|_{L^p} \leq C_p\,|t|^{-n(\frac{1}{2}-\frac{1}{p})}\|\psi_0\|_{L^{p'}} \quad p \geq 2,\ \frac{1}{p}+\frac{1}{p'}=1, \tag{1.1}$$

$$\|e^{-itH}P_c(H)f\|_{L^q_tL^r_x} \leq C_q\|f\|_{L^2} \text{ for } 2 \leq q \leq \infty,\ \frac{n}{r}+\frac{2}{q}=\frac{n}{2}, \quad n \geq 3 \tag{1.2}$$

where $P_c(H)$ is the projection onto the continuous part of the spectrum of the self-adjoint operator H. Its purpose is to prevent the emergence of any bound state, i.e., an L^2 eigenfunctions of H. Under the evolution e^{-itH} such bound states are merely multiplied by oscillating factors and thus do not disperse.

The first approach to the proof of estimates (1.1) was developed by Journé, Soffer, Sogge [JSS]. They used a time dependent method which combined spectral and scattering theory with harmonic analysis. Their method involved splitting solutions into high and low energy parts and using Kato's smoothing and the local energy decay on the corresponding pieces. Later, a stationary method was used by Yajima [Ya1] who proved that the wave operators are L^p-bounded under weaker assumptions on the potential than in [JSS]. His theorem implies the dispersive bounds. See also Weder [We] for results in one dimension $n = 1$ and Yajima [Ya3] for $n = 2$.

Most recently, Rodnianski and Schlag [RS] have addressed the issue of determining the optimal class of potentials $V(x)$ for which the estimates (1.1) and (1.2) hold true. In particular, they obtained $L^1 \to L^\infty$ decay estimates for the scaling invariant "L^p-like" classes of small potentials in dimension $n = 3$. In [NS], Nier and Soffer were able to establish the decay and Strichartz estimates for finite rank perturbations of $H_0 = -\frac{\triangle}{2}$. The Strichartz estimates for the inverse-square potential were obtained by Burq, Planchon, Stalker, and Tahvildar-Zadeh in [BPST] .

The situation in the case of time-dependent potentials $V(t,x)$ becomes more complicated. On a superficial level, this can be attributed to the disappearance of a connection between the decay estimates and spectral theory of a corresponding Hamiltonian. In particular, solutions that remain trapped in a compact region, and thus do not disperse, might still exist. However, they can no longer be easily characterized as corresponding to the bound states of some Hamiltonian as in the case of time independent Hamiltonians.

Let $U(t)$ be the solution operator corresponding to the time-dependent Schrödinger equation

$$\begin{aligned} &\frac{1}{i}\partial_t\psi - \frac{\triangle}{2}\psi + V(t,x)\psi = 0, \\ &\psi|_{t=0} = \psi_0, \end{aligned} \tag{1.3}$$

i.e., $\psi(t,\cdot) = U(t)\psi_0$. Then the desired decay estimates take the form

$$\|U(t)\psi_0\|_{L^p} \leq C_p\,|t|^{-n(\frac{1}{2}-\frac{1}{p})}\|\psi_0\|_{L^{p'}} \quad p \geq 2,\ \frac{1}{p}+\frac{1}{p'}=1 \tag{1.4}$$

holding for an appropriate set of initial data. The first results in this direction were obtained in [RS], where the $L^1 \to L^\infty$ decay estimates for *all* initial data were established in dimension $n = 3$ for a large class of small time-dependent potentials. The smallness assumption rules out the emergence of trapped solutions.

In this paper we prove L^p decay estimates for solutions of a time-dependent Schrödinger equation corresponding to a charge transfer Hamiltonian

$$H(t) = -\frac{1}{2}\triangle + \sum_j V_j(x - \vec{v}_j t), \qquad \vec{v}_j \neq \vec{v}_i,\ i \neq j,$$

see Theorem 2.7 below. Our main motivation comes from the problem of asymptotic stability of noninteracting multi-soliton states. This question refers to solving a NLS

$$i\partial_t \psi + \frac{\triangle}{2}\psi + \beta(|\psi|^2)\psi = 0 \tag{1.5}$$

in $\mathbb{R}^n$, $n \geq 3$ with initial data $\psi_0 = \sum_{j=1}^k w_j(0,\cdot) + R_0$ where w_j are special standing wave solutions called solitons and R_0 is a small perturbation. In our forthcoming paper [RSS] we show that if the solitons are sufficiently separated at time $t = 0$, and if R_0 is sufficiently small in a suitable norm, then the solution ψ evolves like a sum of solitons with time-dependent parameters approaching a limit, plus a radiation term that goes to zero in $L^\infty(\mathbb{R}^3)$. This argument requires linearizing the NLS around the bulk-term, which is given by the sum of the solitons. This leads to the problem of establishing dispersive estimates, which typically means $L^1 \to L^\infty$ decay estimates, for the linear problem. It is well-known that the linearized equation needs to be written as a system in $\delta\psi$ and $\overline{\delta\psi}$ where $\delta\psi$ is the variation of the bulk-term. A first step in the understanding of this problem is to consider the *scalar* model that is closest to the system at hand. It is easy to see that these scalar equations are Schrödinger equations of the charge-transfer type

$$\frac{1}{i}\partial\psi - \frac{\triangle}{2}\psi + \sum_{j=1}^m V_j(\cdot - t\vec{v}_j)\psi = 0, \tag{1.6}$$

with real potentials V_j and distinct vectors $\vec{v}_j \in \mathbb{R}^n$.

This problem has been extensively studied in the literature in connection with the question of asymptotic completeness. Asymptotic completeness for the equation (1.6) had been first established by Yajima [Ya2]. Later, Graf [Gr] has obtained a new proof containing a crucial argument showing that the energy[1] associated with a solution of (1.6) remains bounded in time. For similar results see also the work of Wüller [Wu], and for more general charge transfer models with PDO's, the work of Zielinski [Zi]. Their results imply that for a dense set of initial data, solutions converge to the states represented by radiation and the sum of the bound states, associated with each of the Hamiltonians $H_j = -\frac{\triangle}{2} + V_j(x)$, traveling with velocity $\vec{v}_j$.

In this paper we obtain L^p decay estimates for (1.6), which in particular lead to a very short and self-contained proof of the asymptotic completeness theorem. The idea is to decompose the solution into $k+1$ channels that move (and expand) along with the potentials according to a straight motion. On the j^{th} channel one has well-known dispersive estimates by Journé, Soffer, Sogge [JSS] and Yajima [Ya1] for the equation with the single potential [2] $V_j(\cdot - \vec{v}_j t)$, which dominates on that

[1] Energy here simply means $\|\psi(t)\|_{H^1}$ for a solution $\psi(t)$. While the conservation of the L^2 norm is a trivial consequence of the potentials V_j being real, the energy bound is by no means a simple matter since the problem contains a time-dependent potential.

[2] The results of [JSS] and [Ya1] apply to problems with time-independent potentials. However, the problem with a *single* potential $V_j(x - \vec{v}_j t)$ can be reduced to a problem with the potential $V_j(x)$ by means of a Galilei transform.

channel. One then proceeds to show that the interaction between the channels is weak enough not to destroy this property. This weak interaction is accomplished by distinguishing low from high momenta. In the former case one uses the slow propagation rate, in the latter a variant of Kato's smoothing bound, to derive the desired bounds. As always, we cannot hope to prove dispersive bounds for all initial conditions. In the scalar equation with a single stationary potential this simply means being perpendicular to the bound states. In the charge transfer case one expects this to be replaced with perpendicularity to the (traveling) bound states, or at least that the projection of the solution onto the traveling bound states should go to zero. This is precisely the "asymptotic orthogonality condition" which we use below.

For this type of initial data we establish the decay estimates

$$\|\psi(t)\|_{L^\infty} \le C(n)\,|t|^{-\frac{n}{2}}\|\psi_0\|_{L^1\cap L^2}, \tag{1.7}$$

which, by interpolation with a trivial L^2 bound, also imply the the full range of the L^p estimates

$$\|\psi(t)\|_{L^p} \le C(n,p)\,|t|^{-n(\frac{1}{2}-\frac{1}{p})}\|\psi_0\|_{L^{p'}\cap L^2}, \quad 2 \le p \le \infty \tag{1.8}$$

The presence of the L^2 norm on the right-hand side of (1.7) and (1.8) distinguishes them from the standard $L^{p'} \to L^p$ estimates. This, however, is no more than a technical issue. In fact, our main result is stated as an $L^1 \cap L^2 \to L^2 + L^\infty$ estimate, which, for example, is sufficient for a proof of asymptotic completeness. We then establish a procedure converting this estimate into an $L^1 \cap L^2 \to L^\infty$ bounds, using an argument along the lines of [JSS].

To motivate the results of the second part of this paper, we recall that stability and asymptotic stability of solitons and other coherent structures require analyzing the spectral properties, scattering, and local decay of the linearized matrix operator. A comprehensive description of results in this direction can be found in the book by C. Sulem and P. L. Sulem [SuSu]. In particular, the spectral properties of the linearized operators for NLS solitons were developed in Weinstein [Wein], Shatah, Strauss [ShSt], and Grillakis, Shatah, Strauss [GrShSt]. The linearized theory of other structures was also considered, e.g., vortices in Weinstein, Xin [WeinXi], and Hartree-type solitons in Fröhlich, Tsai, Yau [FTY]. Many of these results are examples to which our L^p theory applies.

On a more technical level, the second part of this paper extends our approach to the full system that one obtains by linearizing (1.5) around a sum of solitons. Several complications arise, even for a system with a single stationary matrix potential: Firstly, these systems are no longer self-adjoint and there is the well-known issue of existence of unstable modes, see Weinstein [Wein]. Secondly, and somewhat related to the aforementioned instabilities, the $L^1 \to L^\infty$ decay estimates are more involved for systems ([JSS] and [Ya1] do not apply).

As far as the latter is concerned, Cuccagna [Cu] recently established the necessary bounds by transferring Yajima's L^p-wave operator approach to the system case. While the wave operators are of course of independent interest, we feel that it is desirable to have a more direct approach to the dispersive estimates for systems. It turns out that one can easily adapt a simple but powerful method of Rauch [R] to systems, which is what we do in Section 7 below, see Theorem 7.4. Rauch's method, which is related to earlier work of Dolph, McLeod, Thoe [DMcLT] and Vainberg [Vai], hinges on analytic continuation of the resolvent across the spectrum as a bounded operator in an exponentially weighted L^2-space. This can only be done for exponentially decreasing potentials, but the potentials arising from linearizing around solitons are of this type (since solitons decay exponentially). It is important to realize, however, that Rauch's method only leads to local L^2-decay, which is inadequate for the nonlinear applications. To obtain L^∞-bounds, we then use an observation of Ginibre [Gin] that allows one to pass from local L^2-decay to L^∞-decay by means of a double application of Duhamel's formula.

After deriving these bounds for systems with a single stationary matrix potentials, we then introduce charge transfer models in the system case, see Definition 8.1. These are of course analogous to the scalar models (1.6) but differ in that one not only needs to translate the potentials according to linear motion, but one also needs to modulate the matrix potentials in a suitable manner. In particular, these matrix potentials have complex off-diagonal entries. We then proceed to apply the aforementioned method of "channel-decomposition" to these matrix charge transfer equations, which yields bounds of the form (1.7). Finally, we extend the L^p decay estimates to the inhomogeneous equation with charge transfer Hamiltonians. We give a general L^p-decay estimate in terms of a properly chosen norm of the inhomogeneous term, see Theorem 6.1.

2 Charge transfer model

Definition 2.1. *By a* charge transfer model *we mean a Schrödinger equation*

$$\frac{1}{i}\partial_t\psi - \frac{1}{2}\triangle\psi + \sum_{\kappa=1}^{m} V_\kappa(x-\vec{v}_\kappa t)\psi = 0 \qquad (2.1)$$
$$\psi|_{t=0} = \psi_0, x\in\mathbb{R}^n,$$

where $\vec{v}_\kappa$ are distinct vectors in $\mathbb{R}^n$, $n\geq 3$, and the real potentials V_κ are such that for every $1\leq\kappa\leq m$,

1. *V_κ has compact support (or fast decay), $V_\kappa, \nabla V_\kappa \in L^\infty$*
2. *0 is neither a zero eigenvalue nor a zero resonance of the operators*

$$H_\kappa = -\frac{1}{2}\triangle + V_\kappa(x).$$

The regularity assumption on the potential can be relaxed considerably without too much effort, but we use the C^1-property for convenience. The assumption $n\geq 3$ will be made throughout. This definition is standard, see [Gr], [Ya2]. The Schrödinger group e^{-itH_κ} is known to satisfy the decay estimates (see Journé, Soffer, Sogge [JSS] and Yajima [Ya1])

$$\|e^{-itH_\kappa}P_c(H_\kappa)\psi_0\|_{L^\infty} \lesssim |t|^{-n/2}\|\psi_0\|_{L^1}. \qquad (2.2)$$

Here $P_c(H_\kappa)$ is the spectral projection onto the continuous spectrum of H_κ. The estimate (2.2) holds under the "no 0 eigenvalue/resonance" assumption 2. above for potentials obeying conditions roughly equivalent to the assumption 1.

Remark 2.2. It follows from the results of Yajima [Ya1] that in dimension $n=3$ the estimate (2.2) holds for any bounded potential with sufficiently fast decay at infinity. In higher dimensions one needs to add assumptions on the derivatives of the potential V_κ:

$$|\nabla^\alpha V_\kappa(x)| \le C_\alpha \langle x\rangle^{-\delta}, \qquad \forall |\alpha| \le \frac{n+1}{2} - 3, \quad \delta > \max\{n+2, \frac{3n}{2} - 2\}, \tag{2.3}$$

$$\mathcal{F}(\langle x\rangle^\sigma V) \in L^{\frac{n-1}{n-2}}, \qquad \sigma > \frac{2(n-2)}{n-1} \tag{2.4}$$

We shall assume that potentials $V_\kappa(x)$ always obey conditions guaranteeing estimate (2.2), e.g. conditions (2.3), (2.4).

Definition 2.3. *Let* $L^2 + L^\infty := \{f : \exists h; g : f = h + g, \|h\|_{L^2} + \|g\|_{L^\infty} < \infty\}$ *with norm*

$$\|f\|_{L^2+L^\infty} := \inf{}_{f=h+g}(\|h\|_{L^2} + \|g\|_{L^\infty})$$

We shall use a weaker version of the decay estimate which follows from (2.2) combined with the unitarity of e^{-itH_κ}, namely

$$\|e^{-itH_\kappa} P_c(H_\kappa)\psi_0\|_{L^\infty+L^2} \lesssim \langle t\rangle^{-n/2} \|\psi_0\|_{L^1\cap L^2} \tag{2.5}$$

where $\langle t\rangle = (1+t^2)^{\frac{1}{2}}$. In the following we shall assume that the number of potentials is $m = 2$ and that the velocities are $\vec{v}_1 = 0, \vec{v}_2 = (1, 0, \dots 0) = \vec{e}_1$. This can be done without loss of generality. An indispensable tool in the study of the charge transfer model are the Galilei transforms

$$\mathfrak{g}_{\vec{v},y}(t) = e^{-i\frac{|\vec{v}|^2}{2}t} e^{-ix\cdot\vec{v}} e^{i(y+t\vec{v})\cdot\vec{p}}, \tag{2.6}$$

cf. [Gr], where $\vec{p} = -i\vec{\nabla}$. These are the quantum analogues of the classical Galilei transforms

$$x \mapsto x - t\vec{v} - y, \quad \vec{p} \mapsto \vec{p} - \vec{v}$$

in the following sense: if f is a Schwartz function, say, such that f and $\hat{f}$ are supported around 0, then $\mathfrak{g}_{\vec{v},y}(t)f$ is supported around $t\vec{v} + y$, and $\widehat{\mathfrak{g}_{\vec{v},y}(t)f}$ is supported around $\vec{v}$. Under $\mathfrak{g}_{\vec{v},y}(t)$ the Schrödinger equation transforms as follows:

$$\mathfrak{g}_{\vec{v},y}(t) e^{it\frac{\triangle}{2}} = e^{it\frac{\triangle}{2}}\, \mathfrak{g}_{\vec{v},y}(0)$$

and moreover, with $H = -\frac{1}{2}\triangle + V$,

$$\psi(t) := \mathfrak{g}_{\vec{v},y}(t)^{-1} e^{-itH}\, \mathfrak{g}_{\vec{v},y}(0)\phi_0, , \qquad \mathfrak{g}_{\vec{v},y}(t)^{-1} = e^{-iy\cdot\vec{v}}\, \mathfrak{g}_{-\vec{v},-y}(t) \tag{2.7}$$

solves

$$\begin{aligned} &\frac{1}{i}\partial_t\psi - \frac{1}{2}\triangle\psi + V(\cdot - t\vec{v} - y)\psi = 0 \\ &\psi|_{t=0} = \phi_0. \end{aligned} \tag{2.8}$$

These properties are not only easy to check, but are of course to be expected in view of the classical interpretation. We will make frequent use of these transformation laws without further notice. Another property that we will use often without further mention is that the transformations

$\mathfrak{g}_{\vec{v},y}(t)$ are isometries on all L^p spaces. Finally, since in our case always $y = 0$, we set $\mathfrak{g}_{\vec{v}}(t) := \mathfrak{g}_{\vec{v},0}(t)$. By (2.7), $\mathfrak{g}_{\vec{e_1}}(t)^{-1} = \mathfrak{g}_{-\vec{e_1}}(t)$ in that case.
We now return to the problem

$$\begin{aligned} &\frac{1}{i}\partial_t\psi - \frac{1}{2}\triangle\psi + V_1\psi + V_2(\cdot - t\vec{e_1})\psi = 0 \\ &\psi|_{t=0} = \psi_0 \end{aligned} \tag{2.9}$$

with V_1, V_2 compactly supported potentials. Let $u_1, \dots, u_m$ and $w_1, \dots, w_\ell$ be the normalized bound states of H_1 and H_2 corresponding to the negative eigenvalues $\lambda_1, \dots, \lambda_m$ and $\mu_1, \dots, \mu_\ell$, respectively (recall that we are assuming that 0 is not an eigenvalue). We denote by $P_b(H_1)$ and $P_b(H_2)$ the corresponding projections onto the bound states of H_1 and H_2, respectively, and let $P_c(H_\kappa) = Id - P_b(H_\kappa)$, $\kappa = 1, 2$. The projections $P_b(H_{1,2})$ have the form

$$P_b(H_1) = \sum_{i=1}^{m} \langle \cdot, u_i \rangle u_i, \qquad P_b(H_2) = \sum_{j=1}^{\ell} \langle \cdot, w_j \rangle w_j.$$

In order to state our main theorem, we need to impose an orthogonality condition in the context of the charge transfer Hamiltonian (2.9).

Definition 2.4. *Let $U(t)\psi_0 = \psi(t, x)$ be the solutions of* (2.9)*. We say that ψ_0 (or also $\psi(t, \cdot)$) is* asymptotically orthogonal *to the bounds states of H_1 and H_2 if*

$$\|P_b(H_1)U(t)\psi_0\|_{L^2} + \|P_b(H_2, t)U(t)\psi_0\|_{L^2} \to 0 \text{ as } t \to +\infty. \tag{2.10}$$

Here

$$P_b(H_2, t) := \mathfrak{g}_{-\vec{e}_1}(t) P_b(H_2)\, \mathfrak{g}_{\vec{e_1}}(t) \tag{2.11}$$

for all times t.

Remark 2.5. Clearly, $P_b(H_2, t)$ is again an orthogonal projection for every t. It gives the projection onto the bound states of H_2 that have been translated to the position of the potential $V_2(\cdot - t\vec{e_1})$. Equivalently, one can think of it as translating the solution of (2.9) from that position to the origin, projection onto the bound states of H_2, and then translating back. The explicit form of $P_b(H_2, t)$ is given by

$$\big(P_b(H_2, t)f\big)(x) = \sum_{j=1}^{\ell} e^{ix_1} w_j(x - t\vec{e_1}) \int_{\mathbb{R}^n} f(y) e^{-iy_1} \overline{w_j(y - t\vec{e_1})}\, dy.$$

We will make little use of this formula, though.

Remark 2.6. It is clear that all ψ_0 that satisfy (2.10) form a closed subspace. This subspace coincides with the space of *scattering states* for the charge transfer problem. The latter is well-defined by Graf's asymptotic completeness result [Gr]. This is discussed in more detail in Section 4.

We now formulate our main result.

Theorem 2.7. *Consider the charge transfer model as in Definition 2.1 with two potentials, cf.* (2.9)*. Let* $U(t)$ *denote the propagator of the equation* (2.9)*. Then for any initial data* $\psi_0 \in L^1 \cap L^2$*, which is asymptotically orthogonal to the bound states of* H_1 *and* H_2 *in the sense of Definition 2.4, one has the decay estimates*

$$\|U(t)\psi_0\|_{L^2+L^\infty} \lesssim \langle t\rangle^{-n/2}\|\psi_0\|_{L^1\cap L^2}. \tag{2.12}$$

An analogous statement holds for any number of potentials, i.e., with arbitrary m *in* (2.1)*.*

We prove this theorem for the case of two potentials, but this is for simplicity only. The same argument also applies to the general case of $m > 2$ (one then needs to split into $m+1$ channels, see below). A more substantial comment has to do with the assumption of compact support of the potentials. Inspection of the argument in the following section reveals that it equally well applies to exponentially decaying potentials, say. But also power decay is allowed, provided it is sufficiently fast. We shall prove (2.12) by means of a bootstrap argument. More precisely, we prove that the *bootstrap assumption*

$$\|U(t)\psi_0\|_{L^2+L^\infty} \le C_0\langle t\rangle^{-n/2}\|\psi_0\|_{L^1\cap L^2} \quad \text{for all} \quad 0 \le t \le T \tag{2.13}$$

implies that

$$\|U(t)\psi_0\|_{L^2+L^\infty} \le \frac{C_0}{2}\langle t\rangle^{-n/2}\|\psi_0\|_{L^1\cap L^2} \quad \text{for all} \quad 0 \le t \le T. \tag{2.14}$$

Here C_0 is some sufficiently large positive constant which is assumed to be much bigger than any universal constant appearing in our calculations (e.g. $\|V_{1,2}\|_{L^1\cap L^\infty}$, implicit constants in Proposition 3.1 below, and decay bounds (2.5)). The logic here is that for arbitrary but fixed T, the assumption (2.13) can be made to hold for some C_0, depending on T. Iterating the implication (2.13) $\Longrightarrow$ (2.14) then yields a constant that does not depend on T, thus proving the theorem. We shall continue to use the notation $\lesssim$ to denote bounds involving multiplicative constants independent of the constant C_0.

Remark 2.8. Since

$$\|U(t)\psi_0\|_{L^2+L^\infty} \le \|U(t)\psi_0\|_{L^2} = \|\psi_0\|_{L^2} \le \|\psi_0\|_{L^1\cap L^2}$$

it suffices to prove (2.14) for $t \ge t_0 := \left(\frac{C_0}{2}\right)^{2/n}$.

Remark 2.9. For the nonlinear applications it might be also useful to have estimates as in Theorem 2.7 for perturbed charge transfer Hamiltonians. It is easy to see that the method of proof is stable under such small perturbations, see Section 3.7 below.

3 Proof of the decay estimates

3.1 Bound states

Our first result concerns the rate of convergence of the projections onto the bound states of solutions which are asymptotically orthogonal to the bound states.

Proposition 3.1. *Let $\psi(t,x) = (U(t)\psi_0)(x)$ be a solution of (2.9) which is asymptotically orthogonal to the bound states of H_1 and H_2 in the sense of Definition 2.4. Then*

$$\|P_b(H_1)U(t)\psi_0\|_{L^2} + \|P_b(H_2,t)U(t)\psi_0\|_{L^2} \lesssim e^{-\alpha t}\|\psi_0\|_{L^2}$$

for some $\alpha > 0$.

Proof. By symmetry it suffices to prove the bound on the first part. More precisely, let $\tilde{U}(t) := \mathfrak{g}_{\vec{e_1}}(t)U(t)$, and $\phi(t) = \tilde{U}(t)\psi_0$. Then $\phi(t)$ solves

$$\begin{aligned} &\frac{1}{i}\partial_t\phi - \frac{1}{2}\triangle\phi + V_1(\cdot + t\vec{e_1})\phi + V_2\phi = 0, \\ &\phi|_{t=0}(x) = (\,\mathfrak{g}_{\vec{e_1}}(0)\psi_0)(x), \end{aligned} \tag{3.1}$$

and

$$\|P_b(H_2,t)U(t)\psi\|_2 = \|P_b(H_2)\,\mathfrak{g}_{\vec{e_1}}(t)U(t)\psi_0\|_2 = \|P_b(H_2)\tilde{U}(t)\psi_0\|_2.$$

This clearly allows one to reduce the treatment of H_2 to that of H_1. Decompose

$$U(t)\psi_0 = \sum_{i=1}^{m} a_i(t)u_i + \psi_1(t,x) \tag{3.2}$$

relative to H_1 so that $\psi_1(t,\cdot)$ lies in the continuous subspace of H_1, i.e., $P_c(H_1)\psi_1 = \psi_1$ and $P_b(H_1)\psi_1 = 0$. By assumption,

$$\sum_{i=1}^{m} |a_i(t)|^2 \to 0 \text{ as } t \to \infty.$$

Substituting (3.2) into (2.9) yields

$$\begin{aligned} &\frac{1}{i}\partial_t\psi_1 - \frac{1}{2}\triangle\psi_1 + V_1\psi_1 + V_2(\cdot - t\vec{e_1})\psi_1 + \\ &+ \sum_{j=1}^{m}\left[\frac{1}{i}\dot{a}_j(t)u_j - \frac{1}{2}\triangle u_j a_j(t) + V_1\,u_j a_j(t) + V_2(\cdot - t\vec{e_1})u_j a_j(t)\right] = 0. \end{aligned} \tag{3.3}$$

Since $P_c(H_1)\psi_1 = \psi_1$ one has

$$(-\frac{1}{2}\triangle + V_1)\psi_1 = H_1\psi_1 = P_c(H_1)H_1\psi_1, \qquad \partial_t\psi_1 = P_c(H_1)\partial_t\psi_1.$$

In particular

$$P_b(H_1)\left(\frac{1}{i}\partial_t\psi_1 - \frac{1}{2}\triangle\psi_1 + V_1\psi_1\right) = 0.$$

Thus taking an inner product of the equation (3.3) with u_κ and using the fact that $\langle u_\kappa, u_j\rangle = \delta_{j\kappa}$ as well as the identity

$$-\frac{1}{2}\triangle u_j + V_1 u_j = \lambda_j u_j$$

we obtain the ODE

$$\frac{1}{i}\dot{a}_\kappa(t) + \lambda_\kappa a_\kappa(t) + \langle V_2(\cdot - t\vec{e_1})\psi_1, u_\kappa\rangle + \sum_{j=1}^{m} a_j(t)\langle V_2(\cdot - t\vec{e_1})u_j, u_\kappa\rangle = 0$$

for each a_κ with the condition that

$$a_\kappa(t) \to 0 \text{ as } t \to +\infty.$$

Recall that u_κ is an eigenfunction of $H_1 = -\frac{1}{2}\triangle + V_1$ with eigenvalue $\lambda_\kappa < 0$. It is well-known (see e.g. Agmon [Ag]) that such eigenfunctions are exponentially localized, i.e.,

$$\int_{\mathbb{R}^n} e^{2\alpha|x|}\,|u_\kappa(x)|^2\,dx \le C = C(V_1, n) < \infty \text{ for some positive } \alpha. \tag{3.4}$$

Therefore,

$$\|V_2(\cdot - t\vec{e_1})u_\kappa\|_2 \lesssim e^{-\alpha t} \quad \text{for all} \quad t \ge 0 \tag{3.5}$$

which follows from the assumption that V_2 has compact support. The implicit constant in (3.5) depends on the size of the support of V_2 and $\|V_2\|_{L^\infty}$. Therefore,

$$f_\kappa(t) := \langle V_2(\cdot - t\vec{e_1})\psi_1, u_\kappa\rangle$$

satisfies

$$|f_\kappa(t)| \lesssim e^{-\alpha t}\|\psi_1\|_{L^2} \lesssim e^{-\alpha t}\|P_c(H_1)U(t)\psi_0\|_{L^2} \lesssim e^{-\alpha t}\|\psi_0\|_2. \tag{3.6}$$

In view of (3.1), a_κ solves the equation

$$\begin{aligned} &\frac{1}{i}\dot{a}_\kappa(t) + \lambda_\kappa a_\kappa(t) + \sum_{j=1}^m a_j(t)C_{j\kappa}(t) + f_\kappa(t) = 0 \\ &a_\kappa(\infty) = 0, \end{aligned} \tag{3.7}$$

where $C_{j\kappa}(t) = C_{\kappa j}(t) = \langle V_2(\cdot - t\vec{e_1})u_j, u_\kappa\rangle$. By (3.5), $\max_{j,\kappa}|C_{j\kappa}(t)| \lesssim e^{-\alpha t}$. Solving (3.7) explicitly we obtain

$$\vec{a}(t) = ie^{-i\int_0^t B(s)\,ds}\int_t^\infty e^{i\int_0^s B(\tau)\,d\tau}\vec{f}(s)\,ds,$$

where $B_{j\kappa}(t) = \lambda_j\delta_{j\kappa} + C_{j\kappa}(t)$. Hence, by unitarity of $e^{i\int_0^s B(\tau)\,d\tau}$ and (3.6), we conclude that

$$|\vec{a}(t)| \le \int_t^\infty |\vec{f}(s)|\,ds \lesssim \alpha^{-1}e^{-\alpha t}\|\psi_0\|_{L^2}.$$

This proves the proposition. □

3.2 The three channels

Fix some small $\delta > 0$. We introduce a partition of unity associated with the sets

$$B_{\delta t}(0) = \{x : |x| \le \delta t\}, \quad B_{\delta t}(t\vec{e_1}) = \{x : |x - t\vec{e_1}| \le \delta t\}$$

and

$$\mathbb{R}^n \setminus (B_{\delta t}(0) \cup B_{\delta t}(t\vec{e_1})).$$

Namely, let $\chi_1(t,x)$ be a cut-off function s.t.

$$\chi_1(t,x) = 1, \quad x \in B_{\delta t}(0) \ \text{ and } \ \chi_1(t,x) = 0, \quad x \in \mathbb{R}^n \setminus B_{2\delta t}(0).$$

Here $t \geq t_0$ and $\delta > 0$ is a fixed small constant. Define

$$\chi_2(t,x) = \chi_1(t, x - t\vec{e_1}), \quad \chi_3(t,x) = 1 - \chi_1(t,x) - \chi_2(t,x).$$

Observe that since $t_0 = \left(\frac{C_0}{2}\right)^{2/n}$ is large, see Remark 2.8, the support of $\chi_1(t,\cdot)$ contains the support of V_1 for all $t \geq t_0$ and the support of $\chi_2(t,\cdot)$ contains the support of $V_2(\cdot - t\vec{e_1})$.
There is the following natural decomposition of the solution $U(t)\psi_0$:

$$\begin{aligned} U(t)\psi_0 &= \chi_1(t,\cdot)P_b(H_1)U(t)\psi_0 + \chi_1(t,\cdot)P_c(H_1)U(t)\psi_0 + \chi_2(t,\cdot)P_b(H_2,t)U(t)\psi_0 \\ &\quad + \chi_2(t,\cdot)P_c(H_2,t)U(t)\psi_0 + \chi_3(t,\cdot)U(t)\psi_0. \end{aligned}$$

The terms $\chi_1 P_b(H_1)U(t)\psi_0$ and $\chi_2 P_b(H_2,t)U(t)\psi_0$ can be estimated immediately using Proposition 3.1. Since $U(t)\psi_0$ is asymptotically orthogonal to the bound states of both H_1 and H_2 by assumption, one concludes that

$$\|\chi_1(t,\cdot)P_b(H_1)U(t)\psi_0\|_{L^2+L^\infty} + \|\chi_2(t,\cdot)P_b(H_2,t)U(t)\psi_0\|_{L^2+L^\infty} \lesssim e^{-\alpha t}\|\psi_0\|_{L^1\cap L^2}.$$

To estimate the remaining three terms

$$\chi_1(t,\cdot)P_c(H_1)U(t)\psi_0, \quad \chi_2(t,\cdot)P_c(H_2,t)U(t)\psi_0, \quad \chi_3(t,\cdot)U(t)\psi_0$$

we use Duhamel's formula. However, on each of the three "channels" we compare $U(t)\psi_0$ to a different group. We claim that on the support of $\chi_1(t,\cdot)$ the solution $U(t)\psi_0$ is best approximated by $e^{-itH_1}\psi_0$, on the support of $\chi_2(t,\cdot)$ by $e^{-itH_2}\psi_0$, and on $\chi_3(t,\cdot)$ by $e^{it\frac{\triangle}{2}}\psi_0$. More precisely, with $\chi_1 = \chi_1(t,x)$,

$$\chi_1 P_c(H_1)U(t)\psi_0 = \chi_1 e^{-itH_1}P_c(H_1)\psi_0 - i\chi_1\int_0^t e^{-i(t-s)H_1}P_c(H_1)V_2(\cdot - s\vec{e_1})U(s)\psi_0\,ds. \tag{3.8}$$

To analyze the second channel, define $\tilde{U}(t) := \mathfrak{g}_{\vec{e_1}}(t)U(t)\,\mathfrak{g}_{-\vec{e_1}}(0)$. Then $\phi(t) := \tilde{U}(t)\phi_0$ solves equation (3.1), where $\phi_0 := \mathfrak{g}_{\vec{e_1}}(0)\psi_0$. The Duhamel formula relative to e^{-itH_2} can now be written in two different ways, namely with respect to a stationary frame (this involves $\tilde{U}(t)$) or with respect to the moving frame (this involves $U(t)$). Indeed,

$$\chi_1 P_c(H_2)\tilde{U}(t)\phi_0 = \chi_1 e^{-itH_2}P_c(H_2)\phi_0 - i\chi_1\int_0^t e^{-i(t-s)H_2}P_c(H_2)V_1(\cdot + s\vec{e_1})\tilde{U}(s)\phi_0\,ds \tag{3.9}$$

$$\begin{aligned} \chi_2 P_c(H_2,t)U(t)\psi_0 &= \chi_2\,\mathfrak{g}_{-\vec{e_1}}(t)e^{-itH_2}P_c(H_2)\,\mathfrak{g}_{\vec{e_1}}(0)\psi_0 \\ &\quad - i\chi_2\,\mathfrak{g}_{-\vec{e_1}}(t)\int_0^t e^{-i(t-s)H_2}P_c(H_2)V_1(\cdot + s\vec{e_1})\,\mathfrak{g}_{\vec{e_1}}(s)U(s)\psi_0\,ds. \end{aligned} \tag{3.10}$$

The stationary formulation (3.9) is basically the same as (3.8), whereas (3.10) follows from (3.9) by means of (2.11) from Definition 2.4. It is evident from (3.8) and (3.9) that it suffices to prove the requisite amount of decay for $\chi_1 P_c(H_1)U(t)\psi_0$. However, we will need (3.10) in the treatment of the first channel, i.e., (3.8). Finally, for the third (the "free") channel, one has

$$\chi_3 U(t)\psi_0 = \chi_3\, e^{it\frac{\triangle}{2}}\psi_0 - i\chi_3\int_0^t e^{i(t-s)\frac{\triangle}{2}}(V_1 + V_2(\cdot - s\vec{e_1}))U(s)\psi_0\,ds. \tag{3.11}$$

3.3 Analysis of $\chi_1 P_c(H_1)U(t)\psi_0$

We start with the following simple lemma.

Lemma 3.2. *Let $W \in L^p \cap L^{\frac{2p}{2-p}}$ for some $p \in [1,2]$. Then*

$$\|Wf\|_{L^p} \lesssim \|W\|_{L^p \cap L^{\frac{2p}{2-p}}} \|f\|_{L^2+L^\infty}.$$

In particular, the dual space of $L^2 + L^\infty$ is $L^1 \cap L^2$.

Fix sufficiently large constants $A, B > 0, B >> A$, which are independent of the constant C_0 and write the following expansion

$$\begin{aligned} \chi_1 P_c(H_1)U(t)\psi_0 &= \chi_1 e^{-itH_1} P_c(H_1)\psi_0 - i\chi_1 \int_0^{t-A} e^{-i(t-s)H_1} P_c(H_1) V_2(\cdot - s\vec{e_1})U(s)\psi_0 \, ds \\ &\quad - i\chi_1 \int_{t-A}^{t} e^{-i(t-s)H_1} P_c(H_1) V_2(\cdot - s\vec{e_1})U(s)\psi_0 \, ds. \end{aligned} \tag{3.12}$$

According to (2.5)

$$\|e^{-itH_1} P_c(H_1)\psi_0\|_{L^2+L^\infty} \lesssim \langle t\rangle^{-n/2} \|\psi_0\|_{L^1\cap L^2}. \tag{3.13}$$

Using (2.5), Lemma 3.2, and the bootstrap assumption (2.13)

$$\begin{aligned} &\Big\| \int_0^{t-A} e^{-i(t-s)H_1} P_c(H_1) V_2(\cdot - s\vec{e_1})U(s)\psi_0 \, ds \Big\|_{L^2+L^\infty} \\ &\lesssim \int_0^{t-A} \frac{1}{\langle t-s\rangle^{n/2}} \|V_2(\cdot - s\vec{e_1})U(s)\psi_0\|_{L^1\cap L^2} \, ds \\ &\lesssim \int_0^{t-A} \frac{1}{\langle t-s\rangle^{n/2}} \Big[\|V_2\|_{L^1\cap L^2} + \|V_2\|_{L^2\cap L^\infty} \Big] \|U(s)\psi_0\|_{L^2+L^\infty} \, ds \\ &\lesssim C_0 \int_A^{t-A} \frac{1}{\langle t-s\rangle^{n/2}} \frac{1}{\langle s\rangle^{n/2}} \, ds \; \|V_2\|_{L^1\cap L^\infty} \|\psi_0\|_{L^1\cap L^2} \\ &\qquad + \int_0^A \frac{1}{\langle t-s\rangle^{n/2}} \, ds \|V_2\|_{L^1\cap L^\infty} \|U(s)\psi_0\|_2 \\ &\lesssim C_0 \, A^{-(n/2-1)} \langle t\rangle^{-n/2} \|V_2\|_{L^1\cap L^\infty} \|\psi_0\|_{L^1\cap L^2} + A\langle t\rangle^{-\frac{n}{2}} \|\psi_0\|_2. \end{aligned} \tag{3.14}$$

Thus, since $n \geq 3$, there exists a choice of a sufficiently large constant A that does not depend on C_0 such that

$$\Big\| \int_0^{t-A} e^{-i(t-s)H_1} P_c(H_1) V_2(\cdot - s\vec{e_1})U(s)\psi_0 \, ds \Big\|_{L^2+L^\infty} \leq 10^{-2} C_0 \langle t\rangle^{-n/2} \|\psi_0\|_{L^1\cap L^2}, \tag{3.15}$$

provided C_0 was chosen sufficiently large. Recall that $t \geq t_0 = (C_0/2)^{2/n}$. We can therefore assume that $t >> A$. The third term on the right hand-side of (3.12) is treated as follows. With

$\chi_1 = \chi_1(t,x)$,

$$\chi_1 \int_{t-A}^t e^{-i(t-s)H_1} P_c(H_1) V_2(\cdot - s\vec{e_1}) U(s)\psi_0 \, ds$$
$$= \chi_1 \int_{t-A}^t e^{-i(t-s)H_1} P_c(H_1) V_2(\cdot - s\vec{e_1}) P_b(H_2,s) U(s)\psi_0 \, ds$$
$$+\chi_1 \int_{t-A}^t e^{-i(t-s)H_1} P_c(H_1) V_2(\cdot - s\vec{e_1}) P_c(H_2,s) U(s)\psi_0 \, ds =: J_b + J_c. \tag{3.16}$$

Since $U(s)\psi_0$ is asymptotically orthogonal to the bound states of both H_1 and H_2 by assumption, Proposition 3.1 implies that

$$\|P_b(H_2,s)U(s)\psi_0\|_{L^2} \lesssim e^{-\alpha s}\|\psi_0\|_{L^2}.$$

Therefore, using unitarity of $e^{-i(t-s)H_1}$ and L^2 boundedness of the spectral projection $P_c(H_1)$ we obtain

$$\begin{aligned}
\|J_b\|_{L^2+L^\infty} &\leq \|J_b\|_{L^2} \lesssim \int_{t-A}^t \|V_2(\cdot - s\vec{e_1}) P_b(H_2,s) U(s)\psi_0\|_{L^2} \, ds \\
&\lesssim \|V_2\|_{L^\infty} \int_{t-A}^t \|P_b(H_2,s)U(s)\psi_0\|_{L^2} \, ds \\
&\lesssim \|V_2\|_{L^\infty} \int_{t-A}^t e^{-\alpha s} ds \|\psi_0\|_{L^2} \lesssim \|V_2\|_{L^\infty} A e^{-\alpha t/2} \|\psi_0\|_{L^1\cap L^2},
\end{aligned} \tag{3.17}$$

assuming as we may that $t - A \geq t/2$. This implies the bound

$$\|J_b\|_{L^2+L^\infty} \lesssim \langle t\rangle^{-n/2}\|\psi_0\|_{L^1\cap L^2}. \tag{3.18}$$

To deal with the term J_c in (3.16) we expand $\chi_2(s,\cdot)P_c(H_2,s)U(s)\psi_0$ via the Duhamel formula (3.10). Then

$$\begin{aligned}
J_c &= \chi_1 \int_{t-A}^t e^{-i(t-s)H_1} P_c(H_1) V_2(\cdot - s\vec{e_1}) \, \mathfrak{g}_{-\vec{e_1}}(s) e^{-isH_2} P_c(H_2) \, \mathfrak{g}_{\vec{e_1}}(0)\psi_0 \, ds \\
&\quad -i\chi_1 \int_{t-A}^t \int_0^s e^{-i(t-s)H_1} P_c(H_1) V_2(\cdot - s\vec{e_1}) \, \mathfrak{g}_{-\vec{e_1}}(s) e^{-i(s-\tau)H_2} P_c(H_2) \\
&\qquad\qquad V_1(\cdot + \tau\vec{e_1}) \, \mathfrak{g}_{\vec{e_1}}(\tau) U(\tau)\psi_0 \, d\tau ds \\
&=: J_c^1 + J_c^2.
\end{aligned} \tag{3.19}$$

Using (2.5) for $\kappa = 2$ and Lemma 3.2 we estimate J_c^1 as follows:

$$\begin{aligned}
\|J_c^1\|_{L^2+L^\infty} &\leq \|J_c^1\|_{L^2} \lesssim \int_{t-A}^t \left\| V_2(\cdot - s\vec{e_1}) \, \mathfrak{g}_{-\vec{e_1}}(s) e^{-isH_2} P_c(H_2) \, \mathfrak{g}_{\vec{e_1}}(0)\psi_0 \right\|_{L^2} ds \\
&\lesssim \|V_2\|_{L^2\cap L^\infty} \int_{t-A}^t \| \, \mathfrak{g}_{-\vec{e_1}}(s) e^{-isH_2} P_c(H_2) \, \mathfrak{g}_{\vec{e_1}}(0)\psi_0\|_{L^2+L^\infty} \, ds \\
&\lesssim \|V_2\|_{L^2\cap L^\infty} \int_{t-A}^t \frac{1}{\langle s\rangle^{n/2}} \|\psi_0\|_{L^1\cap L^2} \, ds \lesssim \|V_2\|_{L^2\cap L^\infty} A \langle t\rangle^{-n/2} \|\psi_0\|_{L^1\cap L^2}.
\end{aligned} \tag{3.20}$$

To pass to (3.20) one uses that

$$\|P_c(H_2)f\|_{L^1\cap L^2} \le \|f\|_{L^1\cap L^2} + \|P_b(H_2)f\|_{L^1\cap L^2} \lesssim \|f\|_{L^1\cap L^2}.$$

Since A is a fixed constant independent of the constant C_0 we conclude that

$$\|J_c^1\|_{L^2+L^\infty} \lesssim \langle t\rangle^{-n/2}\|\psi_0\|_{L^1\cap L^2}. \tag{3.21}$$

We decompose J_c^2 as follows:

$$\begin{aligned} J_c^2 \;=\; & \chi_1(t,\cdot)\int_{t-A}^{t}\int_0^{s-B} e^{-i(t-s)H_1}P_c(H_1)V_2(\cdot - s\vec{e_1})\,\mathfrak{g}_{-\vec{e_1}}(s)e^{-i(s-t)H_2}P_c(H_2) \\ & \qquad V_1(\cdot+\tau\vec{e_1})\,\mathfrak{g}_{\vec{e_1}}(\tau)U(\tau)\psi_0\; d\tau ds \\ & +\chi_1(t,\cdot)\int_{t-A}^{t}\int_{s-B}^{s} e^{-i(t-s)H_1}P_c(H_1)F(|\vec{p}|\le M)V_2(\cdot - s\vec{e_1}) \\ & \qquad \mathfrak{g}_{-\vec{e_1}}(s)e^{-i(s-\tau)H_2}P_c(H_2)V_1(\cdot+\tau\vec{e_1})\,\mathfrak{g}_{\vec{e_1}}(\tau)U(\tau)\psi_0\; d\tau ds \\ & +\chi_1(t,\cdot)\int_{t-A}^{t}\int_{s-B}^{s} e^{-i(t-s)H_1}P_c(H_1)F(|\vec{p}|\ge M)V_2(\cdot - s\vec{e_1}) \\ & \qquad \mathfrak{g}_{-\vec{e_1}}(s)e^{-i(s-\tau)H_2}P_c(H_2)V_1(\cdot+\tau\vec{e_1})\,\mathfrak{g}_{\vec{e_1}}(\tau)U(\tau)\psi_0\; d\tau ds \\ & =: J_c^{2,\tau} + J_c^{2,\text{ low}} + J_c^{2,\text{ high}}. \end{aligned}$$

Here B is a constant independent of C_0 so that $B >> A$, $M >> B$, and $F(|\vec{p}| \le M), F(|\vec{p}| \ge M)$ denote smooth projections onto frequencies $|\vec{p}| \le M, |\vec{p}| \ge M$, respectively. The term $J_c^{2,\tau}$ is estimated similarly to J_1^c above. Indeed, using that $V_1(\cdot+\tau\vec{e_1})\,\mathfrak{g}_{\vec{e_1}}(\tau) = \mathfrak{g}_{\vec{e_1}}(\tau)V_1$, one has

$$\begin{aligned} \|J_c^{2,\tau}\|_{L^2+L^\infty} \;\le\; & \|J_c^{2,\tau}\|_{L^2} \lesssim \int_{t-A}^{t}\int_0^{s-B} \Big\| V_2(\cdot - s\vec{e_1})\,\mathfrak{g}_{-\vec{e_1}}(s)e^{-i(s-\tau)H_2}P_c(H_2) \\ & \qquad V_1(\cdot+\tau\vec{e_1})\,\mathfrak{g}_{\vec{e_1}}(\tau)U(\tau)\psi_0\Big\|_{L^2} d\tau ds \\ \lesssim\; & \int_{t-A}^{t}\int_0^{s-B} \Big\|\,\mathfrak{g}_{-\vec{e_1}}(s)e^{-i(s-\tau)H_2}P_c(H_2)\,\mathfrak{g}_{\vec{e_1}}(\tau)V_1U(\tau)\psi_0\Big\|_{L^2+L^\infty} d\tau ds \\ \lesssim\; & \int_{t-A}^{t}\int_0^{s-B} \frac{1}{\langle s-\tau\rangle^{n/2}}\|V_1U(\tau)\psi_0\|_{L^1\cap L^2}\, d\tau ds \\ \lesssim\; & \int_{t-A}^{t}\int_0^{s-B} \frac{1}{\langle s-\tau\rangle^{n/2}}\|U(\tau)\psi_0\|_{L^2+L^\infty}\, d\tau ds \\ \lesssim\; & C_0\int_{t-A}^{t}\int_B^{s-B} \frac{d\tau}{\langle s-\tau\rangle^{n/2}}\frac{ds}{\langle \tau\rangle^{n/2}}\|\psi_0\|_{L^1\cap L^2} + \int_{t-A}^{t}\int_0^{B}\frac{d\tau}{\langle s-\tau\rangle^{n/2}}\|\psi_0\|_2\, d\tau\, ds \\ \lesssim\; & C_0AB^{-(n/2-1)}\langle t\rangle^{-n/2}\|\psi_0\|_{L^1\cap L^2} + AB\langle t\rangle^{-\frac{n}{2}}\|\psi_0\|_{L^1\cap L^2}. \end{aligned}$$

Since $B >> A$ and $n/2 - 1 > 0$ we can conclude that

$$\|J_c^{2,\tau}\|_{L^2+L^\infty} \le 10^{-2}C_0\langle t\rangle^{-n/2}\|\psi_0\|_{L^1\cap L^2}, \tag{3.22}$$

provided C_0 is large.

3.4 Low velocity estimates

The idea behind the estimate for $J_c^{2,\,\mathrm{low}}$ is that the norm of the operator

$$\chi_1(t,\cdot)e^{-i(t-s)H_1}P_c(H_1)F(|\vec{p}|\leq M)V_2(\cdot - s\vec{e_1})$$

for all s in the interval $[t-A,t]$ is small. This can be explained as follows:
The support of $\chi_1(t,x)$ with respect to x belongs to the ball $B_{2\delta t}(0)$. On the other hand, since V_2 is compactly supported $V_2(\cdot - s\vec{e_1})$ is different from 0 on the set $B_R(s\vec{e_1})$. Here R is the size of the support of V_2, $R << t_0 \leq t$. Since $s \in [t-A_1,t]$, we have that

$$\bigcup_{s\in[t-A_1,t]} B_R \subset B_{R+A}(\ t\vec{e_1})$$

and thus the support of $V_2(\cdot - s\vec{e_1})$ is approximately $(1-\delta)t$ units away from the support of $\chi_1(t,\cdot)$. The operator $e^{-i(t-s)H_1}P_c(H_1)F(|\vec{p}|\leq M)$ can "propagate" the set K into the set L only if $(t-s)M \geq\ \mathrm{dist}(K,L)$ according to the semiclassical picture. Setting $K = \mathrm{sppt}\Big(V_2(\cdot - s\vec{e_1})\Big)$ and $L = \mathrm{sppt}\Big(\chi_1(t,\cdot)\Big)$, we see that this would require that

$$(t-s)M \geq (1-\delta)t.$$

However, $|t-s| \leq A$ and M is fixed so that $A, M << t_0 \leq t$.
We make this argument rigorous in the following lemma.

Lemma 3.3. *Let A, M be large positive constants and $A, M << t$. Then*

$$\sup_{|t-s|\leq A} \|\chi_1(t,\cdot)e^{-i(t-s)H_1}P_c(H_1)F(|\vec{p}|\leq M)V_2(\cdot - s\vec{e_1})\|_{L^2\to L^2} \leq \frac{AM}{\delta t},$$

where δ is the constant from the definition of χ_1, χ_2.

Proof. The proof is a commutator argument. Let $\chi_1 = \chi(t,x)$. Firstly, we claim that

$$\|[\chi_1, P_c(H_1)]\|_{L^2\to L^2} \lesssim e^{-\delta\alpha t}. \tag{3.23}$$

Clearly,

$$[\chi_1, P_c(H_1)] = [\chi_1, I - P_b(H_1)] = -[\chi_1, P_b(H_1)].$$

Recall that $u_1, \ldots, u_m$ are the exponentially decaying eigenvalues of H_1. Therefore,

$$\begin{aligned}
[\chi_1, P_b(H_1)]f &= \sum_{i=1}^{m}(\chi_1 u_i\langle f, u_i\rangle - u_i\langle f\chi_1, u_i\rangle) \\
&= \sum_{i=1}^{m}\left((-1+\chi_1)u_i\langle f, u_i\rangle - u_i\langle f, (\chi_1 - 1)u_i\rangle\right).
\end{aligned}$$

In of the support of $\chi_1 - 1$ we have that

$$\|(1-\chi_1)u_i\|_2 \lesssim e^{-\alpha\delta t}$$

and thus

$$\|[\chi_1, P_b(H_1)]f\|_{L^2} \le e^{-\alpha\delta t}\|f\|_{L^2},$$

as desired. Secondly, we claim that

$$\|[\chi_1, e^{-i(t-s)H_1}]F(|\vec{p}| \le M)\|_{L^2\to L^2} \lesssim \frac{AM}{\delta t}. \tag{3.24}$$

We write

$$[\chi_1, e^{-i(t-s)H_1}] = e^{-i(t-s)H_1}(e^{i(t-s)H_1}\chi_1 e^{-i(t-s)H_1} - \chi_1)$$

and

$$\begin{aligned} e^{i(t-s)H_1}\chi_1\, e^{-i(t-s)H_1} - \chi_1 &= i\int_0^{t-s} e^{i\tau H_1}[H_1,\chi_1]e^{-i\tau H_1}\, d\tau \\ &= i\int_0^{t-s} e^{i\tau H_1}(-\nabla\chi_1\nabla - \frac{1}{2}\triangle\chi_1)e^{-i\tau H_1}\, d\tau. \end{aligned}$$

Observe now that

$$|\nabla\chi_1(t,x)| \lesssim \frac{1}{\delta t}, \quad |\triangle\chi_1(t,x)| \lesssim \frac{1}{(\delta t)^2}.$$

Therefore,

$$\begin{aligned} &\|[\chi_1, e^{-(t-s)H_1}]F(|\vec{p}| \le M)\|_{L^2\to L^2} \\ &\lesssim |t-s|\Big(\|\nabla\chi_1\,\nabla e^{-i\tau H_1}F(|\vec{p}| \le M)\|_{L^2\to L^2} + \|\Delta\chi_1 e^{-i\tau H_1}F(|\vec{p}| \le M)\|_{L^2\to L^2}\Big) \\ &\lesssim \frac{A}{\delta t}\|\nabla e^{-i\tau H_1}F(|\vec{p}| \le M)\|_{L^2\to L^2} + \frac{A}{(\delta t)^2}\|e^{-i\tau H_1}F(|\vec{p}| \le M)\|_{L^2\to L^2}. \end{aligned}$$

Since the potential V_1 is bounded it is standard that

$$\sup_\tau \|\nabla e^{-i\tau H_1}f\|_{L^2} \lesssim \|\nabla f\|_{L^2} + \|f\|_{L^2}. \tag{3.25}$$

Indeed,

$$\sup_\tau \|\triangle e^{-i\tau H_1}f\|_{L^2} \quad \le \quad \sup_\tau \|e^{-i\tau H_1}H_1 f\|_{L^2} + \sup_\tau \|V_1 e^{-i\tau H_1}f\|_{L^2} \lesssim \|\nabla^2 f\|_{L^2} + \|f\|_{L^2},$$

and (3.25) follows by interpolation with L^2. Therefore,

$$\|\nabla e^{-i\tau H_1}F(|\vec{p}| \le M)\|_{L^2\to L^2} \le M.$$

Combining terms we obtain the bound

$$\|[\chi_1, e^{-i(t-s)H_1}]F(|\vec{p}| \le M)\|_{L^2\to L^2} \lesssim \frac{AM}{\delta t} + \frac{A}{(\delta t)^2} \le 2\frac{AM}{\delta t}$$

since $M << t$, which is (3.24). Finally, we invoke one more standard fact, namely

$$\|[\chi_1, F(|\vec{p}| \le M)]\|_{L^2\to L^2} \lesssim M^{-1}\,\|\nabla\chi_1\|_\infty \lesssim \frac{1}{\delta M t}. \tag{3.27}$$

To see this, write $F(|\vec{p}| \le M)f = [\hat{\eta}(\xi/M)\hat{f}(\xi)]^\vee$ with some smooth bump function η. Hence the kernel K of $[\chi_1, F(|\vec{p}| \le M)]$ is

$$K(x,y) = M^n \eta(M(x-y))(\chi_1(x) - \chi_1(y)),$$

and (3.27) follows from Schur's test. One concludes from estimates (3.23), (3.24), (3.27) that

$$\Big\| \chi_1(t,\cdot) e^{-i(t-s)H_1} P_c(H_1) F(|\vec{p}| \le M) V_2(\cdot - s\vec{e_1}) \\ - e^{-i(t-s)H_1} P_c(H_1) F(|\vec{p}| \le M) \chi_1(t,\cdot) V_2(\cdot - s\vec{e_1}) \Big\|_{L^{\to} L^2} \lesssim \frac{AM}{\delta t}.$$

It remains to observe that $\chi_1(t,\cdot)V_2(\cdot - s\vec{e_1}) = 0$ since the supports of $\chi_1(t,\cdot)$ and $V_2(\cdot - s\vec{e_1})$ are disjoint. □

We now estimate the term

$$J_c^{2,\text{ low}} = \chi_1 \int_{t-A}^{t} \int_{s-B}^{s} e^{-i(t-s)H_1} P_c(H_1) F(|\vec{p}| \le M) V_2(\cdot - s\vec{e_1})\, \mathfrak{g}_{-\vec{e_1}}(s) e^{-i(s-\tau)H_2} P_c(H_2) \\ V_1(\cdot + \tau\vec{e_1})\, \mathfrak{g}_{\vec{e_1}}(\tau) U(\tau)\psi_0 \, d\tau ds$$

by means of Lemmas 3.3 and 3.2. Indeed, one has

$$\begin{aligned} &\|J_c^{2,\text{ low}}\|_{L^2+L^\infty} \le \|J_c^{2,\text{ low}}\|_{L^2} \\ &\lesssim \frac{AM}{\delta t} \int_{t-A}^{t} \int_{s-B}^{s} \Big\| \mathfrak{g}_{-\vec{e_1}}(s) e^{-i(s-\tau)H_2} P_c(H_2)\, \mathfrak{g}_{\vec{e_1}}(\tau) V_1 U(\tau)\psi_0 \Big\|_{L^2} d\tau ds \\ &\lesssim \frac{AM}{\delta t} \int_{t-A}^{t} \int_{s-B}^{s} \|V_1 U(\tau)\psi_0\|_{L^2}\, d\tau ds \lesssim \frac{AM}{\delta t} \|V_1\|_{L^2\cap L^\infty} \int_{t-A}^{t} \int_{s-B}^{s} \|U(t)\psi_0\|_{L^2+L^\infty}\, d\tau ds \\ &\lesssim \frac{A^2BM}{\delta t} \|V_1\|_{L^2\cap L^\infty}\, C_0 \langle t\rangle^{-\frac{n}{2}} \|\psi_0\|_{L^1\cap L^2}. \end{aligned}$$

Since $t >> A, M, \|V_1\|_{L^2\cap L^\infty}$ we obtain that

$$\|J_c^{2,\text{ low}}\|_{L^2+L^\infty} \le 10^{-2} C_0 \langle t\rangle^{-n/2} \|\psi_0\|_{L^1\cap L^2}. \tag{3.28}$$

3.5 Large velocity estimates via Kato smoothing

The estimate for $J_c^{2,\text{ high}}$ will require the use of the Kato $\frac{1}{2}$-smoothing estimate. We present a variant of it in the following lemma. It differs from the original version (see [CS1], [KY], [Sj], [Ve]) in so far as we consider $-\frac{1}{2}\triangle + V$ instead of $-\frac{1}{2}\triangle$. Moreover, we employ commutator methods. Both the statement and the proof are known, see e.g. [CS2], [BK], [So], [Do], so we will only sketch the argument.

Lemma 3.4. *Let $H = -\frac{1}{2}\triangle + V$, $\|V\|_\infty < \infty$. Then for all $T, R, M \ge 1$,*

$$\sup_{B_R} \int_0^T \|F(|\vec{p}| \ge M) e^{-itH} f\|_{L^2(B_R)}\, dt \le C(n,V)\, \frac{TR}{M^{\frac{1}{2}}} \|f\|_{L^2}. \tag{3.29}$$

Here the supremum ranges over all balls B_R of radius $R \ge 1$ and $C(n,V)$ is a constant that depends only on $\|V\|_\infty$ and the dimension n.

Proof. We first prove the following estimate, which will then imply the lemma. Let $\psi(t) = e^{-itH}\psi_0$ with $H = -\frac{1}{2}\triangle + V$, $\|V\|_\infty < \infty$. Then for all $T > 0$ and $0 < \alpha$,

$$\sup_{x_0\in\mathbb{R}^n} \int_0^T \int_{\mathbb{R}^n} \frac{|\nabla\langle\nabla\rangle^{-\frac{1}{2}}\psi(x)|^2}{(1+|x-x_0|^\alpha)^{\frac{1}{\alpha}+1}}\,dxdt \le C_{\alpha,n}\,T(1+\|V\|_\infty)\|\psi_0\|_2^2 \tag{3.30}$$

The multiplier $\nabla\langle\nabla\rangle^{-\frac{1}{2}}$ corresponds to the symbol $\xi\langle\xi\rangle^{-\frac{1}{2}} = \xi(1+|\xi|^2)^{-\frac{1}{4}}$. It suffices to prove this with $x_0 = 0$ fixed. The proof is based on taking the commutator of H with $m := w(x)x\cdot\frac{\nabla}{\langle\nabla\rangle}$, where

$$w(x) = (1+|x|^\alpha)^{-\frac{1}{\alpha}}, \qquad \alpha > 0.$$

One has, with $\psi = \psi(t)$ for simplicity,

$$\begin{aligned}
\frac{d}{dt}\langle m\psi,\psi\rangle &= -i\langle[m,H]\psi,\psi\rangle \\
\int_0^T \langle -[m,H]\psi(t),\psi(t)\rangle\,dt &= i\langle m\psi(0),\psi(0)\rangle - i\langle m\psi(T),\psi(T)\rangle \\
\langle -[m,H]\psi,\psi\rangle &= -\langle[m,V]\psi,\psi\rangle + \Big\langle \partial_\ell(w(x)\,x_j)\frac{\partial_j}{\langle\nabla\rangle}\psi, \partial_\ell\psi\Big\rangle + \frac{1}{2}\Big\langle \triangle(w(x)\,x_j)\frac{\partial_j}{\langle\nabla\rangle}\psi,\psi\Big\rangle && (3.31)\\
&= \int_{\mathbb{R}^n} w|\nabla\langle\nabla\rangle^{-\frac{1}{2}}\psi|^2\,dx + \int_{\mathbb{R}^n}(\partial_\ell w)(x)x_j\partial_j\langle\nabla\rangle^{-\frac{1}{2}}\psi\,\partial_\ell\langle\nabla\rangle^{-\frac{1}{2}}\bar\psi\,dx && (3.32)\\
&- \Big\langle[\langle\nabla\rangle^{\frac{1}{2}},w]\frac{\nabla}{\langle\nabla\rangle}\psi,\nabla\langle\nabla\rangle^{-\frac{1}{2}}\psi\Big\rangle - \Big\langle[\langle\nabla\rangle^{\frac{1}{2}},(\partial_\ell w)(x)x_j]\frac{\partial_j}{\langle\nabla\rangle}\psi,\partial_\ell\langle\nabla\rangle^{-\frac{1}{2}}\psi\Big\rangle && (3.33)\\
&+ \frac{1}{2}\Big\langle\triangle(w(x)\,x_j)\frac{\partial_j}{\langle\nabla\rangle}\psi,\psi\Big\rangle - \langle[m,V]\psi,\psi\rangle && (3.34)
\end{aligned}$$

One now checks easily that the two terms in (3.32) satisfy

$$\begin{aligned}
\int_{\mathbb{R}^n} w(x)|\langle\nabla\rangle^{-\frac{1}{2}}\nabla\psi(t,x)|^2\,dx \;&+\; \int_{\mathbb{R}^n}(\partial_\ell w)(x)x_j\langle\nabla\rangle^{-\frac{1}{2}}\partial_j\psi(t,x)\,\langle\nabla\rangle^{-\frac{1}{2}}\partial_\ell\bar\psi(t,x)\,dx \\
&\ge \int_{\mathbb{R}^n}\frac{w(x)}{1+|x|^\alpha}\,|\langle\nabla\rangle^{-\frac{1}{2}}\nabla\psi(t,x)|^2\,dx = \int_{\mathbb{R}^n}\frac{|\langle\nabla\rangle^{-\frac{1}{2}}\nabla\psi(t,x)|^2}{(1+|x|^\alpha)^{\frac{1}{\alpha}+1}}\,dx. && (3.35)
\end{aligned}$$

Notice that (3.35) is precisely the space integral in the desired lower bound from (3.30).

There are several ways to bound the commutators $A_1 := [\langle\nabla\rangle^{\frac{1}{2}},w]$ and $A_2 := [\langle\nabla\rangle^{\frac{1}{2}},(\partial_\ell w)(x)x_j]$. For example, one can use the Kato square root formula as in [JSS]. Alternatively, one can invoke the standard composition formula from ΨDO calculus, see [Tay]. This gives $A_1 = T_{\{\langle\nabla\rangle^{\frac{1}{2}},w\}} + R_{a_1}$ where $\{\langle\nabla\rangle^{\frac{1}{2}},w\}$ is the Poisson bracket of the symbols $\langle\nabla\rangle^{\frac{1}{2}}$ and w. Moreover, $T_{\{\langle\nabla\rangle^{\frac{1}{2}},w\}}$ is an associated ΨDO operator and R_{a_1} another ΨDO operator with symbol $a_1 \in S^{-\frac{3}{2}}$. A similar expression holds for A_2. One checks that $|\{\langle\nabla\rangle^{\frac{1}{2}},w\}(\xi,x)| \lesssim |\nabla w(x)|$ for all x,ξ. Therefore, the two terms in (3.33) both satisfy

$$\begin{aligned}
\Big|\Big\langle A_i\frac{\nabla}{\langle\nabla\rangle}\psi(t),\nabla\langle\nabla\rangle^{-\frac{1}{2}}\psi(t)\Big\rangle\Big| &\lesssim \|\psi(t)\|_2\Big\|\frac{\langle\nabla\rangle^{-\frac{1}{2}}\nabla\psi(t)}{(1+|x|^\alpha)^{\frac{1}{\alpha}+1}}\Big\|_2 + \|\psi(t)\|_2\|R_{a_i}\nabla\langle\nabla\rangle^{-\frac{1}{2}}\psi(t)\|_2 \\
&\le C\|\psi(t)\|_2^2 + \frac{1}{4}\Big\|\frac{\langle\nabla\rangle^{-\frac{1}{2}}\nabla\psi(t)}{(1+|x|^\alpha)^{\frac{1}{\alpha}+1}}\Big\|_2^2. && (3.36)
\end{aligned}$$

Finally, the two terms in (3.34) are bounded by

$$(1+\|V\|_\infty)\|\psi(t)\|_2^2 \le (1+\|V\|_\infty)\|\psi_0\|_2^2. \tag{3.37}$$

In the above estimate we have used the boundedness of the multipliers m and $\triangle(w(x)\,x_j)\frac{\partial_j}{\langle\nabla\rangle}$ on L^2. Integrating (3.35), (3.36), and (3.37) in time, inserting the resulting bounds into (3.31), and finally using that

$$|\langle m\psi(0),\psi(0)\rangle| + |\langle m\psi(T),\psi(T)\rangle| \le 2\|\psi_0\|_2^2,$$

one obtains (3.30). To pass from (3.30) to (3.29), let χ_R be a smooth cut-off to the ball B_R, so that $\widehat{\chi_R}$ has compact support in a ball of size $\sim R^{-1}$. Then, by (3.30) with $\alpha=1$

$$\begin{aligned}
&\int_0^T \|F(|\vec{p}|\ge M)e^{-itH}\psi_0\|_{L^2(B_R)}^2\,dt \le \int_0^T \|\chi_R F(|\vec{p}|\ge M)e^{-itH}\psi_0\|_{L^2}^2\,dt\\
\lesssim\ & \int_0^T \|F(|\vec{p}|\ge M)\chi_R e^{-itH}\psi_0\|_{L^2}^2\,dt + \int_0^T \|[\chi_R, F(|\vec{p}|\ge M)]\|_{2\to2}^2 \|e^{-itH}\psi_0\|_{L^2}^2\,dt\\
\lesssim\ & M^{-1}\int_0^T \|\nabla\langle\nabla\rangle^{-\frac12}F(|\vec{p}|\ge M)\chi_R e^{-itH}\psi_0\|_{L^2}^2\,dt + T(MR)^{-2}\|\psi_0\|_{L^2}^2\\
\lesssim\ & M^{-1}\int_0^T \|F(|\vec{p}|\ge M)\chi_R\nabla\langle\nabla\rangle^{-\frac12}e^{-itH}\psi_0\|_{L^2}^2\,dt + M^{-1}\int_0^T \|[\nabla\langle\nabla\rangle^{-\frac12},\chi_R]\|_{2\to2}^2\|e^{-itH}\psi_0\|_{L^2}^2\\
+\ & T(MR)^{-2}\|\psi_0\|_2^2 \lesssim M^{-1}R^2\int_0^T\int_{\mathbb{R}^n}\frac{|\nabla\langle\nabla\rangle^{-\frac12}e^{-itH}\psi_0|^2}{(1+|x|)^2}dx\,dt + TM^{-1}R^{-1}\|\psi_0\|_{L^2}^2\\
\le\ & C(n,V)\,TM^{-1}R^2\,\|\psi_0\|_{L^2}^2.
\end{aligned}$$

The lemma follows. □

We then have for $J_c^{2,\text{ high}}$

$$\begin{aligned}
&\|J_c^{2,\text{ high}}\|_{L^2+L^\infty} \le \|J_c^{2,\text{ high}}\|_{L^2}\\
&\lesssim \int_{t-A}^t\int_{s-B}^s \|F(|\vec{p}|\ge M)V_2(\cdot - s\vec{e_1})\,\mathfrak{g}_{-\vec{e_1}}(s)e^{-i(s-\tau)H_2}P_c(H_2)V_1(\cdot+\tau\vec{e_1})\,\mathfrak{g}_{\vec{e_1}}(\tau)U(\tau)\psi_0\|_{L^2}\,d\tau ds\\
&\lesssim \int_{t-A}^t\int_{s-B}^s \|F(|\vec{p}|\ge M)\,\mathfrak{g}_{-\vec{e_1}}(s)V_2e^{-i(s-\tau)H_2}P_c(H_2)\,\mathfrak{g}_{\vec{e_1}}(\tau)V_1U(\tau)\psi_0\|_{L^2}\,d\tau ds.
\end{aligned}\tag{3.38}$$

We need to commute the cutoff $F(|\vec{p}|\ge M)$ past the first two terms following it. As for the Galilei transform, one has

$$\|F(|\vec{p}|\ge M)\,\mathfrak{g}_{-\vec{e_1}}(s)f\|_2 = \|F(|\vec{p}|\ge M)e^{ix_1}e^{-ip_1 s}f\|_2 = \|F(|\vec{p}|\ge M)e^{ix_1}f\|_2 = \|F(|\vec{p}-\vec{e_1}|\ge M)f\|_2.$$

Therefore, applying this to the final line of (3.38), one obtains

$$\begin{aligned}
\|J_c^{2,\text{ high}}\|_{L^2+L^\infty} &\lesssim \int_{t-A}^t\int_{s-B}^s \|V_2F(|\vec{p}-\vec{e_1}|\ge M)e^{-i(s-\tau)H_2}P_c(H_2)\,\mathfrak{g}_{\vec{e_1}}(\tau)V_1U(\tau)\psi_0\|_{L^2}\,d\tau ds\\
&+\int_{t-A}^t\int_{s-B}^s \big\|[V_2,F(|\vec{p}-\vec{e_1}|\ge M)]\,e^{-i(s-\tau)H_2}P_c(H_2)\,\mathfrak{g}_{\vec{e_1}}(\tau)V_1U(\tau)\psi_0\big\|_{L^2}\,d\tau ds.
\end{aligned}\tag{3.39}$$

The final two terms in (3.39) are now easily estimated by means of Lemma 3.4 (integrating in the variable $s-\tau$) and (3.27), respectively. To apply Lemma 3.4 choose a ball B_R around the origin that contains sppt(V_2). The conclusion is that

$$\begin{aligned}\|J_c^{2,\text{ high}}\|_{L^2+L^\infty} &\lesssim \|V_2\|_{L^\infty}\frac{BR}{M^{\frac12}}\int_{t-B-A}^t \|P_c(H_2)\,\mathfrak{g}_{\vec{e_1}}(\tau)V_1U(\tau)\psi_0\|_{L^2}\,d\tau \\ &\quad + M^{-1}\int_{t-A}^t\int_{s-B}^s \big\|e^{-i(s-\tau)H_2}P_c(H_2)\,\mathfrak{g}_{\vec{e_1}}(\tau)V_1U(\tau)\psi_0\big\|_{L^2}\,d\tau ds \\ &\lesssim \Big(\|V_2\|_{L^\infty}\frac{BR}{M^{\frac12}}B + M^{-1}B^2\Big)\|V_1\|_{L^1\cap L^\infty}C_0\langle t\rangle^{-n/2}\|\psi_0\|_{L^1\cap L^2}.\end{aligned}$$

Thus for a sufficiently large constant M one obtains

$$\|J_c^{2,\text{ high}}\|_{L^2+L^\infty} \le 10^{-2}C_0\langle t\rangle^{-n/2}\|\psi_0\|_{L^1\cap L^2}. \tag{3.40}$$

Combining (3.15), (3.18), (3.21), (3.22), (3.28), (3.40) yields

$$\|\chi_1(t,\cdot)P_cU(t)\psi_0\|_{L^2+L^\infty} \le \frac{C_0}{10}\langle t\rangle^{-n/2}\|\psi_0\|_{L^1\cap L^2}. \tag{3.41}$$

This is precisely the desired bound on the first channel. Since the second channel can be treated by the very same method, see the discussion following (3.10), only the third channel remains.

3.6 The third channel

By (3.11),

$$\chi_3\,U(t)\psi_0 = \chi_3\,e^{it\frac{\triangle}{2}}\phi_0 - i\chi_3\int_0^t e^{i(t-s)\frac{\triangle}{2}}(V_1+V_2(\cdot-s\vec{e_1}))U(s)\psi_0\,ds$$

where $\chi_3=\chi_3(t,x)$. By the standard decay estimates for $e^{it\frac{\triangle}{2}}$,

$$\|e^{it\frac{\triangle}{2}}\psi_0\|_{L^2+L^\infty} \lesssim \langle t\rangle^{-n/2}\|\psi_0\|_{L^1\cap L^2}.$$

Thus

$$\|\chi_3\,U(t)\psi_0\|_{L^2+L^\infty} \lesssim \langle t\rangle^{-n/2}\|\psi_0\|_{L^1\cap L^2} + \Big\|\int_0^t \chi_3\,e^{i(t-s)\frac{\triangle}{2}}(V_1+V_2(\cdot-s\vec{e_1}))U(s)\psi_0\,ds\Big\|_{L^2+L^\infty}.$$

By Lemma 3.2 and the bootstrap assumption (2.13),

$$\begin{aligned}&\|(V_1+V_2(\cdot-s\vec{e_1}))U(s)\psi_0\|_{L^1\cap L^2} \le (\|V_1\|_{L^1\cap L^\infty}+\|V_2\|_{L^1\cap L^\infty})\|U(s)\psi_0\|_{L^2+L^\infty}\\ &\lesssim C_0\,(\|V_1\|_{L^1\cap L^\infty}+\|V_2\|_{L^1\cap L^\infty})\langle t\rangle^{-n/2}\|\psi_0\|_{L^1\cap L^2}.\end{aligned}$$

Therefore, by the same calculation that lead to (3.14),

$$\begin{aligned}\|\chi_3\,U(t)\psi_0\|_{L^2+L^\infty} &\lesssim \langle t\rangle^{-n/2}\|\psi_0\|_{L^1\cap L^2} + C_0\,A^{-n/2+1}\langle t\rangle^{-n/2}\|\psi_0\|_{L^1\cap L^2} \\ &\quad + \Big\|\int_{t-A}^t \chi_3\,e^{i(t-s)\frac{\triangle}{2}}(V_1+V_2(\cdot-s\vec{e_1}))U(s)\psi_0\,ds\Big\|_{L^2+L^\infty}.\end{aligned}$$

Since A is sufficiently large we only need to estimate the last term, which we can split as follows:

$$\int_{t-A}^{t} \chi_3\, e^{i(t-s)\frac{\triangle}{2}} V_1 U(s)\psi_0\, ds + \int_{t-A}^{t} \chi_3\, e^{i(t-s)\frac{\triangle}{2}} V_2(\cdot - s\vec{e_1}) U(s)\psi_0\, ds.$$

It suffices to consider the first term, which is split further by means of the projections $P_b(H_1)$ and $P_c(H_1) = Id - P_b(H_1)$:

$$\begin{aligned} \int_{t-A}^{t} \chi_3\, e^{i(t-s)\frac{\triangle}{2}} V_1 U(s)\psi_0\, ds \;&=\; \int_{t-A}^{t} \chi_3\, e^{i(t-s)\frac{\triangle}{2}} V_1 P_b(H_1) U(s)\psi_0\, ds \\ &\quad + \int_{t-A}^{t} \chi_3\, e^{i(t-s)\frac{\triangle}{2}} V_1 P_c(H_1) U(s)\psi_0\, ds =: \mathcal{L}_1 + \mathcal{L}_2. \end{aligned} \tag{3.42}$$

Since we are assuming asymptotic orthogonality of $U(t)\psi_0$ to the bound states, see Definition 2.4, Proposition 3.1 implies that

$$\begin{aligned} \|\mathcal{L}_1\|_{L^2+L^\infty} \;&\le\; \|\mathcal{L}_1\|_{L^2} \le \int_{t-A}^{t} \|V_1\|_{L^2\cap L^\infty} \|P_b(H_1)U(s)\psi_0\|_{L^2}\, ds \\ &\lesssim\; A\|V_1\|_{L^1\cap L^\infty} e^{-\alpha t/2} \|\psi_0\|_{L^2} \lesssim \langle t\rangle^{-n/2} \|\psi_0\|_{L^1\cap L^2}. \end{aligned} \tag{3.43}$$

It is now natural to expand $\mathcal{L}_2$ further, cf. (3.19):

$$\mathcal{L}_2 \;=\; \chi_3 \int_{t-A}^{t} e^{i(t-s)\frac{\triangle}{2}} V_1 P_c(H_1) e^{-isH_1} \psi_0\, ds \tag{3.44}$$

$$\qquad -i\chi_3 \int_{t-A}^{t}\int_0^s e^{i(t-s)\frac{\triangle}{2}} V_1 P_c(H_1) e^{-i(s-\tau)H_1} V_2(\cdot - \tau\vec{e_1}) U(\tau)\psi_0\, d\tau ds. \tag{3.45}$$

The first integral (3.44) is controlled by means of the decay estimate for the evolution $e^{-itH_1}P_c(H_1)$:

$$\begin{aligned} \Big\|\chi_3 \int_{t-A}^{t} e^{i(t-s)\frac{\triangle}{2}} V_1 P_c(H_1) e^{-isH_1}\psi_0\, ds\Big\|_{L^2+L^\infty} &\lesssim \|V_1\|_{L^2\cap L^\infty} \int_{t-A}^{t} \|e^{-isH_1} P_c(H_1)\psi_0\|_{L^2+L^\infty}\, ds \\ &\lesssim \|V_1\|_{L^2\cap L^\infty} A\, \langle t\rangle^{-n/2} \|\psi_0\|_{L^1\cap L^2} \lesssim \langle t\rangle^{-n/2} \|\psi_0\|_{L^1\cap L^2}. \end{aligned} \tag{3.46}$$

For the second integral (3.45) one also invokes the bootstrap assumption (2.13), as well as Lemma 3.2:

$$\begin{aligned} &\Big\|\chi_3 \int_{t-A}^{t}\int_0^{s-B} e^{i(t-s)\frac{\triangle}{2}} V_1 e^{-i(s-\tau)H_1} P_c(H_1) V_2(\cdot - \tau\vec{e_1}) U(\tau)\psi_0\, d\tau ds\Big\|_{L^2+L^\infty} \\ &\lesssim \|V_1\|_{L^2\cap L^\infty} \int_{t-A}^{t}\int_0^{s-B} \Big\| e^{-i(s-\tau)H_1} P_c(H_1) V_2(\cdot - \tau\vec{e_1}) U(\tau)\psi_0\, d\tau ds\Big\|_{L^2+L^\infty} \\ &\lesssim \|V_1\|_{L^2\cap L^\infty} \|V_2\|_{L^1\cap L^\infty} \int_{t-A}^{t}\int_0^{s-B} \frac{1}{\langle s-\tau\rangle^{n/2}} \|U(\tau)\psi_0\|_{L^2+L^\infty}\, d\tau ds \\ &\lesssim \|V_1\|_{L^2\cap L^\infty} \|V_2\|_{L^1\cap L^\infty} \Big\{ C_0 \int_{t-A}^{t}\int_B^{s-B} \frac{d\tau}{\langle s-\tau\rangle^{n/2}} \frac{ds}{\langle \tau\rangle^{n/2}} \|\psi_0\|_{L^1\cap L^2}\, d\tau ds \\ &\qquad\qquad + \int_{t-A}^{t}\int_0^{B} \frac{d\tau}{\langle s-\tau\rangle^{n/2}} \|\psi_0\|_{L^2}\, d\tau ds \Big\} \\ &\lesssim \|V_1\|_{L^2\cap L^\infty} \|V_2\|_{L^1\cap L^\infty} \Big(AB^{-n/2+1} C_0\, \|\psi_0\|_{L^1\cap L^2} + AB \langle t\rangle^{-\frac{n}{2}} \|\psi_0\|_{L^2} \Big) \\ &\le 10^{-2} C_0 \|\psi_0\|_{L^1\cap L^2} \end{aligned}$$

since $B >> A$, and provided C_0 is large. As in the proof of the decay estimates for the first channel, we split the remaining expression, which is the most difficult part, into low and high momenta:

$$\chi_3 \int_{t-A}^{t} \int_{s-B}^{s} e^{-i(t-s)\frac{\triangle}{2}} V_1 e^{-i(s-\tau)H_1} P_c(H_1) V_2(\cdot - \tau \vec{e_1}) U(\tau)\psi_0 \, d\tau ds = \mathcal{L}_1^{\text{low}} + \mathcal{L}_1^{\text{high}}$$

where we define

$$\begin{aligned} \mathcal{L}_1^{\text{low}} &= \chi_3 \int_{t-A}^{t} \int_{s-B}^{s} e^{-i(t-s)\frac{\triangle}{2}} F(|\vec{p}| < M) V_1 e^{-i(s-\tau)H_1} P_c(H_1) V_2(\cdot - \tau \vec{e_1}) U(\tau)\psi_0 \, d\tau ds \\ \mathcal{L}_1^{\text{high}} &= \chi_3 \int_{t-A}^{t} \int_{s-B}^{s} e^{-i(t-s)\frac{\triangle}{2}} F(|\vec{p}| \geq M) V_1 e^{-i(s-\tau)H_1} P_c(H_1) V_2(\cdot - \tau \vec{e_1}) U(\tau)\psi_0 \, d\tau ds. \end{aligned}$$

As in Lemma 3.3 one has

$$\sup_{|t-s|\leq A} \|\chi_3 \, e^{-i(t-s)\frac{\triangle}{2}} F(|\vec{p}| \leq M) V_1\|_{L^2\to L^2} \lesssim \frac{AM}{\delta t}. \tag{3.47}$$

This is just a consequence of the following:

1. $\text{sppt}(\chi_3(t,\cdot)) \cap \text{sppt}(V_1) = \emptyset$ for $t \geq t_0 = \left(\frac{C_0}{2}\right)^{2/n}$
2. $|\vec{\nabla}\chi_3(t,\cdot)| \lesssim \frac{1}{\delta t}$.

The first property holds since

$$\chi_3(t,x) = 1 - \chi_1(t,x) - \chi_2(t,x) = 0$$

on the set

$$\{x : |x| \leq \delta t\} \cup \{x : |x - r\vec{e_1}| \leq \delta t\}.$$

The bound (3.47) leads to the following estimate on $\mathcal{L}_1^{\text{low}}$

$$\begin{aligned} \|\mathcal{L}_1^{\text{low}}\|_{L^2+L^\infty} &\leq \|\mathcal{L}_1^{\text{low}}\|_{L^2} \\ &\lesssim \frac{AM}{\delta t}\|V_1\|_{L^\infty}\|V_2\|_{L^2\cap L^\infty} \int_{t-A}^{t}\int_{s-B}^{s} \|U(\tau)\psi_0\|_{L^2+L^\infty} \, d\tau ds \\ &\lesssim \frac{A^2MB}{\delta t}\|V_1\|_{L^\infty}\|V_2\|_{L^2\cap L^\infty} C_0 \langle t\rangle^{-n/2}\|\psi_0\|_{L^1\cap L^2}. \end{aligned}$$

Since $A, M < B$ are large but universal constants and $t \geq t_0 = \left(\frac{C_0}{2}\right)^{2/n}$, this implies that

$$\|\mathcal{L}_1^{\text{low}}\|_{L^2+L^\infty} \leq 10^{-2} C_0 \langle t\rangle^{-n/2}\|\psi_0\|_{L^1\cap L^2}\|U(\tau)\psi_0\|_{L^2+L^\infty}. \tag{3.48}$$

Finally, the high energy part is treated by means of Lemma 3.4 as before. More precisely, one has

$$\begin{aligned} \mathcal{L}_1^{\text{high}} &= \chi_3 \int_{t-A}^{t}\int_{s-B}^{s} e^{-i(t-s)\frac{\triangle}{2}} [F(|p| \geq M), V_1] e^{-i(\tau-s)H_1} P_c(H_1) V_2(\cdot - \tau\vec{e_1}) U(\tau)\psi_0 \, d\tau ds \\ &\quad + \chi_3 \int_{t-A}^{t}\int_{s-B}^{s} e^{-i(t-s)\frac{\triangle}{2}} V_1 F(|p| \geq M) e^{-i(s-\tau)H_1} P_c(H_1) V_2(\cdot - \tau\vec{e_1}) U(\tau)\psi_0 \, d\tau ds \\ &= \mathcal{L}_{2,1}^{\text{high}} + \mathcal{L}_{2,2}^{\text{high}}. \end{aligned}$$

The commutator estimate, see (3.27),

$$\|[F(|p|\geq M),V_1]\|_{L^2\to L^2}\lesssim \frac{1}{M}\|\vec{\nabla}V_1\|_{L^\infty},$$

implies that

$$\|\mathcal{L}_{2,1}^{\text{high}}\|_{L^2+L^\infty} \lesssim \frac{AB}{M}C_0\,\langle t\rangle^{-n/2}\|\psi_0\|_{L^1\cap L^2}\lesssim 10^{-2}C_0\,t^{-n/2}\|\psi_0\|_{L^1\cap L^2}. \tag{3.49}$$

By the smoothing estimate of Lemma 3.4 applied to the integration variable $u=s-\tau$ and with R sufficiently large depending on the support of V_1,

$$\begin{aligned}&\|\mathcal{L}_{2,2}^{\text{high}}\|_{L^2+L^\infty}\lesssim \int_{t-A-B}^{t}\int_0^B \|V_1F(|p|\geq M)e^{-iuH_1}P_c(H_1)V_2(\cdot-\tau\vec{e_1})U(\tau)\psi_0\|_{L^2}\,dud\tau\\ &\lesssim \frac{\|V_1\|_\infty}{M^{\frac12}}BR\|V_2\|_{L^1\cap L^\infty}\int_{t-A-B}^{t}\|U(\tau)\psi_0\|_{L^2+L^\infty}\,d\tau\lesssim \|V_1\|_{L^\infty}\|V_2\|_{L^1\cap L^\infty}\frac{BR}{M^{\frac12}}C_0\,\langle t\rangle^{-n/2}\|\psi_0\|_{L^1\cap L^2}.\end{aligned}$$

Therefore, with M large,

$$\|\mathcal{L}_{2,2}^{\text{high}}\|_{L^2+L^\infty}\leq 10^{-2}C_0\|\psi_0\|_{L^1\cap L^2}. \tag{3.50}$$

Combining (3.43) - (3.50) we obtain

$$\|\chi_3\,U(t)\psi_0\|_{L^2+L^\infty}\leq 10^{-2}C_0\,\langle t\rangle^{-n/2}\|\psi_0\|_{L^1\cap L^2},$$

which concludes the proof of (2.14), and thus of Theorem 2.7.

3.7 Perturbations of charge transfer models

It is easy to see that the argument proving Theorem 2.7 also yields the following result.

Theorem 3.5. *Consider the equation*

$$\begin{aligned}&\frac{1}{i}\partial_t\psi-\frac{1}{2}\triangle\psi+\sum_{\kappa=1}^{m}V_\kappa(x-\vec{v}_\kappa t)\psi+V_0(t,x)\psi=0\\ &\psi|_{t=0}=\psi_0,x\in\mathbb{R}^n,\end{aligned} \tag{3.51}$$

where the charge transfer part is an in Definition 2.1 and the perturbation satisfies

$$\sup_t\|V_0(t,\cdot)\|_{1\cap\infty}<\varepsilon.$$

Let $\tilde{U}(t)$ denote the propagator of the equation (3.51). Then for any initial data $\psi_0\in L^1\cap L^2$, which is asymptotically orthogonal to the bound states of H_j in the sense of Definition 2.4 (with $\tilde{U}(t)$ instead of $U(t)$), one has the decay estimates

$$\|\tilde{U}(t)\psi_0\|_{L^2+L^\infty}\lesssim\langle t\rangle^{-n/2}\|\psi_0\|_{L^1\cap L^2}$$

provided $0<\varepsilon<\varepsilon_0$ is sufficiently small with ε_0 independent of ψ_0.

The proof of this theorem is basically identical with the proof of Theorem 2.7. For example, consider again the case of two potentials $m = 2$. Then the Duhamel formula relative to the first channel, see (3.8), becomes

$$\begin{aligned}\chi_1 P_c(H_1)\tilde{U}(t)\psi_0 &= \chi_1 e^{-itH_1} P_c(H_1)\psi_0 - i\chi_1 \int_0^t e^{-i(t-s)H_1} P_c(H_1) V_2(\cdot - s\vec{e_1})\tilde{U}(s)\psi_0 \, ds \\ &\quad -i\chi_1 \int_0^t e^{-i(t-s)H_1} P_c(H_1) V_0(s,\cdot)\tilde{U}(s)\psi_0 \, ds. \qquad (3.53)\end{aligned}$$

Impose the bootstrap assumption (2.13) on the evolution $\tilde{U}$. Then the new term (3.53) satisfies

$$\begin{aligned}\|(3.53)\|_{2+\infty} &\leq \int_0^t C \langle t-s\rangle^{-\frac{3}{2}} \sup_{0\leq s\leq t} \|V_0(s,\cdot)\|_{1\cap\infty} C_0 \langle s\rangle^{-\frac{3}{2}} \|\psi_0\|_{1\cap 2} \, ds \\ &\leq C\,\varepsilon\, C_0 \langle t\rangle^{-\frac{3}{2}} \|\psi_0\|_{1\cap 2}.\end{aligned}$$

Modifying the other two channels (3.9) and (3.11) in the same way, one concludes that the only change from the estimates in the previous section is the addition of a term $C\,\varepsilon\, C_0\langle t\rangle^{-\frac{3}{2}}\|\psi_0\|_{1\cap 2}$. Choosing ε small, one again arives at the improved bootstrap assumption (2.14) for the evolution $\tilde{U}(t)$.

4 Asymptotic completeness

Our next goal is to provide the following version of the asymptotic completeness for the charge transfer model (see [Ya2], [Gr]). In this section we require more regularity of the potential, see Lemma 4.2 below.

Theorem 4.1. *Let, as before, $u_1, \ldots, u_m$ and $w_1, \ldots, w_\ell$ be the eigenfunctions of $H_1 = -\frac{\triangle}{2} + V_1(x)$ and $H_2 = -\frac{\triangle}{2} + V_2(x)$, respectively, corresponding to the negative eigenvalues $\lambda_1, \cdots, \lambda_m$ and $\mu_1, \cdots, \mu_\ell$. Then for any initial data $\psi_0 \in L^1 \cap L^2$ the solution $U(t)\psi_0$ of the charge transfer problem* (2.9) *can be written in the form*

$$U(t)\psi_0 = \sum_{r=1}^{m} A_r e^{-i\lambda_r t} u_r + \sum_{s=1}^{\ell} B_s e^{-i\mu_s t}\, \mathfrak{g}_{-\vec{e}_1}(t) w_s + e^{-it\frac{\triangle}{2}}\phi_0 + \mathcal{R}(t),$$

for some choice of the constants A_r, B_s and the function ϕ_0. The remainder term $\mathcal{R}(t)$ satisfies the estimate

$$\|\mathcal{R}(t)\|_{L^2} \longrightarrow 0, \quad \text{as } t \to \infty$$

Proof. Decompose

$$\psi(t) := U(t)\psi_0 = P_b(H_1)U(t)\psi_0 + P_b(H_2,t)U(t)\psi_0 + R(t).$$

By construction we clearly have

$$\begin{aligned}&P_b(H_2,t)U(t)\psi_0 + R(t) \in \mathrm{Ran}(P_c(H_1)), \qquad (4.3)\\ &P_b(H_1)U(t)\psi_0 + R(t) \in \mathrm{Ran}(P_c(H_2,t))\end{aligned}$$

We further write

$$P_b(H_1)U(t)\psi_0 = \sum_{r=1}^{m} e^{-i\lambda_r t} a_r(t) u_r(x)$$

for some choice of unkown functions $a_r(t)$. Using (4.3) we obtain, similar to (3.1),

$$\dot{a}_r + i\,\langle V_2(\cdot - t\vec{e}_1)\psi(t), u_r\rangle = 0 \quad \text{for all} \quad 1 \le r \le m.$$

The exponential localization of u_r implies that $|\langle V_2(\cdot - t\vec{e}_1)\psi(t), u_r\rangle| \lesssim e^{-\alpha t}$. Therefore, $a_r(t)$ has a limit $a_r(t) \to A_r, \quad$ as $t \to +\infty$ and thus,

$$\Big\| P_b(H_1)U(t)\psi_0 - \sum_{r=1}^{m} A_r e^{-i\lambda_r t} u_r \Big\|_{L^2} \to 0, \quad t \to +\infty. \tag{4.4}$$

We next define the functions $v_r = \lim_{t\to+\infty} U(-t)e^{-i\lambda_r t}u_r$. The existence of v_r is guaranteed by the existence of the wave operators (see [Gr] and for an independent proof, Lemma 4.2)

$$\Omega^1_- = s - \lim_{t\to+\infty} U(-t)e^{-itH_1}P_b(H_1)$$

In this notation, $v_r = \Omega^1_- u_r$. In particular,

$$\Big\| U(t)\big(\sum_{r=1}^{m} A_r v_r\big) - \sum_{r=1}^{m} A_r e^{-i\lambda_r t} u_r \Big\|_{L^2} \to 0, \quad t \to +\infty. \tag{4.5}$$

We then infer from (4.4) that

$$\Big\| U(t)\big(\sum_{r=1}^{m} A_r v_r\big) - P_b(H_1)U(t)\psi_0 \Big\|_{L^2} \to 0, \quad t \to +\infty. \tag{4.6}$$

Similar arguments apply to $P_b(H_2,t)U(t)\psi_0$. More precisely, we write

$$U(t)\psi_0 = P_b(H_2,t)U(t)\psi_0 + \Gamma(t) = \mathfrak{g}_{-\vec{e}_1}(t)P_b(H_2)\,\mathfrak{g}_{\vec{e}_1}(t)U(t)\psi_0 + \Gamma(t).$$

Therefore,

$$\mathfrak{g}_{\vec{e}_1}(t)U(t)\psi_0 = P_b(H_2)\,\mathfrak{g}_{\vec{e}_1}(t)U(t)\psi_0 + \mathfrak{g}_{\vec{e}_1}(t)\Gamma(t) \tag{4.7}$$

Recall that the function

$$\tilde{\psi}(t) = \mathfrak{g}_{\vec{e}_1}(t)U(t)\psi_0$$

is a solution of the problem

$$\frac{1}{i}\partial_t\tilde{\psi} - \frac{\triangle}{2}\tilde{\psi} + V_2(x)\tilde{\psi} + V_1(x + t\vec{e}_1)\tilde{\psi} = 0, \qquad \tilde{\psi}|_{t=0} = \mathfrak{g}_{\vec{e}_1}(0)\psi_0 \tag{4.8}$$

According to (4.7), $\tilde{\psi}(t) = P_b(H_2)\tilde{\psi}(t) + \Gamma_1(t)$, where $\Gamma_1(t) = \mathfrak{g}_{\vec{e}_1}(0)\Gamma(t)$. In particular,

$$\Gamma_1(t) \in P_c(H_2)L^2.$$

Decompose

$$P_b(H_2)\tilde{\psi}(t) = \sum_{s=1}^{\ell} b_s(t)e^{-i\mu_s t} w_s$$

for some choice of unknown functions $b_s(t)$. After substituting the decomposition in (4.8) we obtain the equations

$$\dot{b}_s(t) + i\,\langle V_1(\cdot + t\vec{e}_1)\tilde{\psi}, w_s\rangle = 0 \quad \text{for all} \quad 1 \le s \le \ell.$$

Using exponential localization of w_s we conclude the existence of the limit $b_s(t) \to B_s$ as $t \to +\infty$. Thus $\|P_b(H_2)\tilde{\psi}(t) - \sum_{s=1}^{\ell} B_s e^{-i\mu_s t} w_s\|_{L^2} \to 0, \quad t \to \infty$. Equivalently, after applying $\mathfrak{g}_{-\vec{e}_1}(t)$, we have

$$\Big\| P_b(H_2,t)U(t)\psi_0 - \sum_{s=1}^{\ell} B_s e^{-i\mu_j t}\, \mathfrak{g}_{-\vec{e}_1}(t) w_s \Big\|_{L^2} \to 0. \tag{4.9}$$

We now invoke the existence of the wave operator (see [Gr] and Lemma 4.2)

$$\Omega_2^- = s - \lim_{t\to+\infty} U(-t)\, \mathfrak{g}_{-\vec{e}_1}(t) e^{-itH_2} P_b(H_2)$$

which allows us to define $\omega_s := \Omega_2^- w_s$. Moreover,

$$\Big\| U(t)\big(\sum_{s=1}^{\ell} B_s \omega_s\big) - \sum_{s=1}^{\ell} B_s e^{-i\mu_s t}\, \mathfrak{g}_{-\vec{e}_1}(t) w_s \Big\|_{L^2} \to 0, \quad t \to +\infty. \tag{4.10}$$

It then follows from (4.9) that

$$\|P_b(H_2,t)U(t)\psi_0 - U(t)\big(\sum_{s=1}^{\ell} B_s\omega_s\big)\|_{L^2} \to 0, \quad t \to +\infty. \tag{4.11}$$

We now define the function

$$\phi := \psi_0 - \sum_{r=1}^{m} A_r v_r - \sum_{s=1}^{\ell} B_s \omega_s, \tag{4.12}$$

which will lead to the initial data ϕ_0 for the free channel. We have that

$$P_b(H_1)U(t)\phi = P_b(H_1)U(t)\psi_0 - P_b(H_1)U(t)\big(\sum_{r=1}^{m} A_r v_r\big) - P_b(H_1)U(t)\big(\sum_{s=1}^{\ell} B_s\omega_s\big).$$

It follows from (4.6) and the identity $P_b^2(H_1) = P_b(H_1)$ that

$$\Big\| P_b(H_1)U(t)\psi_0 - P_b(H_1)U(t)\big(\sum_{r=1}^{m} A_r v_r\big)\Big\|_{L^2} \to 0. \qquad \text{as } t \to +\infty \tag{4.13}$$

Furthermore,

$$P_b(H_1)\sum_{s=1}^{\ell} B_s e^{-i\mu_s t}\, \mathfrak{g}_{-\vec{e}_1}(t) w_j = \sum_{r=1}^{m}\sum_{s=1}^{\ell} B_s e^{-i\mu_s t} \langle\, \mathfrak{g}_{-\vec{e}_1}(t) w_j, u_r\rangle\, u_r \to 0 \tag{4.14}$$

in the L^2 sense as $t \to +\infty$, due to the exponential localization of the eigenfunctions u_r. One concludes from (4.13), (4.10), and (4.14) that $\|P_b(H_1)U(t)\phi\|_{L^2} \to 0$. Similarly, we can show that $\|P_b(H_2,t)U(t)\phi\|_{L^2} \to 0$. Thus, $U(t)\phi$ is asymptotically orthogonal to the bound states of H_1 and H_2 and therefore, according to Theorem 2.7, satisfies the estimate

$$\|U(t)\phi\,\|_{L^2+L^\infty} \lesssim \langle t\rangle^{-\frac{n}{2}}\|\phi\|_{L^1\cap L^2}. \tag{4.15}$$

In order to be able to apply estimate (4.15) one needs to verify that $\phi \in L^1 \cap L^2$. By the assumption of the theorem the function $\psi_0 \in L^1 \cap L^2$. Thus it remains to check this property for the functions v_r, $r = 1,..,m$ and ω_s, $s = 1,..,\ell$. Since $v_r = \Omega_1^- u_r$ and $\omega_s = \Omega_2^- w_s$ the L^2 property follows immediately. The L^1 property, on the other hand, is guaranteed by Lemma 4.2 below. Assuming this lemma for the moment, we now consider the expression

$$e^{-it\frac{\triangle}{2}}U(t)\phi = \phi - i\int_0^t e^{-is\frac{\triangle}{2}}\,(V_1(x) + V_2(x - s\vec{e}_1))\,U(s)\phi\,ds.$$

The estimate

$$\int_t^\infty \|e^{-is\frac{\triangle}{2}}\,(V_1(x) + V_2(x - t\vec{e}_1))U(s)\phi\|_{L^2} \lesssim (\|V_1\|_{L^2\cap L^\infty} + \|V_2\|_{L^2\cap L^\infty})\int_t^\infty \|U(s)\phi\|_{L^2+L^\infty}ds$$
$$\lesssim t^{-\frac{n}{2}+1}\,(\|V_1\|_{L^2\cap L^\infty} + \|V_2\|_{L^2\cap L^\infty}) \to 0, \quad \text{as} \quad t \to +\infty$$

allows us to show the existence of the limit

$$\phi_0 := \lim_{t\to\infty} e^{it\frac{\triangle}{2}}U(t)\phi.$$

It follows that

$$\|U(t)\phi - e^{it\frac{\triangle}{2}}\phi_0\|_{L^2} \to 0, \quad t \to +\infty. \tag{4.16}$$

Combining (4.5), (4.10), (4.12), and (4.12) we infer that

$$\Big\|U(t)\psi_0 - \sum_{r=1}^m A_r e^{-i\lambda_r t}u_r - \sum_{s=1}^\ell B_s e^{-i\mu_s t}\,\mathfrak{g}_{-\vec{e}_1}(t)w_s - e^{it\frac{\triangle}{2}}\phi_0\Big\|_{L^2} \to 0, \quad \text{as} \quad t \to +\infty,$$

as claimed. □

Lemma 4.2. *Assume that the potentials $V_1(x), V_2(x)$ satisfy the following conditions*

$$\sum_{0\le|\gamma|\le n+2} |\partial_x^\gamma V_\kappa(x)| \le \frac{c(\kappa)}{\langle x\rangle^{3[\frac{n}{2}]+5}}, \tag{4.17}$$

for some positive constants $c(\kappa)$, $\kappa = 1, 2$. Then the range of the wave operators $\Omega_{1,2}^-$ is contained in the space of L^1 functions.

Proof. Without loss of generality we only consider the wave operator

$$\Omega_1^- = s - \lim_{t\to+\infty} U(-t)e^{-itH_1}P_b(H_1).$$

Therefore, for an arbitrary L^2 function f

$$\Omega_1^- f = \sum_{r=1}^m f_r \lim_{t\to+\infty} U(-t)e^{-itH_1}u_r,$$

where $P_b(H_1)f = \sum_{r=1}^m f_r u_r$ for some constants f_r. It follows from the Duhamel formula for the equation (2.9) generating the evolution operator $U(t)$ that

$$\begin{aligned} U(-t)e^{-itH_1}u_r =& u_r + i\int_0^t U(-s)V_2(\cdot - s\vec{e}_1)e^{-isH_1}u_r\,ds \\ =& u_r + i\int_0^t U(-s)V_2(\cdot - s\vec{e}_1)e^{-i\lambda_r s}u_r\,ds \end{aligned} \tag{4.18}$$

with (4.18) follows since u_r is an eigenfunction of H_1 corresponding to an eigenvalue λ_r. The function u_r is exponentially localized in L^2 together with its $n+2$ derivatives [3]

$$\sum_{0\le|\gamma|\le n+2} \int_{\mathbb{R}^n} e^{2\alpha|x|}|\partial_x^\gamma u_r(x)|^2\,dx \le C$$

for some positive constant α appearing in (3.4). It follows from (4.17) that the function

$$G_r(s,x) := e^{-i\lambda_r s}V_2(x - s\vec{e}_1)u_r(x)$$

has the property that for any $k\ge 0$ and multi-index γ, $0\le|\gamma|\le n+2$

$$\|\langle x\rangle^k \partial_x^\gamma G_r(s,\cdot)\|_{L^2_x} \le c(r,|\gamma|,k)\langle s\rangle^{-3[\frac{n}{2}]-5}.$$

To prove the desired conclusion it would then suffice to show that there exists a positive constant k such that for any function $g(x)$

$$\|\langle x\rangle^{[\frac{n}{2}]+1}U(t)g\|_{L^2} \lesssim \langle t\rangle^{3[\frac{n}{2}]+3} \sum_{|\beta|\le n+2} \|\langle x\rangle^k \partial_x^\beta g\|_{L^2}, \qquad \forall t\ge 0. \tag{4.19}$$

We note here that $U(t)g$ denotes the solution of the problem

$$\frac{1}{i}\partial_t\psi - \frac{\triangle}{2}\psi + V(t,x)\psi = 0, \qquad V(t,x) = V_1(x) + V_2(x - t\vec{e}_1) \tag{4.20}$$

evaluated at time t with initial data function g given at time $t=0$. The required estimate should, however, involve the expression $U(-t)g$ which stands for the solution of the above problem at time $t=0$ with initial data g given at time t. Yet it is not difficult to see that the two problems are almost equivalent. Therefore, for simplicity we shall prove estimate (4.19). We note that the estimates of the type (4.19) for problems with time independent potentials are well-known. They have been proved in the paper by Hunziker [H]. In the time-dependent case the argument is essentially the same. More precisely, define the functions

$$\Phi_{j,|\gamma|}(t) := \sum_{j'=0}^{j} \sum_{|\gamma'|=0}^{|\gamma|} \|\langle x\rangle^{j'}\partial_x^{\gamma'}U(t)g\|_{L^2}$$

[3] The localization of higher derivatives of u_r follows from the localization of u_r stated in (3.4) and the equation $-\frac{\triangle}{2}u_r + V_1(x)u_r = \lambda_r u_r$ with potential $V_1(x)$ which is bounded together with all its derivatives of order $\le (n+2)$.

for any index $j \geq 0$ and any multi-index γ. Using equation (4.20) we obtain that

$$\frac{d}{dt}\|\langle x\rangle^j \partial_x^\gamma U(t)g\|_{L^2}^2 = i(-1)^{|\gamma|}\langle [\frac{\triangle}{2} - V(t,x), \langle x\rangle^j \partial_x^\gamma \langle x\rangle^j \partial_x^\gamma] U(t)g, U(t)g\rangle$$

Computing the commutator we obtain the recurrence relation

$$\Phi_{j,|\gamma|}(t) \lesssim \Phi_{j,|\gamma|}(0) + \langle t\rangle^2 \sum_{|\gamma'|\leq 2|\gamma|} \Big\|\frac{\langle x\rangle}{\langle t\rangle}\partial_x^{\gamma'} V\Big\|_{L^\infty_{t,x}} \sup_{0\leq\tau\leq t} \Phi_{j-1,|\gamma|+1}(\tau) \leq$$

$$C(V)\Big(\sum_{k=0}^{j-1}\langle t\rangle^{2k}\Phi_{j-k,|\gamma|+k}(0) + \langle t\rangle^{2j}\Phi_{0,|\gamma|+j}(\tau)\Big),$$

where $C(V)$ is a constant depending on

$$\sum_{|\gamma'|\leq 2(|\gamma|+j-1)} \Big\|\frac{\langle x\rangle}{\langle t\rangle}\partial_x^{\gamma'} V\Big\|_{L^\infty_{t,x}} \tag{4.21}$$

In addition, differentiating the equation (4.20) $(|\gamma| + j)$ times with respect to x and using the standard L^2 estimate, we have

$$\Phi_{0,|\gamma|+j}(\tau) \leq C(V)(1 + |\tau|^{|\gamma|+j})\Phi_{0,|\gamma|+j}(0)$$

Therefore,

$$\Phi_{j,|\gamma|}(t) \leq C(V)(1 + |t|)^{3j+|\gamma|}\Phi_{j,|\gamma|+j}(0).$$

Now setting $j = [\frac{n}{2}] + 1$ and $|\gamma| = 0$ we obtain the desired estimate (4.19) with $k = [\frac{n}{2}] + 1$. Observe that the assumption (4.17) controls the constant $C(V)$ in (4.21) for the potential $V(t,x) = V_1(x) + V_2(x - t\vec{e}_1)$. □

5 L^∞ estimates

In this section we develop a general simple scheme allowing to convert the $L^2 + L^\infty$ estimates developed above into the true dispersive L^∞ estimates.

Proposition 5.1. *Let ψ be a solution of the Schrödinger equation*

$$\frac{1}{i}\partial_t\psi - \frac{\triangle}{2}\psi + V(t,x)\psi = 0, \qquad \psi|_{t=0} = \psi_0$$

with a time-dependent potential $V(t,x)$ satisfying the condition

$$\sup_t \|V(t,\cdot)\|_{L^1\cap L^2} < \infty, \qquad \sup_t \|\hat{V}(t,\cdot)\|_{L^1} < \infty. \tag{5.2}$$

Here $\hat{V}(t,\cdot)$ denotes the Fourier transform of V only with respect to the spatial variable x. Assume that ψ obeys the estimate

$$\|\psi(t)\|_{L^2+L^\infty} \lesssim \langle t\rangle^{-\frac{n}{2}}\|\psi_0\|_{L^1\cap L^2}. \tag{5.3}$$

Then ψ also satisfies the L^∞ estimate

$$\|\psi(t)\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}}\|\psi_0\|_{L^1\cap L^2}. \tag{5.4}$$

Proof. For simplicity consider the case of dimension $n = 3$. The $n > 3$ dimensionial case can be treated by considering $k = [\frac{n}{2}] + 2$ terms in the Duhamel expansion and repeatedly exploiting the cancellation property below (see [JSS]).
We have

$$\psi(t) = e^{it\frac{\triangle}{2}}\psi_0 - i\int_0^t e^{i(t-s)\frac{\triangle}{2}}V(s,\cdot)\psi(s)\,ds$$

$$= e^{it\frac{\triangle}{2}}\psi_0 - i\int_0^t e^{i(t-s)\frac{\triangle}{2}}V(s,\cdot)e^{is\frac{\triangle}{2}}\psi_0\,ds - \int_0^t\int_0^s e^{i(t-s)\frac{\triangle}{2}}V(s,\cdot)e^{i(s-\tau)\frac{\triangle}{2}}V(\tau,\cdot)\psi(\tau)\,d\tau\,ds. \tag{5.5}$$

We recall the cancellation property

$$\sup_s \|e^{is\frac{\triangle}{2}}V(t,\cdot)e^{-is\frac{\triangle}{2}}f\|_{L^p} \lesssim \|\hat{V}(t,\cdot)\|_{L^1}\|f\|_{L^p}, \quad \forall p\in[1,\infty] \tag{5.6}$$

which was used in [JSS]. This property can be checked directly for pure exponentials, and then one writes the potential as superposition of those. We can then easily estimate the first two terms in (5.5) by the desired bound $|t|^{-\frac{n}{2}}\|\psi_0\|_{L^1}$. For the second one, split the integration according to $0 < s < 1$, $1 < s < t-1$, and $t-1 < s < t$. For the values $t \geq 2$ we split the term

$$\int_0^t\int_0^s e^{i(t-s)\frac{\triangle}{2}}V(s,\cdot)e^{i(s-\tau)\frac{\triangle}{2}}V(\tau,\cdot)\psi(\tau)\,d\tau\,ds =$$

$$\Big(\int_0^{t-2}\int_{\tau+1}^{t-1} + \int_0^{t-2}\int_\tau^{\tau+1} + \int_0^{t-2}\int_{t-1}^{t} + \int_{t-2}^{t}\int_\tau^{t}\Big)e^{i(t-s)\frac{\triangle}{2}}V(s,\cdot)e^{i(s-\tau)\frac{\triangle}{2}}V(\tau,\cdot)\psi(\tau)\,ds\,d\tau. \tag{5.7}$$

With the help of the assumption (5.3) we estimate

$$\begin{aligned}\|\int_0^{t-2}\int_{\tau+1}^{t-1} & e^{i(t-s)\frac{\triangle}{2}}V(s,\cdot)e^{i(s-\tau)\frac{\triangle}{2}}V(\tau,\cdot)\psi(\tau)\,ds\,d\tau\|_{L^\infty}\\ &\lesssim \int_0^t\int_\tau^t \frac{\sup_s\|V(s,\cdot)\|_{L^1}}{\langle t-s\rangle^{\frac{3}{2}}}\frac{\sup_\tau\|V(\tau,\cdot)\|_{L^1\cap L^2}}{\langle s-\tau\rangle^{\frac{3}{2}}}\langle\tau\rangle^{-\frac{3}{2}}ds\,d\tau\|\psi_0\|_{L^1\cap L^2}\\ &\lesssim \langle t\rangle^{-\frac{3}{2}}\|\psi_0\|_{L^1\cap L^2}.\end{aligned} \tag{5.8}$$

Furthermore, in view of (5.6), the decay of the free evolution, and (5.3),

$$\begin{aligned}\Big\|\Big(\int_0^{t-2}\int_\tau^{\tau+1} + \int_0^{t-2}\int_{t-1}^{t}\Big)e^{i(t-s)\frac{\triangle}{2}}V(s,\cdot)e^{i(s-\tau)\frac{\triangle}{2}}V(\tau,\cdot)\psi(\tau)\,ds\,d\tau\Big\|_{L^\infty}&\\ \lesssim \int_0^t \sup_s\|\hat{V}(s,\cdot)\|_{L^1}\frac{\sup_\tau\|V(\tau,\cdot)\|_{L^1}}{\langle t-\tau\rangle^{\frac{3}{2}}}\langle\tau\rangle^{-\frac{3}{2}}\,d\tau\|\psi_0\|_{L^1\cap L^2}&\\ \lesssim \langle t\rangle^{-\frac{3}{2}}\|\psi_0\|_{L^1\cap L^2}.&\end{aligned}$$

Finally,

$$\begin{aligned}\Big\|\int_{t-2}^t\int_\tau^t e^{i(t-s)\frac{\triangle}{2}}V(s,\cdot)e^{i(s-\tau)\frac{\triangle}{2}}V(\tau,\cdot)\psi(\tau)\,ds\,d\tau\Big\|_{L^\infty}&\\ \lesssim \langle t\rangle^{-\frac{3}{2}}\sup_s\|\hat{V}(s,\cdot)\|_{L^1}\sup_\tau\|V(\tau,\cdot)\|_{L^1\cap L^2}\int_{t-2}^t\int_\tau^t|t-\tau|^{-\frac{3}{2}}ds\,d\tau\,\|\psi_0\|_{L^1\cap L^2}&\\ \lesssim \langle t\rangle^{-\frac{3}{2}}\|\psi_0\|_{L^1\cap L^2}.&\end{aligned} \tag{5.9}$$

The remaining estimate in the case $t < 2$ can be carried out precisely in the same fashion as in (5.9) with an integral lower limit $t-2$ replaced by 0. $\square$

6 Estimates for the inhomogeneous equation

Inspection of the proof of Theorem 2.7 indicates that we can extend the L^∞ estimates also to the case of the inhomogeneous equation

$$\frac{1}{i}\partial_t\psi - \frac{\triangle}{2}\psi + \sum_{\kappa=1}^{m} V(x-\vec{v}_\kappa t)\psi = F(t,x), \qquad \psi|_{t=0} = \psi_0. \tag{6.1}$$

This will be particularly useful in the forthcoming application [RSS] to the question of stability of multi-soliton states. Again, for simplicity consider the case of two potentials, $m = 2$. Then we have the following

Theorem 6.1. *Let V_κ be as in Definition 2.1, with the additional requirement that*

$$\max_{0\le m\le 2\left\lfloor\frac{n-2}{4}\right\rfloor} \|\partial^m V_\kappa\|_{L^\infty} < \infty. \tag{6.2}$$

Assume also that Condition 5.2 of Proposition 5.1 is satisfied for $V(t,x) = V_1(x) + V_2(x - t\vec{e}_1)$. Let ψ be a solution of equation (6.1) *with the right-hand side F satisfying the condition*

$$|||F||| := \sup_t \Big\{ \int_0^t \|F(\tau,\cdot)\|_{L^1}\,d\tau + \langle t\rangle^{\frac{n}{2}} \|F(t,\cdot)\|_{W^{\frac{n-2}{2}\pm,\frac{2n}{n+2}\mp}(\mathbb{R}^n)} \Big\} < \infty. \tag{6.3}$$

We assume the following decay of the projections of ψ onto the bound states corresponding to the Hamiltonians H_1 and H_2:

$$\|P_b(H_1)\psi(t,\cdot)\|_{L^2} + \|P_b(H_2,t)\psi(t,\cdot)\|_{L^2} \lesssim \langle t\rangle^{-\frac{n}{2}}, \tag{6.4}$$

cf. Definition 2.4. Then the L^∞ estimate

$$\|\psi(t,\cdot)\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}} \Big(\|\psi_0\|_{L^1\cap L^2} + |||F||| \Big) \tag{6.5}$$

holds.

Proof. The proof requires repeating the arguments from Sections 3 and 5. As in Theorem 2.7, we introduce the three channel decomposition and represent the solution via the respective Duhamel formulas. The novel terms generated by the inhomogeneous term F can be written in the form

$$I_1(t) = \chi_1 \int_0^t e^{-i(t-s)H_1} P_c(H_1) F(s,\cdot)\,ds$$
$$I_2(t) = \chi_2\, \mathfrak{g}_{-\vec{e}_1}(t) \int_0^t e^{-i(t-s)H_2} P_c(H_2)\, \mathfrak{g}_{\vec{e}_1}(s) F(s,\cdot)\,ds$$
$$I_3(t) = \chi_3 \int_0^t e^{-i(t-s)\frac{\triangle}{2}} F(s,\cdot)\,ds,$$

which should be compared with (3.8), (3.10), and (3.11), respectively. Observe that by our assumption (6.4) the projections of ψ onto the bound states corresponding to H_1 and H_2 already satisfy the desired decay. The terms I_1, I_2, I_3 are estimated in a similar manner. Consider for instance the L^∞ norm of $I_1(t)$:

$$\|I_1(t)\|_{L^\infty} \lesssim \int_0^{\frac{t}{2}} |t|^{-\frac{n}{2}} \|F(s,\cdot)\|_{L^1}\, ds + \int_{\frac{t}{2}}^{t} \|e^{-i(t-s)H_1} P_c(H_1) F(s,\cdot)\|_{W^{\frac{n-2}{2}\pm,\frac{2n}{n-2}\pm}} ds, \tag{6.6}$$

where the last terms appears as a consequence of the Sobolev imbedding $W^{\frac{n-2}{2}\pm,\frac{2n}{n-2}\pm} \subset L^\infty$. We now claim that for any $\alpha \geq 0$,

$$\Big\|(1-\triangle)^\alpha e^{itH_1} P_c(H_1) f\Big\|_{L^{p'}} \leq C(s,p)\, |t|^{-n(\frac{1}{p}-\frac{1}{2})} \Big\|(1-\triangle)^\alpha f\Big\|_{L^p}, \tag{6.7}$$

for $1 \leq p \leq 2$, and similarly for H_2. If $2k < \alpha \leq 2(k+1)$, $k \in Z_+$, then one also needs to assume that

$$\max_{0\leq m\leq 2k} \|\partial^m V_1\|_{L^\infty} < \infty. \tag{6.8}$$

Let C_1 be large. By the resolvent identity

$$(H_1 + iC_1)^{-1} = (-\triangle + iC_1)^{-1}\big(1 + (-\triangle + iC_1)^{-1} V_1\big)^{-1},$$

which holds for bounded V_1. The final inverse is then defined as a convergent Neumann series on L^q for any $1 \leq q \leq \infty$. This implies that $(H_1 + iC_1)^{-1}$ is bounded $L^q \to W^{2,q}$ and inductively, that $(H_1 + iC_1)^{-m}$ is bounded $L^q \to W^{2m,q}$ for any positive integer m and $1 < q < \infty$. For the latter one needs (6.8) if $2k < m \leq 2(k+1)$. Since it is clear that $(H_1 + iC_1)^m$ is bounded $W^{2m,q} \to L^q$ for any positive integer m under the same assumptions, one concludes by means of the standard decay estimate for H_1 as in [JSS] that (6.7) holds for all integers $\alpha \geq 0$. Interpolation finally yields (6.7) for all $\alpha \geq 0$. We now choose $p = \frac{2n}{n+2}\pm$ in (6.7), which leads to the decay $|t|^{-1\pm}$. Therefore,

$$\begin{aligned}\int_{\frac{t}{2}}^{t} \|e^{-i(t-s)H_1} P_c(H_1) F(s,\cdot)\|_{W^{\frac{n-2}{2}\pm,\frac{2n}{n-2}\pm}}\, ds &\lesssim \int_{\frac{t}{2}}^{t} \min\{|t-s|^{-1+}, |t-s|^{-1-}\} \|F(s,\cdot)\|_{W^{\frac{n-2}{2}\pm,\frac{2n}{n+2}\mp}}\, ds \\ &\lesssim t^{-\frac{n}{2}}\big(\sup s^{\frac{n}{2}} \|F(s,\cdot)\|_{W^{\frac{n-2}{2}\pm,\frac{2n}{n+2}\mp}}\big).\end{aligned}$$

We now easily obtain from (6.6) that $\|I_1(t)\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}} |||F|||$ provided the potential V_1 satisfies the necessary smoothness as given by (6.8) with $\alpha = \frac{n-2}{2}\pm$, and thus in particular if (6.2) holds. □

7 Decay estimates: Systems with a single matrix potential

7.1 Three dimensions

In the following sections we develop the decay estimates for the charge transfer model with matrix potentials. These systems arise in the study of stability of multi-soliton states on nonlinear Schrödinger equations [RSS].

We first investigate the question of the decay estimates for problems with a single time-independent matrix potential. We show that the approach developed by Rauch [R] for scalar equations can be naturally extended to systems. The method relies on meromorphic continuation of the resolvent of the Hamiltonian across the spectrum and requires exponential decay of the potential at infinity. Then, following the ideas we have developed in the previous sections for the scalar charge transfer models, we establish the decay estimates for solutions of the matrix charge transfer problem satisfying appropriate "asymptotic orthogonality" conditions. Our approach to dispersive estimates for systems is more direct than the one of Cuccagna [Cu], who extended Yajima's L^p-boundedness result for the wave operators to the case of systems. On the other hand, [Cu] does not require exponential decay of the potentials as we do.

In this section we consider the case of three dimensions, $n = 3$. The higher dimensional situation is discussed in the next section.

In this section $H = -\frac{1}{2}\triangle + \mu$ where $\mu > 0$ and H is an operator on $L^2(\mathbb{R}^3)$. Set

$$B = \begin{pmatrix} 0 & H \\ -H & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & V_1 \\ -V_2 & 0 \end{pmatrix}, \quad A = B + V = \begin{pmatrix} 0 & H + V_1 \\ -H - V_2 & 0 \end{pmatrix}.$$

By means of the matrix $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ one can also write

$$B = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} J, \quad A = \begin{pmatrix} H + V_1 & 0 \\ 0 & H + V_2 \end{pmatrix} J.$$

Since $B^* = -B$ it follows that $\mathrm{spec}(B) \subset i\mathbb{R}$. One checks that for $\Re z \neq 0$

$$\begin{aligned} (B - z)^{-1} &= -(B + z) \begin{pmatrix} (H^2 + z^2)^{-1} & 0 \\ 0 & (H^2 + z^2)^{-1} \end{pmatrix} \\ &= -\begin{pmatrix} (H^2 + z^2)^{-1} & 0 \\ 0 & (H^2 + z^2)^{-1} \end{pmatrix} (B + z) \end{aligned} \tag{7.1}$$

$$(A - z)^{-1} = (B - z)^{-1} - (B - z)^{-1} W_1 \Big[1 + W_2 J (B - z)^{-1} W_1\Big]^{-1} W_2 J (B - z)^{-1}, \tag{7.2}$$

where (7.2) also requires the expression in brackets to be invertible, and with

$$W_1 = \begin{pmatrix} |V_1|^{\frac{1}{2}} & 0 \\ 0 & |V_2|^{\frac{1}{2}} \end{pmatrix}, \quad W_2 = \begin{pmatrix} |V_1|^{\frac{1}{2}}\mathrm{sign}(V_1) & 0 \\ 0 & |V_2|^{\frac{1}{2}}\mathrm{sign}(V_2) \end{pmatrix}.$$

It follows from (7.1) that $\mathrm{spec}(B) = (-i\infty, -i\mu] \cup [i\mu, i\infty) \subset i\mathbb{R}$. We will assume that there exists $\varepsilon_0 > 0$ so that for all $x \in \mathbb{R}^3$

$$|V_j(x)| \leq Ce^{-\varepsilon_0 |x|} \quad \text{for} \quad j = 1, 2. \tag{7.3}$$

Then it follows from Weyl's theorem, see Theorem XIII.14 in [RS4], and the representation (7.2) for the resolvent of B, that $\mathrm{spec}_{ess}(A) = \mathrm{spec}_{ess}(B) = (-i\infty, -i\mu] \cup [i\mu, i\infty) \subset i\mathbb{R}$. This was observed by Grillakis [Gr]. Moreover, (7.2) implies via the analytic Fredholm alternative that $(A - z)^{-1}$ is a meromorphic function in $\mathbb{C} \setminus (-i\infty, -i\mu] \cup [i\mu, i\infty)$. We need to make several further assumptions on A that are motivated by applications to NLS, but we first switch to a different (but equivalent) way of writing systems. Let ψ be a solution of some NLS. Studying stability questions for ψ, as we do in [RSS], leads to a system for the variation $\delta\psi$ that can be written either for the vector of real and imaginary parts of $\delta\psi$, or for the vector $\binom{\delta\psi}{\delta\bar\psi}$. For the purposes of obtaining (7.2) it was convenient for us to follow the former convention, but from now on it will be advantageous to use the latter. It is clear that these two different ways of writing the system are equivalent by means of a change of coordinates in $\mathbb{C}^2$. More precisely, one has the following lemma whose proof is left to the reader.

Lemma 7.1. *Let*

$$\binom{\psi_1}{\psi_2} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \binom{f_1}{f_2}.$$

Then, with $H = -\frac{1}{2}\triangle + \mu$,

$$\partial_t \binom{f_1}{f_2} + \begin{pmatrix} 0 & -H - V_1 \\ H + V_2 & 0 \end{pmatrix} \binom{f_1}{f_2} = 0 \tag{7.4}$$

holds if and only if

$$\frac{1}{i}\partial_t \binom{\psi_1}{\psi_2} + \begin{pmatrix} H + U & -W \\ W & -H - U \end{pmatrix} \binom{\psi_1}{\psi_2} = 0 \tag{7.5}$$

where $U = \frac{1}{2}(V_1 + V_2)$, $W = \frac{1}{2}(V_1 - V_2)$.

Abusing notation, we will from now on write

$$B = \begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix}, \quad V = \begin{pmatrix} U & -W \\ W & -U \end{pmatrix}, \quad A = B + V = \begin{pmatrix} H + U & -W \\ W & -H - U \end{pmatrix}. \tag{7.6}$$

with U, W real-valued. We now specify the spectral assumptions we impose on A.

Definition 7.2. *Let* $A = B + V$ *with* V *exponentially decaying. We call the operator* A *on* $\mathcal{H} := L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ admissible *provided*

- $\mathrm{spec}(A) \subset \mathbb{R}$ *and* $\mathrm{spec}(A) \cap (-\mu, \mu) = \{\omega_\ell \mid 0 \le \ell \le M\}$, *where* $\omega_0 = 0$ *and all* ω_j *are distinct eigenvalues. There are no eigenvalues in* $\mathrm{spec}_{ess}(A)$.
- *For* $1 \le \ell \le M$, $L_\ell := \ker(A - \omega_\ell)^2 = \ker(A - \omega_\ell)$, *and* $\ker(A) \subsetneq \ker(A^2) = \ker(A^3) =: L_0$. *Moreover, these spaces are finite dimensional.*
- *The ranges* $\mathrm{Ran}(A - \omega_\ell)$ *for* $1 \le \ell \le M$ *and* $\mathrm{Ran}(A^2)$ *are closed.*
- *The spaces* L_ℓ *are spanned by exponentially decreasing functions in* $\mathcal{H}$ *(say with bound* $e^{-\varepsilon_0|x|}$*).*
- *All these assumptions hold as well for the adjoint* A^*. *We denote the corresponding (generalized) eigenspaces by* L_ℓ^*.
- *The points* $\pm\mu$ *are not resonances of* A *(we will define a resonance below, see Remark 7.10).*

Definition 7.2 is motivated by applications to the questions of stability of soliton solutions of NLS (see e.g. [BP1], [RSS]).
Henceforth it will be automatically assumed that A is admissible.

Lemma 7.3. *There is a direct sum decomposition*

$$\mathcal{H} = \sum_{j=0}^{M} L_j + \Big(\sum_{j=0}^{M} L_j^*\Big)^{\perp} \tag{7.7}$$

This means that the individual summands are linearly independent. The decomposition (7.7) is invariant under A. Let P_c denote the projection onto $\Big(\sum_{j=0}^{M} L_j^*\Big)^{\perp}$ which is induced by the splitting (7.7), and set $P_b = Id - P_c$. Then $AP_c = P_cA$, and there exist numbers c_{ij} so that

$$P_b f = \sum_{i,j} \phi_j c_{ij} \langle f, \psi_i \rangle \quad \textit{for all} \quad f \in \mathcal{H} \tag{7.8}$$

where ϕ_j, ψ_i are exponentially decreasing functions.

Proof. One has

$$\Big(\sum_{j=0}^{M} L_j^*\Big)^{\perp} = \mathrm{Ran}(A^2) \cap \bigcap_{j=1}^{M} \mathrm{Ran}(A - \omega_j) \tag{7.9}$$

by the assumption that the ranges are closed. We verify first that the summands in (7.7) are linearly independent. To this end suppose

$$f = \sum_{j=0}^{M} f_j, \quad f_j \in L_j,\ 0 \le j \le M, \quad f \in \Big(\sum_{j=0}^{M} L_j^*\Big)^{\perp}. \tag{7.10}$$

Then by (7.9), $f = A^2 g_0$ and $f = (A - \omega_j) g_j$ for some $g_0 \in \mathrm{Dom}(A^2)$, $g_j \in \mathrm{Dom}(A)$. Note that for $j \ge 1$, $f_j \in \ker(A - \omega_j) \Longrightarrow f_j \in \mathrm{Dom}(A^s)$ for all $s \ge 1$. Thus (7.10) implies that

$$A^2 f = \sum_{j=1}^{M} A^2 f_j, \quad \text{and} \quad A^2 \prod_{j=1}^{M} (A - \omega_j) f = 0.$$

Therefore, by our assumption on (generalized) eigenspaces

$$(A - \omega_\ell)^2 \prod_{j \ne \ell} (A - \omega_j) A^2 g_\ell = 0 \Longrightarrow (A - \omega_\ell) \prod_{j \ne \ell} (A - \omega_j) A^2 g_\ell = 0,$$

and similarly,

$$A^4 \prod_{j=1}^{M} (A - \omega_j) g_0 = 0 \Longrightarrow A^2 \prod_{j=1}^{M} (A - \omega_j) g_0 = 0.$$

Hence f satisfies the equations

$$\prod_{j=1}^{M} (A - \omega_j) f = 0, \quad A^2 \prod_{j \ne \ell} (A - \omega_j) f = 0 \quad \text{for all} \quad 1 \le \ell \le M.$$

Using that $L_j \cap L_k = \emptyset$ for $j \neq k$, one concludes inductively that $\prod_{j>j_0}(A-\omega_j)f = 0$ for any j_0. Indeed, setting $A_0 := A^2$ and $A_j := A - \omega_j$, we just showed that $\prod_{j\neq\ell} A_j f = 0$ for all $0 \leq \ell \leq M$. Thus also $\prod_{j\neq\ell,k} A_j f = 0$ for any $\ell \neq k$. Continuing inductively one finds that $f = 0$. By the linear independence of the spaces $\{L_j\}_{j=0}^M$ this implies that $f_0 = f_1 = \ldots = f_M = 0$. Thus,

$$\Big(\sum_{j=0}^M L_j\Big) \cap \Big(\sum_{j=0}^M L_j^*\Big)^\perp = \{0\}.$$

Dually, one finds that

$$\Big(\sum_{j=0}^M L_j\Big)^\perp \cap \Big(\sum_{j=0}^M L_j^*\Big) = \{0\}$$

which means that the right-hand side of (7.7) is dense in $\mathcal{H}$. Since the sum of a closed space with a finite dimensional space is again closed, we have proved that (7.7) is indeed a direct sum decomposition. Hence every $f \in \mathcal{H}$ can be written as $f = g + h$ where $g \in \sum_{j=0}^M L_j$, and $h \in \Big(\sum_{j=0}^M L_j^*\Big)^\perp$. We set $P_c f = h$, $P_b f = g$. Since $AL_j \subset L_j$ for $0 \leq j \leq M$, and $A\Big(\Big(\sum_{j=0}^M L_j^*\Big)^\perp\Big) \subset \Big(\sum_{j=0}^M L_j^*\Big)^\perp$, one has $AP_c = P_c A$. Denote the orthogonal projection onto $\sum_{j=0}^M L_j^*$ by Q. Then there is an isomorphism

$$S : \begin{cases} Qf \mapsto P_b f \\ \sum_{j=0}^M L_j^* \to \sum_{j=0}^M L_j \end{cases}$$

and $P_b = SQ$ by construction. As a map between finite dimensional spaces S is given by a matrix $\{c_{ij}\}$. By assumption, these finite dimensional subspaces of $\mathcal{H}$ are spanned by exponentially decaying functions, and (7.8) follows. □

Observe that (7.2) implies that there exists $\lambda > 0$ large so that

$$\|(A - (i\lambda + z))^{-1}\| \leq |\Im z|^{-1} \quad \text{for all} \quad \Im z \geq 0.$$

Unless indicated otherwise, $\|\cdot\|$ refers to the operator norm on $\mathcal{H}$. This allows one to define the (quasi-bounded) semigroup e^{itA} by means of the Hille–Yoshida theorem and $\|e^{itA}\| \leq e^{t\lambda}$. We now make one more requirement for A to be admissible, namely the *stability assumption*

$$\sup_{t\in\mathbb{R}} \|e^{itA} P_c\| < \infty. \tag{7.11}$$

Under this assumption the semigroup is bounded on all functions that have no component in the space L_0 under the splitting (7.7). Otherwise, it can grow at most like t.
Assumption (7.11) is related to the notion of *linear stability* in the context of stability of soliton solutions of NLS (see e.g. [Wein], [St]).
Our goal in this section is to prove the following theorem.

Theorem 7.4.

- *Let A be admissible as in Definition 7.2 together with the stability assumption* (7.11)*. Then*

$$\|e^{itA}P_c f\|_{L^2+L^\infty} \lesssim \langle t\rangle^{-\frac{3}{2}}\|f\|_{L^1\cap L^2} \tag{7.12}$$

  for all t.

- *If in addition the matrix potential satisfies $\|\hat{V}\|_{L^1} < \infty$, then one has the bound*

$$\|e^{itA}P_c f\|_{L^\infty} \lesssim |t|^{-\frac{3}{2}}\|f\|_{L^1\cap L^2} \tag{7.13}$$

  for all t.

The norms here are given by Definition 2.3

$$\|f\|_{L^2+L^\infty} := \inf_{f=h+g}(\|h\|_{L^2} + \|g\|_{L^\infty})$$

with the dual space $(L^2 + L^\infty)^* = L^1 \cap L^2$.

The proof of Theorem 7.4 will follow an old strategy of Rauch [R], see also [DMcLT] and [Vai], that is based on representing e^{itA} as the inverse Laplace transform of the resolvent, and then to move the contour across the spectrum. This only gives local L^2 decay, and we then use an observation of Ginibre [Gin] to pass from this to the bound (7.12). In the following we prepare the proof of Theorem 7.4 by means of several technical lemmas. We will assume from now on that A is as in Theorem 7.4.
The next lemma is a version of von Neumann's ergodic theorem.

Lemma 7.5. *For any $\omega \in \mathbb{R}$,*

$$\frac{1}{T}\int_0^T e^{-i\omega t}e^{itA}P_c\,dt \to 0 \quad \textit{as } T \to \infty \tag{7.14}$$

in the strong sense.

Proof. Let $\{\omega_j\}_{j=0}^M$ be as in Definition 7.2. We first prove (7.14) for ω equal to one of the ω_j, $j \neq 0$. Let $f \in \mathrm{Ran}P_c$. Then for every $1 \le j \le M$ there is $g_j \in \mathrm{Dom}(A)$ so that $f = (A - \omega_j)g_j$, see (7.9). Note that g_j is unique up to addition of an element of L_j. Thus

$$\frac{1}{T}\int_0^T e^{-i\omega_j t}e^{itA} f\,dt = \frac{1}{iT}\Big(e^{iT(A-\omega_j)}g_j - g_j\Big). \tag{7.15}$$

It remains to show that we can choose g_j so that (7.11) applies. To do so, use (7.7) to write $g_j = \sum_{k=0}^M h_k + \tilde{h}$ where $h_k \in L_k$ and $\tilde{h} \in (\sum_{k=0}^M L_k^*)^\perp =: \tilde{L}$. We may assume that $h_j = 0$. Then $Ag_j = \sum_{k=0}^M Ah_k + A\tilde{h}$ and by A-invariance of (7.7), $Ah_k \in L_k$, $A\tilde{h} \in \tilde{L}$. On the other hand,

$$Ag_j = f + \omega_j g_j = \sum_{k=0}^M \omega_j\, h_k + \omega_j\tilde{h} + f.$$

The last two terms lie in $\mathrm{Ran}(P_c) = \tilde{L}$, whereas clearly $\omega_j\, h_k \in L_k$ for all $0 \le k \le M$. By uniqueness of such a representation, $Ah_k = \omega_j h_k$ for all $0 \le k \le M$. But then

$$\begin{aligned} h_k &\in \ker(A-\omega_j)\cap\ker(A-\omega_k) = \{0\} \ \text{ for all } \ k \neq j, \\ h_0 &\in \ker(A-\omega_j)\cap\ker(A^2) = \{0\}. \end{aligned}$$

Since $h_j = 0$ by assumption, it follows that $g_j = \tilde{h} \in \mathrm{Ran}(P_c)$, as desired. Hence the right–hand side of (7.15) is $O(T^{-1})$ and we are done with this case. Now consider the case $\omega = \omega_0 = 0$. We write $f = A^2 g$ for some $g \in \mathrm{Dom}(A^2)$. By (7.7) one has as before $Ag = \sum_{k=0}^{M} h_k + \tilde{h}$ where $h_k \in L_k$ and $\tilde{h} \in \tilde{L}$. We claim that $-h_0 + Ag \in \mathrm{Ran}(P_c)$ and $Ah_0 = 0$. If so, then one has $f = A(-h_0 + Ag)$ which implies that

$$\frac{1}{T}\int_0^T e^{itA} f\, dt = \frac{1}{iT}\Big(e^{iTA}(-h_0 + Ag) - (-h_0 + Ag)\Big),$$

cf. (7.15), and the right–hand side is again $O(T^{-1})$ by (7.11). To prove the claim, simply observe that

$$f = A^2 g = \sum_{k=0}^{M} Ah_k + A\tilde{h} \Longrightarrow \sum_{k=0}^{M} Ah_k \in \tilde{L}.$$

Since $Ah_k \in L_k$ one concludes from Lemma 7.3 that $Ah_k = 0$ for all $0 \le k \le M$. If $1 \le k \le M$, then $h_k \in L_0 \cap L_k = \{0\}$ so that $h_k = 0$. But this is precisely the claim, and we are done with the case $\omega = 0$ as well.

It remains to consider $\omega \in \mathbb{R} \setminus \{\omega_j\}_{j=0}^{M}$. Firstly, if $\omega \in (-\mu, \mu)$, then $(A - \omega)^{-1}$ is a bounded operator on $\mathcal{H}$ by assumption. Hence,

$$\frac{1}{T}\int_0^T e^{-i\omega t} e^{itA} P_c f\, dt = (A - \omega)^{-1}\frac{1}{iT}\Big(e^{iT(A-\omega)} P_c f - P_c f\Big),$$

and the right–hand side is $O(T^{-1})$ by (7.11). Finally, assume that $|\omega| \ge \mu$ so that ω belongs to the essential spectrum of A. This case is different from the previous ones because one cannot expect to obtain the $O(T^{-1})$ bound. Since

$$\sup_T \left\| \frac{1}{T}\int_0^T e^{-i\omega t} e^{itA} P_c f\, dt \right\| < \infty$$

for all f by (7.11) it suffices to prove that

$$\frac{1}{T}\int_0^T e^{-i\omega t} e^{itA} f\, dt \to 0$$

for all f belonging to a dense subspace of $\mathrm{Ran}(P_c)$. In view of the preceding, it therefore suffices to show that for all $|\omega| \ge \mu$,

$$\overline{(A - \omega)\mathrm{Ran}(P_c)} = \mathrm{Ran}(P_c). \tag{7.17}$$

By the A-invariance of $\mathrm{Ran}(P_c)$ it is clear that $\overline{(A - \omega)\mathrm{Ran}(P_c)} \subset \mathrm{Ran}(P_c)$. By Lemma 7.3 we can write a direct sum decomposition $\mathcal{H} = L' + \tilde{L}$ where $L' = \sum_{j=0}^{M} L_j$ and $\tilde{L}$ is the space from above with finite codimension. Now $\ker(A^* + \omega) = \{0\}$ implies that

$$\mathcal{H} = \overline{(A - \omega)\mathcal{H}} = \overline{(A - \omega)\tilde{L}} + (A - \omega)L'. \tag{7.18}$$

Note that by A-invariance of the splitting both summands here lie again in the spaces $\tilde{L}$ and L', respectively. In particular, the sum on the right-hand side is again direct. Since $\ker(A - \omega) = \{0\}$, one has $\dim((A-\omega)L') = \dim(L') < \infty$ so that $(A-\omega)L' = L'$. By the directness of the sum in (7.18) therefore

$$\overline{(A-\omega)\tilde{L}} = \tilde{L},$$

as desired. $\square$

The last of the abstract statements is a bound on the resolvents.

Lemma 7.6. *For all $z \in \mathbb{C}$ with $\Re z > 0$ one has*

$$\|(iA-z)^{-1}P_c\| \leq \frac{C}{\Re z}, \tag{7.19}$$

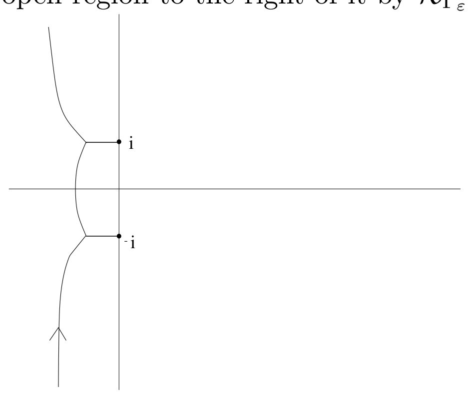
for some constant C.

Proof. Since for $\Re z > 0$,

$$(iA-z)^{-1}P_c = -\int_0^\infty e^{-tz}e^{itA}P_c\,dt,$$

(7.19) is an immediate consequence of (7.11) with $C = \sup_{t\in\mathbb{R}} \|e^{itA}P_c\|$. $\square$

Following Rauch [R], we now show how to continue the resolvents $(iA-z)^{-1} = -i(A+iz)^{-1}$ meromorphically across the spectrum as bounded operators in an exponentially weighted L^2 space. Set $(E_\varepsilon f)(x) := e^{-\varepsilon\rho(x)}f(x)$ where ρ is smooth, $\rho(x) = |x|$ for large x and $\varepsilon > 0$ will be some small number (compared to ε_0 in (7.3) above). Our goal now is to prove that $E_\varepsilon(iA-z)^{-1}E_\varepsilon$ has a meromorphic continuation to the region which lies to the right of the curve shown below. From now on, we set $\mu = 1$, so that $H = -\frac{1}{2}\triangle + 1$. Moreover, we denote the curve depicted in the following figure by Γ_ε and the open region to the right of it by $\mathcal{R}_{\Gamma_\varepsilon}$.



We start with the resolvent of B. By (7.6) one has

$$(B+iz)^{-1} = \begin{pmatrix} (H+iz)^{-1} & 0 \\ 0 & -(H-iz)^{-1} \end{pmatrix}. \tag{7.20}$$

The resolvents

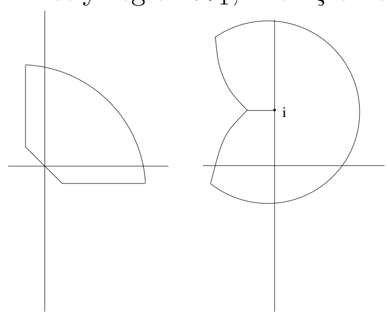
$$(H \pm iz)^{-1}(x,y) = (-\frac{1}{2}\triangle + 1 \pm iz)^{-1}(x,y) = \frac{e^{\sqrt{2(1\pm iz)}\,|x-y|}}{2\pi|x-y|} \tag{7.21}$$

have singularities at the points $\pm i$, respectively. We will resolve these singularities by means of the transformations $U_1 : \zeta \mapsto z = i - i\zeta^2$, and $U_2 : \zeta \mapsto -i + i\zeta^2$. For the domain of U_1 we will choose either of the regions

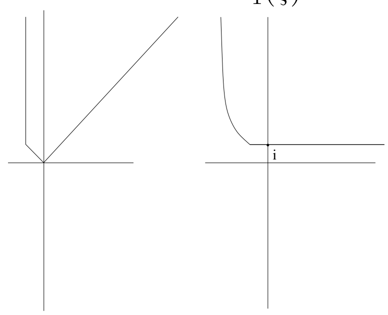
$$\mathcal{R}_1 = \mathcal{R}_1(\varepsilon) := \Big\{\zeta \in \mathbb{C} \,|\, \Re\zeta > -\varepsilon,\ \Im\zeta > -\varepsilon,\ -\frac{\pi}{4} \le \operatorname{Arg}\zeta \le \frac{3\pi}{4},\ |\zeta| \le r\Big\} \tag{7.22}$$

$$\mathcal{R}_2 = \mathcal{R}_2(\varepsilon) := \Big\{\zeta \in \mathbb{C} \,|\, \Re\zeta > -\varepsilon,\ \frac{\pi}{4} \le \operatorname{Arg}\zeta \le \frac{3\pi}{4}\Big\}. \tag{7.23}$$

Here $\varepsilon > 0$ is a small parameter, and r will be chosen to be on the order of 1. These two regions are shown in the following figures. Firstly region $\mathcal{R}_1$, with ζ on the left and $z = U_1(\zeta)$ on the right,



and secondly region $\mathcal{R}_2$, with ζ on the left and $z = U_1(\zeta)$ on the right:



The curved (but not circular) pieces of the boundaries that are shown in the z-plane are parabolic arcs which can be of course written down exactly. For the map U_2 we choose the domains $\mathcal{R}_1^*$ and $\mathcal{R}_2^*$ which are the reflections of $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively, across the real axis (i.e., conjugation; note that $U_2(\zeta) = \overline{U_1(\bar\zeta)}$). One now defines $\mathcal{R}_{\Gamma_\varepsilon} \cap \{z \in \mathbb{C} \,|\, \Re z \le 1/2\}$ by means of four separate regions, namely by $U_1(\mathcal{R}_1)$, $U_1(\mathcal{R}_2)$, $U_2(\mathcal{R}_1^*)$, $U_2(\mathcal{R}_2^*)$ with r sufficiently large, say $r = \sqrt{5}/2$ (by taking ε small we can still ensure that we move only a very short distance into the negative half plane $\Re z < 0$, at least on compact sets). This choice of r is made to ensure that the two disks intersect the real axis without coming close to each others centers.

Lemma 7.7. *Let $0 < \varepsilon$ be small. Then the weighted resolvent $E_\varepsilon(iB - z)^{-1}E_\varepsilon$, originally defined on $\Re z > 0$, admits an analytic continuation to the region $\mathcal{R}_{\Gamma_{\varepsilon/4}}$ with values in the compact operators on $\mathcal{H}$. This continuation, which we denote by $F_\varepsilon(z)$, satisfies the estimate $\|F_\varepsilon(z)\| \le C_\varepsilon(1+|z|)^{-\frac{1}{2}}$ for all $z \in \mathcal{R}_{\Gamma_{\varepsilon/4}}$. Furthermore, both $F_\varepsilon(i - i\zeta^2)$ and $F_\varepsilon(-i + i\zeta^2)$ are analytic for $|\zeta| < \frac{\varepsilon}{4}$.*

Proof. It was shown by Rauch that (recall that $H = -\frac{1}{2}\triangle + 1$)

$$\zeta \mapsto E_\varepsilon(H + i(i - i\zeta^2))^{-1}E_\varepsilon = E_\varepsilon(-\frac{1}{2}\triangle + \zeta^2)^{-1}E_\varepsilon$$

has an analytic extension to the domain $\Re\zeta > -\varepsilon/4$ with a norm bound of $(1+|\zeta|)^{-1}$ (see equation (2.12) in [R]). Similarly for $\zeta \mapsto E_\varepsilon(H - i(-i+i\zeta^2))^{-1}E_\varepsilon = E_\varepsilon(-\frac{1}{2}\triangle + \zeta^2)^{-1}E_\varepsilon$. This shows that both $E_\varepsilon(H \pm iz)^{-1}E_\varepsilon$ have analytic continuations to the region $\mathcal{R}_{\Gamma_{\varepsilon/4}}$ satisfying the norm bound $(1+|z|)^{-\frac{1}{2}}$ as $|z| \to \infty$ in $\mathcal{R}_{\Gamma_{\varepsilon/4}}$. The lemma follows. □

We can now formulate our meromorphic continuation result for the perturbed resolvent.

Corollary 7.8. *Let $0 < \varepsilon$ be small depending on ε_0 from Definition 7.2. Then for some $\delta > 0$, the weighted resolvents $E_\varepsilon(iA - z)^{-1}P_cE_\varepsilon$, originally defined on $\Re z > 0$, admit a meromorphic continuation to the region $\mathcal{R}_{\Gamma_\delta}$ with values in the compact operators on $\mathcal{H}$. This continuation, which we denote by $G_\varepsilon(z)$, has only finitely many poles in $\mathcal{R}_{\Gamma_\delta}$ all of which lie in $\Re z \le 0$. Moreover, it has continuous limits on the two horizontal parts of $\Gamma_\delta \setminus \{\pm i\}$ from above and below. Finally, for large $|z|$ there are no poles and one has the estimate*

$$\|G_\varepsilon(z)\| \le \frac{C_\varepsilon}{\sqrt{|z|}} \tag{7.24}$$

for all large $z \in \mathcal{R}_{\Gamma_\delta}$.

Proof. Since $P_c = Id - P_b$ and one has the explicit representation (7.8) with exponentially decaying ϕ_j, it suffices to prove the same statement for $E_\varepsilon(iA - z)^{-1}E_\varepsilon = -iE_\varepsilon(A+iz)^{-1}E_\varepsilon$. By the resolvent identity,

$$\begin{aligned} E_\varepsilon(A+iz)^{-1}E_\varepsilon &= E_\varepsilon(B+iz)^{-1}E_\varepsilon - E_\varepsilon(B+iz)^{-1}E_\varepsilon\, E_\varepsilon^{-1}VE_\varepsilon^{-1}E_\varepsilon(A+iz)^{-1}E_\varepsilon \\ E_\varepsilon(A+iz)^{-1}E_\varepsilon &= \Big(1 + E_\varepsilon(B+iz)^{-1}E_\varepsilon\, E_\varepsilon^{-2}V\Big)^{-1} E_\varepsilon(B+iz)^{-1}E_\varepsilon. \end{aligned}$$

Lemma 7.7 and the analytic Fredholm alternative imply that $E_\varepsilon(A+iz)^{-1}E_\varepsilon$ has a meromorphic extension to $\mathcal{R}_{\Gamma_{\varepsilon/4}}$ which is analytic if $|z|$ is large. Moreover,

$$G_\varepsilon(z) = \Big(1 + iF_\varepsilon(z)\, E_\varepsilon^{-2}V\Big)^{-1} F_\varepsilon(z) \tag{7.25}$$

for all $z \in \mathcal{R}_{\Gamma_{\varepsilon/4}}$ at which the inverse is defined. Hence (7.24) follows from the previous lemma. The poles can only accumulate on the boundary $\Gamma_{\varepsilon/4}$ of $\mathcal{R}_{\Gamma_{\varepsilon/4}}$. If $\delta < \frac{\varepsilon}{4}$, then the poles of G_ε can therefore only accumulate on the two horizontal pieces of Γ_δ. We claim that reducing δ even further, one can eliminate any accumulation on these horizontal pieces as well. To see this, "resolve" the singularities at the points $\pm i$ by means of the maps $z = U_1(\zeta) = i - i\zeta^2$ and $z = U_2(\zeta) = -i + i\zeta^2$, respectively. Clearly,

$$G_\varepsilon(U_1(\zeta)) = \Big(1 + F_\varepsilon(U_1(\zeta))\, E_\varepsilon^{-2}V\Big)^{-1} F_\varepsilon(U_1(\zeta)) \tag{7.26}$$

as long as $U_1(\zeta) \subset \mathcal{R}_{\Gamma_{\varepsilon/4}}$. In particular, this holds for all $\zeta \in \mathcal{R}(\varepsilon/4)$. In view of the final statement of Lemma 7.7, one can apply the analytic Fredholm alternative to (7.26) with $|\zeta| < \frac{\varepsilon}{4}$. Thus $G_\varepsilon(U_1(\zeta))$ is a meromorphic function for $|\zeta| < \frac{\varepsilon}{4}$. It follows that for $\delta > 0$ small, $G_\varepsilon(U_1(\zeta))$ is analytic on $|\zeta| < \delta$ with the possible exception of a pole at $\zeta = 0$. In the z variable this means that $G_\varepsilon(z)$ has continuous limits (from above and below) on the horizontal piece of Γ_δ emanating from $+i$, with a possible pole at $z = i$. The same statements also hold at $-i$. □

We are now ready to prove the local L^2-decay.

Proposition 7.9. *Let A be as in Definition 7.2 together with* (7.11)*. Then for $\varepsilon > 0$ small one has*

$$\|E_\varepsilon e^{itA} P_c E_\varepsilon\| \lesssim \langle t \rangle^{-\frac{3}{2}}. \tag{7.27}$$

for all t.

Proof. By the inversion formula for the Laplace transform (to be justified below),

$$E_\varepsilon(e^{itA} - I)P_c E_\varepsilon = \frac{-1}{\pi i} \int_{a-i\infty}^{a+i\infty} e^{t\tau} \Big(E_\varepsilon (iA - \tau)^{-1} P_c\, E_\varepsilon + E_\varepsilon P_c E_\varepsilon \tau^{-1} \Big)\, d\tau$$

for any $a > 0$. The subtraction of the identity serves to make the integral absolutely convergent. Indeed, $(iA - \tau)(iA - \tau)^{-1} = Id$ implies that

$$E_\varepsilon (iA - \tau)^{-1} P_c E_\varepsilon + \frac{1}{\tau} E_\varepsilon P_c E_\varepsilon = \frac{1}{\tau} E_\varepsilon (iA - \tau)^{-1} P_c E_\varepsilon E_\varepsilon^{-1} iA E_\varepsilon.$$

By means of a commutator calculation one checks that $E_\varepsilon^{-1} A E_\varepsilon$ is a bounded operator $W^{2,2} \times W^{2,2} \to \mathcal{H}$. Hence, by (7.24), the left-hand side decays like $|\tau|^{-\frac{3}{2}}$. We now deform the contour to Γ_δ as in Corollary 7.8. This gives

$$\begin{aligned} E_\varepsilon(e^{itA} - I)P_c E_\varepsilon \;&=\; \frac{-1}{\pi i} \int_{\Gamma_\delta} e^{t\tau} \Big(G_\varepsilon(\tau) + E_\varepsilon P_c E_\varepsilon \tau^{-1} \Big)\, d\tau \\ &\quad -2 \sum_{\tau_j \text{ poles}} \operatorname{Res}\Big(e^{t\tau} \Big(G_\varepsilon(\tau) + E_\varepsilon P_c E_\varepsilon \tau^{-1} \Big); \tau = \tau_j \Big). \end{aligned}$$

This can be simplified to

$$E_\varepsilon e^{itA} P_c E_\varepsilon = \frac{-1}{\pi i} \int_{\Gamma_\delta} e^{t\tau}\, G_\varepsilon(\tau)\, d\tau - 2 \sum_{\tau_j \text{ poles}} \operatorname{Res}\Big(e^{t\tau} G_\varepsilon(\tau); \tau = \tau_j \Big). \tag{7.29}$$

The integral over Γ_δ converges by (7.24). Clearly,

$$\sum_{\tau_j \text{ poles}} \operatorname{Res}\Big(e^{t\tau} G_\varepsilon(\tau); \tau = \tau_j \Big) = \sum_{\tau_j \text{ poles}} p_j(t) e^{t\tau_j},$$

where $p_j(t)$ is a polynomial with coefficients given by compact operators of degree equal to the order of the pole minus one. As far as the integral over Γ_δ is concerned, only the horizontal "thermometers" can contribute much. "Thermometer" here refers to the fact that one the horizontal pieces one first moves to the right on the lower edge, then around a loop encircling $\pm i$, and then to the left on the upper edge. It is clear that the parabolic arcs give an exponentially small contribution. Denote the two thermometers around i and $-i$ by $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. To determine their contributions, we need to again use the ζ-variables. Indeed, setting $z = U_1(\zeta) = i - i\zeta^2$ we showed above that $G_\varepsilon(i - i\zeta^2)$ is analytic for small ζ up to possibly a pole at 0. Writing the Laurent expansion $G_\varepsilon(i-i\zeta^2)=\sum_{j=-N}^\infty B_j\zeta^j$, one deduces from Lemma 7.6 that $N\le 2$. Also, our assumption on the absence of resonances means that $B_{-1}=0$. Therefore,

$$\frac{1}{\pi i}\int_{\mathcal{T}_1} e^{t\tau}\,G_\varepsilon(\tau)\,d\tau=\frac{1}{\pi i}\int_{\mathcal{T}_1} e^{t\tau}\,\frac{B_{-2}}{\tau-i}\,d\tau+\frac{1}{\pi i}\int_{\mathcal{T}_1} e^{t\tau}\,C(\tau)\,d\tau, \tag{7.30}$$

where $\|C(\tau)\|\lesssim|\tau-i|^{\frac12}$ so that the contribution of the second integral is $O(|t|^{-\frac32})$. The first is equal to the compact operator $e^{it}\,B_{-2}$. The conclusion is that (7.29) now takes the form of the estimate

$$\Big\|E_\varepsilon e^{itA}P_cE_\varepsilon+\sum_j p_j(t)e^{it\eta_j}\Big\|\lesssim|t|^{-\frac32} \tag{7.31}$$

where we have only retained the purely imaginary poles $\tau_j=e^{i\eta_j}$ with distinct η_j. In view of (7.11) it follows that for large t,

$$\sup_{t\in\mathbb{R}}\Big\|\sum_j p_j(t)e^{it\eta_j}\Big\|\lesssim 1$$

which implies that necessarily each $p_j=\text{const}$. Finally, the ergodic average lemma, Lemma 7.5, shows that $p_j=0$. Hence we are left with

$$\Big\|E_\varepsilon e^{itA}P_cE_\varepsilon\Big\|\lesssim|t|^{-\frac32},$$

for large t as desired. □

Remark 7.10. According to Corollary 7.8 (and its proof), $G_\varepsilon(i-i\zeta^2)$ and $G_\varepsilon(-i+i\zeta^2)$ can at most have poles of second order at $\zeta=0$. It is well-known that in generic situation there will be no singularity at the origin. In fact, a second order pole means that $\pm i$ are eigenvalues, whereas a first order pole is what one calls a resonance. For more on this, see for example [R].

We now pass from local L^2-decay to global decay.

Proof of Theorem 7.4. We start with the first statement in the theorem. By Duhamel's formula, and with A,B,V as in (7.6),

$$e^{itA}P_c = e^{itB}P_c-i\int_0^t e^{i(t-s)B}\,Ve^{isA}P_c\,ds=e^{itB}P_c-i\int_0^t e^{i(t-s)B}\,VP_ce^{isA}\,ds \tag{7.32}$$

$$P_ce^{isB} = P_ce^{isA}-i\int_0^s P_ce^{i(s-\tau)A}\,Ve^{i\tau B}\,d\tau=P_ce^{isA}-i\int_0^s e^{i(s-\tau)A}\,P_cVe^{i\tau B}\,d\tau \tag{7.33}$$

Inserting (7.33) into (7.32) yields

$$e^{itA}P_c=e^{itB}P_c-i\int_0^t e^{i(t-s)B}\,VP_ce^{isB}\,ds+\int_0^t\int_0^s e^{i(t-s)B}\,Ve^{i(s-\tau)A}\,P_cVe^{i\tau B}\,d\tau.$$

Therefore, applying Proposition 7.9 to the middle term in the double integral, yields

$$\begin{aligned}\|e^{itA}P_c\vec\psi_0\|_{L^2+L^\infty} &\lesssim \langle t\rangle^{-\frac32}\|P_c\|_{L^1\cap L^2\to L^1\cap L^2}\,\|\vec\psi_0\|_{L^1\cap L^2}\\ &\quad+\int_0^t\langle t-s\rangle^{-\frac32}\langle s\rangle^{-\frac32}\,ds\,\|VP_c\|_{L^2+L^\infty\to L^1\cap L^2}\,\|\vec\psi_0\|_{L^1\cap L^2}\\ &\quad+\int_0^t\int_0^s\langle t-s\rangle^{-\frac32}\langle s-\tau\rangle^{-\frac32}\langle\tau\rangle^{-\frac32}\,d\tau ds\,\|E_\varepsilon^{-1}V\|^2_{L^2\to L^1\cap L^2}\,\|\vec\psi_0\|_{L^1\cap L^2}\\ &\lesssim \langle t\rangle^{-\frac32}\|\vec\psi_0\|_{L^1\cap L^2}.\end{aligned} \tag{7.35}$$

To pass to (7.35) one uses that

$$\|VP_c\|_{L^2+L^\infty\to L^1\cap L^2}<\infty,\quad \|P_c\|_{L^1\cap L^2\to L^1\cap L^2}<\infty,\quad \|VE_\varepsilon^{-1}\|_{L^2\to L^1\cap L^2}<\infty,$$

which follow from (7.8) and the assumption that V,ϕ_j,ψ_i are exponentially decaying.
To remove L^2 we use the same device from [JSS] that already appeared in Section 5 for the same purpose. However, some care is needed for the case of systems since the analogue of (5.6), viz.

$$\sup_{1\le p\le\infty}\|e^{itB}Ve^{-itB}\|_{p\to p}<\infty$$

does not hold now. Indeed, by (7.6), and with $H=-\frac{1}{2}\triangle+\mu$,

$$e^{itB}Ve^{-itB}=\begin{pmatrix} e^{itH}Ue^{-itH} & -e^{itH}We^{itH}\\ e^{-itH}We^{-itH} & -e^{-itH}Ue^{itH}\end{pmatrix}, \tag{7.36}$$

and the off-diagonal terms are not bounded on L^p if $p\neq 2$, $W\neq 0$. As in (5.5),

$$\begin{aligned} e^{itA}\vec\psi_0 &= e^{itB}\vec\psi_0 - i\int_0^t e^{i(t-s)B}Ve^{itA}\vec\psi_0\,ds\\ &= e^{itB}\vec\psi_0 - i\int_0^t e^{i(t-s)B}Ve^{isB}\vec\psi_0\,ds - \int_0^t\int_0^s e^{i(t-s)B}Ve^{i(s-\tau)B}Ve^{i\tau A}\vec\psi_0\,d\tau\,ds. \end{aligned} \tag{7.37}$$

In what follows it suffices to assume $U=0$, since the diagonal terms in (7.36) have the same cancellation as in the scalar case, see (5.6). Consider one of the off-diagonal entries of the second term in (7.37) for times $t\ge\frac{1}{2}$. Thus,

$$\begin{aligned} \Big\|\int_0^t e^{i(t-s)H}We^{-isH}\vec\psi_0\,ds\Big\|_\infty &\le \Big\|\int_0^{\frac14} e^{i(t-s)H}We^{-isH}\vec\psi_0\,ds\Big\|_\infty + \Big\|\int_{\frac14}^{t-\frac14} e^{i(t-s)H}We^{-isH}\vec\psi_0\,ds\Big\|_\infty\\ &\quad + \Big\|\int_{t-\frac14}^{t} e^{i(t-s)H}We^{-isH}\vec\psi_0\,ds\Big\|_\infty\\ &\lesssim \int_0^{\frac14}|t-2s|^{-\frac32}\,\|e^{isH}We^{-isH}\|_{1\to1}\|\vec\psi_0\|_1\,ds + \int_{\frac14}^{t-\frac14}|t-s|^{-\frac32}\,\|W\|_1\,|s|^{-\frac32}\|\vec\psi_0\|_1\,ds\\ &\quad + \int_{t-\frac14}^{t}\|e^{i(t-s)H}We^{-i(t-s)H}\|_{\infty\to\infty}\|e^{i(t-2s)H}\vec\psi_0\|_\infty\,ds\\ &\lesssim (\|W\|_1+\|\hat W\|_1)\,|t|^{-\frac32}\|\vec\psi_0\|_1, \end{aligned} \tag{7.38}$$

where we have employed the cancellation property (5.6) twice. For small times $0<t$ one obtains

similarly

$$\Big\| \int_0^t e^{i(t-s)H} W e^{-isH} \vec{\psi}_0 \, ds \Big\|_\infty \le \Big\| \int_0^{\frac{1}{2}t-t^2} e^{i(t-s)H} W e^{-isH} \vec{\psi}_0 \, ds \Big\|_\infty$$
$$+ \Big\| \int_{|\frac{t}{2}-s|<t^2} e^{i(t-s)H} W e^{-isH} \vec{\psi}_0 \, ds \Big\|_\infty + \Big\| \int_{\frac{1}{2}t+t^2}^t e^{i(t-s)H} W e^{-isH} \vec{\psi}_0 \, ds \Big\|_\infty$$
$$\lesssim \int_0^{\frac{1}{2}t-t^2} |t-2s|^{-\frac{3}{2}} \, \|e^{isH} W e^{-isH}\|_{1\to 1} \|\vec{\psi}_0\|_1 \, ds + \int_{|\frac{t}{2}-s|<t^2} |t-s|^{-\frac{3}{2}} \, \|W\|_1 \, |s|^{-\frac{3}{2}} \|\vec{\psi}_0\|_1 \, ds$$
$$+ \int_{\frac{1}{2}t+t^2}^t \|e^{i(t-s)H} W e^{-i(t-s)H}\|_{\infty\to\infty} \|e^{i(t-2s)H} \vec{\psi}_0\|_\infty \, ds$$
$$\lesssim (\|W\|_1 + \|\hat{W}\|_1) \, |t|^{-1} \|\vec{\psi}_0\|_1. \tag{7.39}$$

We now deal with the third term in (7.37), and we select one representative off-diagonal term of the form

$$\int_0^t \int_0^s e^{i(t-s)H} W e^{i(\tau-s)H} U \psi_1(\tau) \, d\tau ds, \tag{7.40}$$

where $\psi_1(\tau)$ is the first component of the solution $\vec{\psi}(\tau) := e^{i\tau A}\vec{\psi}_0$. For $t \ge 4$, introducing the new variable $\tau' := 2s - \tau$, we rewrite (7.40) in the form

$$\int_0^t \int_s^{2s} e^{i(t-s)H} W e^{i(s-\tau')H} U \psi_1(2s-\tau') \, d\tau' ds =$$
$$\Big(\int_0^{t-1} \int_s^{2s} + \int_{t-1}^t \int_s^{2t-s} + \int_{t-1}^t \int_{2t-s}^{2s} \Big) e^{i(t-s)H} W e^{i(s-\tau')H} U \psi_1(2s-\tau') \, d\tau' ds \tag{7.41}$$

The estimate for the first term in (7.41) is straightforward

$$\| \int_0^{t-1} \int_s^{2s} e^{i(t-s)H} W e^{i(s-\tau')H} U \psi_1(2s-\tau') \, d\tau' ds \|_\infty \lesssim$$
$$\int_0^{t-1} \int_s^{2s} \langle t-s \rangle^{-\frac{3}{2}} \|W\|_{1\cap 2} \langle s-\tau' \rangle^{-\frac{3}{2}} \|U\|_{1\cap 2} \langle 2s-\tau' \rangle^{-\frac{3}{2}} \|\vec{\psi}_0\|_{1\cap 2} \, d\tau' ds \lesssim$$
$$\langle t \rangle^{-\frac{3}{2}} \|\vec{\psi}_0\|_{1\cap 2} \tag{7.42}$$

The second term in (7.41) is dealt with in the following manner:

$$\| \int_{t-1}^t \int_s^{2t-s} e^{i(t-s)H} W e^{i(s-\tau')H} U \psi_1(2s-\tau') \, d\tau' ds \|_\infty \lesssim$$
$$\int_{t-1}^t \int_s^{2t-s} |t-s|^{-\frac{3}{2}} \|W\|_2 \|U\|_{2\cap\infty} \langle 2s-\tau' \rangle^{-\frac{3}{2}} \|\vec{\psi}_0\|_{1\cap 2} \, d\tau' ds$$
$$\lesssim \langle t \rangle^{-\frac{3}{2}} \|W\|_2 \|U\|_{2\cap\infty} \int_{t-1}^t \int_s^{2t-s} |t-s|^{-\frac{3}{2}} \|\vec{\psi}_0\|_{1\cap 2} \, d\tau' ds \lesssim$$
$$\langle t \rangle^{-\frac{3}{2}} \int_{t-1}^t |t-s|^{-\frac{1}{2}} \|\vec{\psi}_0\|_{1\cap 2} \, ds \lesssim t^{-\frac{3}{2}} \|\vec{\psi}_0\|_{1\cap 2} \tag{7.43}$$

The last term in (7.41) requires further splitting

$$\int_{t-1}^{t}\int_{2t-s}^{2s} e^{i(t-s)H}We^{i(s-\tau')H}U\psi_1(2s-\tau')\,d\tau' ds =$$

$$\left(\int_{t-1}^{t}\int_{2t-s}^{t+\frac{s}{2}} + \int_{t-1}^{t}\int_{t+\frac{s}{2}}^{2s}\right)e^{i(t-s)H}We^{i(s-\tau')H}U\psi_1(2s-\tau')\,d\tau' ds \tag{7.44}$$

Observe that in the region $(s,\tau')\in[t-1,t]\times[2t-s,t+\frac{s}{2}]$ we have $2s-\tau'\geq\frac{t-3}{2}$. On the other hand, in the region $(s,\tau')\in[t-1,t]\times[t+\frac{s}{2},2s]$ we have $|t-\tau'|\geq\frac{t}{2}-\frac{1}{2}$. Therefore,

$$\|\int_{t-1}^{t}\int_{2t-s}^{t+\frac{s}{2}} e^{i(t-s)H}We^{i(s-\tau')H}U\psi_1(2s-\tau')\,d\tau' ds\|_\infty =$$

$$\int_{t-1}^{t}\int_{2t-s}^{t+\frac{s}{2}} \|e^{i(t-s)H}We^{-i(t-s)H}\|_{\infty\to\infty}\|e^{i(t-\tau')H}U\psi_1(2s-\tau')\,d\tau' ds\|_\infty\,d\tau' ds \lesssim$$

$$\|\hat{W}\|_1\|U\|_{1\cap 2}\int_{t-1}^{t}\int_{2t-s}^{t+\frac{s}{2}} |t-\tau'|^{-\frac{3}{2}}\langle 2s-\tau'\rangle^{-\frac{3}{2}}\|\vec{\psi}_0\|_{1\cap 2}\,d\tau' ds \lesssim$$

$$\langle t\rangle^{-\frac{3}{2}}\int_{t-1}^{t}\int_{2t-s}^{t+\frac{s}{2}} |t-\tau'|^{-\frac{3}{2}}\|\vec{\psi}_0\|_{1\cap 2}\,d\tau' ds \lesssim$$

$$\langle t\rangle^{-\frac{3}{2}}\int_{t-1}^{t}\left(|t-s|^{-\frac{1}{2}}-|s/2|^{-\frac{1}{2}}\right)\|\vec{\psi}_0\|_{1\cap 2}\,ds \lesssim t^{-\frac{3}{2}}\|\vec{\psi}_0\|_{1\cap 2} \tag{7.45}$$

Similarly,

$$\|\int_{t-1}^{t}\int_{t+\frac{s}{2}}^{2s} e^{i(t-s)H}We^{i(s-\tau')H}U\psi_1(2s-\tau')\,d\tau' ds\|_\infty \lesssim$$

$$\|\hat{W}\|_1\|U\|_{1\cap 2}\int_{t-1}^{t}\int_{t+\frac{s}{2}}^{2s} |t-\tau'|^{-\frac{3}{2}}\langle 2s-\tau'\rangle^{-\frac{3}{2}}\|\vec{\psi}_0\|_{1\cap 2}\,d\tau' ds \lesssim$$

$$t^{-\frac{3}{2}}\int_{t-1}^{t}\int_{t+\frac{s}{2}}^{2s} \langle 2s-\tau'\rangle^{-\frac{3}{2}}\|\vec{\psi}_0\|_{1\cap 2}\,d\tau' ds \lesssim t^{-\frac{3}{2}}\|\vec{\psi}_0\|_{1\cap 2} \tag{7.46}$$

For the values of $t<4$ we decompose (7.40), with the variable $\tau'=2s-\tau$, as follows.

$$\int_{0}^{t}\int_{s}^{2s} e^{i(t-s)H}We^{i(s-\tau')H}U\psi_1(2s-\tau')\,d\tau' ds =$$

$$\left(\int_{0}^{\frac{3t}{4}}\int_{s}^{2s} + \int_{\frac{3t}{4}}^{t}\int_{s}^{2t-s} + \int_{\frac{3t}{4}}^{t}\int_{2t-s}^{2s}\right)e^{i(t-s)H}We^{i(s-\tau')H}U\psi_1(2s-\tau')\,d\tau' ds \tag{7.47}$$

Note that $(2t-s) \le 2s$ provided that $s \ge \frac{2t}{3}$. In particular, this holds for $s \ge \frac{3t}{4}$. We have

$$\begin{aligned}
&\| \int_0^{\frac{3t}{4}} \int_s^{2s} e^{i(t-s)H} W e^{i(s-\tau')H} U \psi_1(2s-\tau')\, d\tau' ds \|_\infty \lesssim \\
&\qquad \int_0^{\frac{3t}{4}} \int_s^{2s} |t-s|^{-\frac{3}{2}} \|W\|_2 \|U\|_2 \langle 2s-\tau' \rangle^{-\frac{3}{2}} \|\vec{\psi}_0\|_{1\cap 2}\, d\tau' ds \\
&\qquad \lesssim t^{-\frac{3}{2}} \|W\|_2 \|U\|_2 \int_0^{\frac{3t}{4}} \int_s^{2s} \langle 2s-\tau' \rangle^{-\frac{3}{2}} \|\vec{\psi}_0\|_{1\cap 2}\, d\tau' ds \lesssim \\
&\qquad t^{-\frac{3}{2}} \|\vec{\psi}_0\|_{1\cap 2} \qquad (7.48)
\end{aligned}$$

Now observe that in the region $(s,\tau') \in [\frac{3t}{4}, t] \times [s, 2t-s]$ the value $(2s-\tau') \ge \frac{t}{4}$. Therefore, for the second term in (7.47) we obtain that

$$\begin{aligned}
&\| \int_{\frac{3t}{4}}^{t} \int_s^{2t-s} e^{i(t-s)H} W e^{i(s-\tau')H} U \psi_1(2s-\tau')\, d\tau' ds \|_\infty \lesssim \\
&\qquad \int_{\frac{3t}{4}}^{t} \int_s^{2t-s} |t-s|^{-\frac{3}{2}} \|W\|_2 \|U\|_2 \langle 2s-\tau' \rangle^{-\frac{3}{2}} \|\vec{\psi}_0\|_{1\cap 2}\, d\tau' ds \\
&\qquad \lesssim \langle t \rangle^{-\frac{3}{2}} \|W\|_2 \|U\|_2 \int_{\frac{3t}{4}}^{t} \int_s^{2t-s} |t-s|^{-\frac{3}{2}} \|\vec{\psi}_0\|_{1\cap 2}\, d\tau' ds \lesssim \\
&\qquad \langle t \rangle^{-\frac{3}{2}} \int_{\frac{3t}{4}}^{t} |t-s|^{-\frac{1}{2}} \|\vec{\psi}_0\|_{1\cap 2}\, ds \lesssim t^{-\frac{3}{2}} \|\vec{\psi}_0\|_{1\cap 2} \qquad (7.49)
\end{aligned}$$

Similarly to (7.44) we split the last term in (7.47)

$$\begin{aligned}
&\int_{\frac{3t}{4}}^{t} \int_{2t-s}^{2s} e^{i(t-s)H} W e^{i(s-\tau')H} U \psi_1(2s-\tau')\, d\tau' ds = \\
&\left(\int_{\frac{3t}{4}}^{t} \int_{2t-s}^{t+\frac{s}{2}} + \int_{\frac{3t}{4}}^{t} \int_{t+\frac{s}{2}}^{2s} \right) e^{i(t-s)H} W e^{i(s-\tau')H} U \psi_1(2s-\tau')\, d\tau' ds \qquad (7.50)
\end{aligned}$$

Observe that in the region $(s,\tau') \in [\frac{3t}{4}, t] \times [2t-s, t+\frac{s}{2}]$ we have $2s-\tau' \ge \frac{t}{8}$. On the other hand, in the region $(s,\tau') \in [\frac{3t}{4}, t] \times [t+\frac{s}{2}, 2s]$ we have $|t-\tau'| \ge \frac{3t}{8}$. Therefore,

$$\begin{aligned}
&\| \int_{\frac{3t}{4}}^{t} \int_{2t-s}^{t+\frac{s}{2}} e^{i(t-s)H} W e^{i(s-\tau')H} U \psi_1(2s-\tau')\, d\tau' ds \|_\infty = \\
&\qquad \int_{\frac{3t}{4}}^{t} \int_{2t-s}^{t+\frac{s}{2}} \|e^{i(t-s)H} W e^{-i(t-s)H}\|_{\infty\to\infty} \|e^{i(t-\tau')H} U \psi_1(2s-\tau')\|_\infty\, d\tau' ds \lesssim \\
&\qquad \|\hat{W}\|_1 \|U\|_{1\cap 2} \int_{\frac{3t}{4}}^{t} \int_{2t-s}^{t+\frac{s}{2}} |t-\tau'|^{-\frac{3}{2}} \langle 2s-\tau' \rangle^{-\frac{3}{2}} \|\vec{\psi}_0\|_{1\cap 2}\, d\tau' ds \lesssim \\
&\qquad \langle t \rangle^{-\frac{3}{2}} \int_{\frac{3t}{4}}^{t} \int_{2t-s}^{t+\frac{s}{2}} |t-\tau'|^{-\frac{3}{2}} \|\vec{\psi}_0\|_{1\cap 2}\, d\tau' ds \lesssim \\
&\qquad \langle t \rangle^{-\frac{3}{2}} \int_{\frac{3t}{4}}^{t} \left(|t-s|^{-\frac{1}{2}} - |s/2|^{-\frac{1}{2}} \right) \|\vec{\psi}_0\|_{1\cap 2}\, ds \lesssim t^{-\frac{3}{2}} \|\vec{\psi}_0\|_{1\cap 2}
\end{aligned}$$

Similarly,

$$\begin{aligned}\| \int_{\frac{3t}{4}}^{t}\int_{t+\frac{s}{2}}^{2s} e^{i(t-s)H}We^{i(s-\tau')H}U\psi_1(2s-\tau')\,d\tau'ds\|_\infty \lesssim \\ \|\hat{W}\|_1\|U\|_{1\cap 2}\int_{\frac{3t}{4}}^{t}\int_{t+\frac{s}{2}}^{2s}|t-\tau'|^{-\frac{3}{2}}\langle 2s-\tau'\rangle^{-\frac{3}{2}}\|\vec{\psi}_0\|_{1\cap 2}\,d\tau'ds \lesssim \\ t^{-\frac{3}{2}}\int_{\frac{3t}{4}}^{t}\int_{t+\frac{s}{2}}^{2s}\langle 2s-\tau'\rangle^{-\frac{3}{2}}\|\vec{\psi}_0\|_{1\cap 2}\,d\tau'ds \lesssim t^{-\frac{3}{2}}\|\vec{\psi}_0\|_{1\cap 2}\end{aligned}$$

The conclusion from the preceding is that

$$\|e^{itA}P_c\vec{\psi}_0\|_\infty = \|P_ce^{itA}\vec{\psi}_0\|_\infty \lesssim |t|^{-\frac{3}{2}}\|\vec{\psi}_0\|_{1\cap 2}. \tag{7.51}$$

as desired. □

Remark 7.11. Dualizing (7.51) yields

$$\|e^{itA^*}P_c^*\vec{\psi}_0\|_{2+\infty} \lesssim |t|^{-\frac{3}{2}}\|\vec{\psi}_0\|_1.$$

By construction, $\mathrm{Ran}(P_c^*) = (\ker P_c)^\perp = \left(\sum_{j=1}^k L_j\right)^\perp$, $\ker(P_c^*) = (\mathrm{Ran}P_c)^\perp = \sum_{j=1}^k L_j^*$, and $(P_c^*)^2 = P_c^*$. It follows from these properties that P_c^* is precisely the projection onto $\left(\sum_{j=1}^k L_j\right)^\perp$ corresponding to the decomposition (7.7) with the roles of L_j and L_j^* interchanged. Therefore, one has the corresponding estimate for e^{itA} as well, namely

$$\|e^{itA}P_c\vec{\psi}_0\|_{2+\infty} \lesssim |t|^{-\frac{3}{2}}\|\vec{\psi}_0\|_1.$$

Remark 7.12. We have made no use of the wave operators $\Omega := \lim_{t\to\infty} e^{-itA}e^{itB}$ in this section. It is easy to see from the dispersive bound for the free evolution e^{itB} and Cook's method that this limit exists. There is also a corresponding asymptotic completeness statement, namely that Ω is an isomorphism from $\mathcal{H}$ onto $\mathrm{Ran}P_c$. A first step in this direction is to check the existence of the limit $\lim_{t\to\infty} e^{-itB}e^{itA}P_c$. This is nontrivial, and can be obtained by Cook's method using Theorem 7.4. Then there is an additional issue of showing that $\mathrm{Ran}(\Omega) = \mathrm{Ran}(P_c)$. For this and much more on the wave operators see [Cu]. Let us just mention that Lemmas 7.5 and 7.6 are both immediate consequences of this asymptotic completeness. Indeed, Lemma 7.6 follows from

$$(A-z)^{-1}P_c = \Omega(B-z)^{-1}(P_c\Omega)^{-1}P_c,$$

whereas Lemma 7.5 is an immediate consequence of the fact that

$$\frac{1}{T}\int_0^T e^{-i\omega t}e^{tA}P_c\,dt = \Omega\frac{1}{T}\int_0^T e^{-i\omega t}e^{tB}(P_c\Omega)^{-1}P_c\,dt$$

via the usual ergodic theorem for the unitary group e^{tB}. But since the proof of asymptotic completeness requires Theorem 7.4, we needed to give a direct proof of these lemmas.

7.2 The case of higher dimensions

The proof that was just presented for $n = 3$ also works in odd dimensions $n \geq 3$. We do not address the situation in even dimensions here since it is not clear at the moment whether these methods can be extended to that case.
There are three places where the dimension becomes relevant in the previous proof. The first instance is Lemma 7.7, which establishes the analytic continuation of the weighted free resolvent $E_\varepsilon(iB - z)^{-1}E_\varepsilon$ from $\Re z > 0$ to the region $\mathcal{R}_{\Gamma_{\frac{\varepsilon}{4}}}$, together with the bound $(1 + |z|)^{-\frac{1}{2}}$. Since

$$E_\varepsilon(B + iz)^{-1}E_\varepsilon = \begin{pmatrix} E_\varepsilon(-\frac{1}{2}\triangle + 1 + iz)^{-1}E_\varepsilon & 0 \\ 0 & E_\varepsilon(\frac{1}{2}\triangle - 1 + iz)^{-1}E_\varepsilon \end{pmatrix},$$

this reduces to the same statements for the *scalar* operators $(-\frac{1}{2}\triangle + 1 + iz)^{-1} = (-\frac{1}{2}\triangle + \zeta^2)^{-1}$ and $(\frac{1}{2}\triangle - 1 + iz)^{-1} = (-\frac{1}{2}\triangle - \zeta^2)^{-1}$, where we have used the maps $z = U_1(\zeta) = i - i\zeta^2$, and $z = U_2(\zeta) = -i + i\zeta^2$, respectively. As far as the region is concerned, observe that $\mathcal{R}_{\Gamma_{\frac{\varepsilon}{4}}} \subset U_1(\mathcal{R}_1 \cup \mathcal{R}_2) \cap U_2(\mathcal{R}_1^* \cup \mathcal{R}_2^*)$. Rauch proves the desired bound on the analytic continuation of the free resolvent by means of the sharp Huyghens principle in dimension $n = 3$. The very same argument also applies to any odd dimension $n \geq 3$. Indeed, one writes[4]

$$\int_0^\infty e^{-\zeta t}\, E_\varepsilon \cos(t\sqrt{-\triangle})E_\varepsilon\, dt = \zeta E_\varepsilon(-\triangle + \zeta^2)^{-1}E_\varepsilon, \qquad \Re\zeta > 0. \tag{7.52}$$

If n is odd, then the sharp Huyghens principle implies that $\|E_\varepsilon \cos(t\sqrt{-\triangle})E_\varepsilon\|_{2\to 2} \lesssim e^{-\varepsilon t/2}$ for large t, so that (7.52) allows one to analytically continue the right-hand side to $\Re\zeta > -\frac{\varepsilon}{2}$ together with the desired bound. See [R] for details.
The second place where the dimension becomes relevant is in the proof of local decay, see the term involving $C(\tau)$ in (7.30). In general (odd) dimensions, there is still the Laurent expansion

$$G_\varepsilon(z) = \sum_{j=-2}^{\infty} B_j(z - i)^{\frac{j}{2}},$$

and similarly about the point $z = -i$. By our assumption on absence of eigenvalue and resonance at i one has $B_{-2} = B_{-1} = 0$. In order to obtain the desired decay estimate on the evolution e^{itA} one needs to show that the coefficients B_j of odd powers j vanish for $1 \leq j < n - 2$. This follows from the corresponding property of the "free" function $F_\varepsilon(z)$. Indeed, by the previous paragraph,

$$F_\varepsilon(z) = \sum_{j=0}^{\infty} C_j\,(z - i)^{\frac{j}{2}} \quad \text{for} \quad |z - i| < \frac{\varepsilon^2}{16}. \tag{7.53}$$

[4]The analytic continuation of the free resovent in the region $|\zeta| < \frac{\epsilon}{4}$, which avoids the introduction of the pole at $\zeta = 0$, can be constructed via the formula

$$\int_0^\infty e^{-\zeta t}\, E_\varepsilon \frac{\sin(t\sqrt{-\triangle})}{\sqrt{-\triangle}} E_\varepsilon\, dt = E_\varepsilon(-\triangle + \zeta^2)^{-1}E_\varepsilon.$$

The energy estimate $\|\partial_t \frac{\sin(t\sqrt{-\triangle})}{\sqrt{-\triangle}} f\|_2 \leq \|f\|_2$ together with the sharp Huyghens principle lead to the bound $\|\, E_\varepsilon \frac{\sin(t\sqrt{-\triangle})}{\sqrt{-\triangle}} E_\varepsilon\|_{2\to 2} \lesssim e^{-\varepsilon t/2}$, which is sufficient for analytic continuation.

where the coefficients are compact operators in L^2. Indeed, in Lemma 7.7 (and its generalization to $n \geq 3$ odd) we observed that $F_\varepsilon(i - i\zeta^2)$ is analytic for $|\zeta| < \frac{\varepsilon}{4}$. Thus, the series in (7.53) is absolutely convergent in the slit region $U_1(\mathcal{R}_1)$ for small z. Moreover, since the rate of decay for the free evolution e^{itB} is known to be $|t|^{-\frac{n}{2}}$, the argument from the proof of Proposition 7.9, applied to the free evolution, yields that all $C_j = 0$ for $1 \leq j < n-2$ odd. Finally, by the resolvent identity one has, see (7.25)

$$
\begin{aligned}
G_\varepsilon(z) &= \Big(1 + F_\varepsilon(z)\, E_\varepsilon^{-2} V\Big)^{-1} F_\varepsilon(z) \\
&= \Big(1 + F_\varepsilon(i)\, E_\varepsilon^{-2} V\Big)^{-1} \Big[1 + (F_\varepsilon(z) - F_\varepsilon(i))\, E_\varepsilon^{-2} V \Big(1 + F_\varepsilon(i)\, E_\varepsilon^{-2} V\Big)^{-1}\Big]^{-1} F_\varepsilon(z). \qquad (7.54)
\end{aligned}
$$

This is justified, since by our assumption of $\pm i$ being neither eigenvalue nor resonance of A the inverse $\Big(1 + F_\varepsilon(i)\, E_\varepsilon^{-2} V\Big)^{-1}$ exists as a bounded operator on L^2, cf. (7.26). Furthermore, the inverse in square brackets in (7.54) exists as an L^2 convergent Neuman series. Inserting (7.53) into (7.54) reveals that all coefficients of the entire right-hand side in (7.54) corresponding to powers $(z-i)^{\frac{j}{2}}$ with $1 \leq j < n-2$ odd, have to vanish. Therefore, (7.30) holds with $\|C(\tau)\| \lesssim |\tau - i|^{\frac{n-2}{2}}$, and (7.27) thus holds with $\langle t \rangle^{-\frac{n}{2}}$.
The third place where the dimension becomes relevant is the argument allowing for L^2 removal in Theorem 7.4. As in the scalar case, a tedious calculation involving further Duhamel expansion would extend the 3-d argument to higher dimensions.

8 Matrix charge transfer model

Definition 8.1. *By a* matrix charge transfer model *we mean a system*

$$
\begin{aligned}
&\frac{1}{i}\partial_t \vec{\psi} + \begin{pmatrix} -\frac{1}{2}\triangle & 0 \\ 0 & \frac{1}{2}\triangle \end{pmatrix} \vec{\psi} + \sum_{j=1}^{\nu} V_j(\cdot - \vec{v}_j t)\vec{\psi} = 0 \qquad (8.1)\\
&\vec{\psi}|_{t=0} = \vec{\psi}_0,
\end{aligned}
$$

where $\vec{v}_j$ are distinct vectors in $\mathbb{R}^3$, and V_j are matrix potentials of the form

$$
V_j(t,x) = \begin{pmatrix} U_j(x) & -e^{i\theta_j(t,x)}\, W_j(x) \\ e^{-i\theta_j(t,x)}\, W_j(x) & -U_j(x) \end{pmatrix},
$$

where $\theta_j(t,x) = (|\vec{v}_j|^2 + \alpha_j^2)t + 2x \cdot \vec{v}_j + \gamma_j$, $\alpha_j, \gamma_j \in \mathbb{R}$, $\alpha_j \neq 0$. Furthermore, we require that each

$$
H_j = \begin{pmatrix} -\frac{1}{2}\triangle + \frac{1}{2}\alpha_j^2 + U_j & -W_j \\ W_j & \frac{1}{2}\triangle - \frac{1}{2}\alpha_j^2 - U_j \end{pmatrix}
$$

be admissible in the sense of Definition 7.2 and that it satisfy the stability condition (7.11)*.*

In comparison to the simple definition of a charge transform model in the scalar case, Definition 8.1 might seem unnatural. However, it is natural by virtue of being the only Galilei invariant definition. In our context we need to use the following vector-valued Galilei transform (which should explain why we are using (7.5) rather than (7.4)):

$$\mathcal{G}_{\vec v,y}(t)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} := \begin{pmatrix}\mathfrak{g}_{\vec v,y}(t)\psi_1\\ \overline{\mathfrak{g}_{\vec v,y}(t)\overline{\psi_2}}\end{pmatrix}, \tag{8.2}$$

where $\mathfrak{g}_{\vec v,y}$ are as in Section 2. Clearly, the transformations $\mathcal{G}_{\vec v,y}(t)$ are isometries on all L^p spaces. Since in our case always $y=0$, we set $\mathcal{G}_{\vec v}(t) := \mathcal{G}_{\vec v,0}(t)$. By (2.7), $\mathcal{G}_{\vec v}(t)^{-1} = \mathcal{G}_{-\vec v}(t)$ in that case. There is now the following analogue of (2.7), (2.8). In contrast to the scalar case, the transformation law of Lemma 8.2 also involves a modulation $\mathcal{M}(t)$. It should be compared with Definition 2.1.

Lemma 8.2. *Let $\alpha\in\mathbb{R}$ and set*

$$A := \begin{pmatrix} -\frac12\triangle + \frac12\alpha^2 + U & -W \\ W & \frac12\triangle - \frac12\alpha^2 - U\end{pmatrix}$$

with real-valued U, W. Moreover, let $\vec v\in\mathbb{R}^3$, $\theta(t,x) = (|\vec v\,|^2+\alpha^2)t + 2x\cdot\vec v + \gamma$, $\gamma\in\mathbb{R}$, and define

$$H(t) := \begin{pmatrix} -\frac12\triangle + U(\cdot-\vec v t) & -e^{i\theta(t,\cdot-\vec v t)}W(\cdot-\vec v t) \\ e^{-i\theta(t,\cdot-\vec v t)}W(\cdot-\vec v t) & \frac12\triangle - U(\cdot-\vec v t)\end{pmatrix}.$$

Let $\mathcal{S}(t)$, $\mathcal{S}(0)=Id$, denote the propagator of the system

$$\frac1i\partial_t\mathcal{S}(t) + H(t)\mathcal{S}(t) = 0.$$

Finally, let

$$\mathcal{M}(t) = \mathcal{M}_{\alpha,\gamma}(t) = \begin{pmatrix} e^{-i\omega(t)/2} & 0 \\ 0 & e^{i\omega(t)/2}\end{pmatrix} \tag{8.3}$$

where $\omega(t) = \alpha^2 t + \gamma$. Then

$$\mathcal{S}(t) = \mathcal{G}_{\vec v}(t)^{-1}\mathcal{M}(t)^{-1}e^{-itA}\mathcal{M}(0)\mathcal{G}_{\vec v}(0). \tag{8.4}$$

Proof. One has

$$\frac1i\partial_t\mathcal{M}(t)\mathcal{G}_{\vec v}(t)\mathcal{S}(t) = \begin{pmatrix} -\frac12\dot\omega & 0 \\ 0 & \frac12\dot\omega\end{pmatrix}\mathcal{M}(t)\mathcal{G}_{\vec v}(t)\mathcal{S}(t) + \mathcal{M}(t)\frac1i\dot{\mathcal{G}}_{\vec v}(t)\mathcal{S}(t) - \mathcal{M}(t)\mathcal{G}_{\vec v}(t)H(t)\mathcal{S}(t). \tag{8.5}$$

Let $\rho(t,x) = t|\vec v\,|^2 + 2x\cdot\vec v$. One now checks the following properties by differentiation:

$$\begin{aligned}
\mathcal{M}(t)\frac1i\dot{\mathcal{G}}_{\vec v}(t) &= \begin{pmatrix} \frac12|\vec v\,|^2 + \vec v\cdot\vec p & 0 \\ 0 & -\frac12|\vec v\,|^2 + \vec v\cdot\vec p\end{pmatrix}\mathcal{M}(t)\mathcal{G}_{\vec v}(t)\\
\mathcal{M}(t)\mathcal{G}_{\vec v}(t)H(t) &= \begin{pmatrix} -\frac12\triangle + U & -e^{i(\theta-\rho-\omega)}W \\ e^{-i(\theta-\rho-\omega)}W & \frac12\triangle - U\end{pmatrix}\mathcal{M}(t)\mathcal{G}_{\vec v}(t)\\
&\quad + \begin{pmatrix} \frac12|\vec v\,|^2 + \vec v\cdot\vec p & 0 \\ 0 & -\frac12|\vec v\,|^2 + \vec v\cdot\vec p\end{pmatrix}\mathcal{M}(t)\mathcal{G}_{\vec v}(t).
\end{aligned} \tag{8.6}$$

In view of our definitions, $\theta-\rho-\omega=0$. Since $\dot\omega = \alpha^2$, the lemma follows by inserting (8.6) into (8.5). ∎

We now return to the matrix charge transfer problem (8.1). In order to state our main theorem, we need to impose an orthogonality condition in the context of the charge transfer, analogous to the one used in the scalar case. To do so, let $P_c(H_j)$ and $P_b(H_j)$ be the projectors as in Lemma 7.3. Abusing terminology somewhat, we refer to $\mathrm{Ran}(P_b(H_j))$ as the *bound states* of H_j. Since P_b and P_c are no longer orthogonal projections, we use "scattering states" instead of "asymptotically orthogonal to the bound states" in the following definition.

Definition 8.3. *Let $U(t)\vec{\psi}_0 = \vec{\psi}(t,\cdot)$ be the solution of (8.1). We say that $\vec{\psi}_0$ is a* scattering state *relative to H_j if*

$$\|P_b(H_j,t)U(t)\vec{\psi}_0\|_{L^2} \to 0 \text{ as } t \to +\infty.$$

Here

$$P_b(H_j,t) := \mathcal{G}_{\vec{v}_j}(t)^{-1}\mathcal{M}_j(t)^{-1}P_b(H_j)\,\mathcal{M}_j(t)\mathcal{G}_{\vec{v}_j}(t) \tag{8.7}$$

with $\mathcal{M}_j(t) = \mathcal{M}_{\alpha_j,\gamma_j}(t)$ as in (8.3).

The formula (8.7) is of course motivated by (8.4). Clearly, $P_b(H_j,t)$ is the projection onto the bound states of H_j that have been translated to the position of the matrix potential $V_j(\cdot - t\vec{v}_j)$. Equivalently, one can think of it as translating the solution of (8.1) from that position to the origin, projecting onto the bound states of H_j, and then translating back.
We now formulate our decay estimate for matrix charge transfer models.

Theorem 8.4. *Consider the matrix charge transfer model as in Definition 8.1. Let $U(t)$ denote the propagator of the equation (8.1). Then for any initial data $\vec{\psi}_0 \in L^1 \cap L^2$, which is a scattering state relative to each H_j in the sense of Definition 8.3, one has the decay estimates*

$$\|U(t)\vec{\psi}_0\|_{L^2+L^\infty} \lesssim \langle t\rangle^{-\frac{3}{2}}\|\vec{\psi}_0\|_{L^1\cap L^2}. \tag{8.8}$$

If in addition the matrix potentials V_j satisfy $\|\hat{V}_j\|_{L^1} < \infty$ for all $j = 1,..,\nu$ then one also has the bound

$$\|U(t)\vec{\psi}_0\|_{L^\infty} \lesssim |t|^{-\frac{3}{2}}\|\vec{\psi}_0\|_{L^1\cap L^2}. \tag{8.9}$$

Remark 8.5. Theorem 8.4 also holds in the case of higher dimensions $n \geq 3$. In particular, we have the corresponding estimate

$$\|U(t)\vec{\psi}_0\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}}\|\vec{\psi}_0\|_{L^1\cap L^2}.$$

As in the case of *scalar* charge transfer models the proof of Theorem 8.4 mainly relies on the decay estimates for the corresponding matrix problems H_j with a *single* time-independent potential $V_j(x)$

$$\|e^{itH_j}P_c(H_j)\,f\|_{L^2+L^\infty} \lesssim \langle t\rangle^{-\frac{n}{2}}\|f\|_{L^1\cap L^2} \tag{8.10}$$

We have established such estimates for the admissible Hamiltonians H_j in Theorem 7.4 for $n = 3$. In section 7.2 we have also observed that our method for deriving (8.10) can be generalized to treat the case of an arbitrary odd dimension $n \geq 3$. While our method leaves the even dimensional case open at the moment, one can, in principle, substitute Cuccagna's dispersive estimates [Cu] instead.

Remark 8.6. The conclusions of Theorem 8.4 are also valid for *perturbed* matrix charge transfer Hamiltonians. This refers to equations of the type

$$\frac{1}{i}\partial_t\vec{\psi} + \begin{pmatrix} -\frac{1}{2}\triangle & 0 \\ 0 & \frac{1}{2}\triangle \end{pmatrix}\vec{\psi} + \sum_{j=1}^{\nu} V_j(\cdot - \vec{v_j}t)\vec{\psi} + V_0(t,\cdot)\vec{\psi} = 0$$
$$\vec{\psi}|_{t=0} = \vec{\psi}_0,$$

where the charge transfer part is as in Definition 8.1 and the perturbation satisfies

$$\sup_t \|V_0(t,\cdot)\|_{1\cap\infty} < \varepsilon.$$

See the scalar case in Section 3.7 for an exact formulation.

As in the scalar case, (8.8) is proved by means of a bootstrap argument with the same assumption (2.13). Since the argument is basically identical with the scalar case, we do not write it out in full detail. As in the scalar case, the estimate (8.9) is a consequence of the $L^2 + L^\infty$ bound (8.8). This follows by combining the machinery developed in Proposition 5.1 with the cancellation property for a matrix problem with a *single* potential established in the proof of the estimate (7.13) in Theorem 7.4.
The fact that systems have generalized eigenspaces rather than just eigenspaces leads to some minor changes from the scalar case. But this really only affects the proof of Proposition 3.1. We show below that the statement still remains the same.

In the following we shall assume that the number of potentials is $\nu = 2$ and that the velocities are $\vec{v_1} = 0, \vec{v_2} = (1, 0, \ldots 0) = \vec{e_1}$. This can be done without loss of generality.

8.1 Bound states

We now estimate the rate of convergence of the projections onto the bound states of solutions which evolve from scattering states.

Proposition 8.7. *Let $\vec{\psi}(t,x) = (U(t)\vec{\psi}_0)(x)$ be a solution of* (8.1) *where $\vec{\psi}_0$ is a scattering state relative to H_1 and H_2 in the sense of Definition 8.3. Then*

$$\|P_b(H_1,t)U(t)\vec{\psi}_0\|_{L^2} + \|P_b(H_2,t)U(t)\vec{\psi}_0\|_{L^2} \lesssim e^{-\alpha t}\|\vec{\psi}_0\|_{L^2}$$

for some $\alpha > 0$.

Proof. By symmetry it suffices to prove the bound on the first part (one can switch the roles of V_1 and V_2 by means of a Galilei transform and a modulation). In view of Lemma 7.3 one decomposes

$$\vec{\phi}(t) := \mathcal{M}_{\alpha_1,\gamma_1}(t)U(t)\vec{\psi}_0 = \sum_{j=0}^{M} \vec{f_j}(t) + \vec{\phi}_1(t,\cdot) \tag{8.11}$$

relative to H_1 so that $\vec{\phi}_1(t,\cdot)$ lies in the continuous subspace of H_1, i.e., $P_c(H_1)\vec{\phi}_1 = \vec{\phi}_1$ and $P_b(H_1)\vec{\phi}_1 = 0$ for all times. Furthermore, $\vec{f_j}(t) \in L_j$ where L_j are the spaces from Lemma 7.3 for the operator H_1. By assumption,

$$\sum_{j=0}^{M} |\vec{f_j}(t)|^2 \to 0 \text{ as } t \to \infty. \tag{8.12}$$

One checks that $\vec{\phi}$ satisfies the equation

$$\frac{1}{i}\partial_t\vec{\phi} + H_1\vec{\phi} + \tilde{V}_2(\cdot - \vec{e}_1 t)\vec{\phi} = 0 \tag{8.13}$$

where $\tilde{V}_2 = \mathcal{M}_{\alpha_1,\gamma_1}(t)V_2\mathcal{M}_{\alpha_1,\gamma_1}(t)^{-1}$. Substituting (8.11) into (8.13) yields

$$\begin{aligned}&\frac{1}{i}\partial_t\vec{\phi} + H_1\vec{\phi} + \tilde{V}_2(\cdot - t\vec{e_1})\vec{\phi}\\ &= \sum_{j=0}^{M}\frac{1}{i}\dot{\vec{f_j}} + \frac{1}{i}\dot{\vec{\phi}}_1(t) + + \sum_{j=0}^{M} H_1\vec{f_j}(t) + H_1\vec{\phi}_1 + \tilde{V}_2(\cdot - t\vec{e_1})\vec{\phi} = 0\end{aligned} \tag{8.14}$$

Since $P_c(H_1)\vec{\phi}_1 = \vec{\phi}_1$ and $\mathrm{Ran}(P_c(H_1))$ is closed by assumption, one has

$$H_1\vec{\phi}_1 = P_c(H_1)H_1\vec{\phi}_1, \qquad \partial_t\vec{\phi}_1 = P_c(H_1)\partial_t\vec{\phi}_1.$$

In particular

$$P_b(H_1)\left(\frac{1}{i}\partial_t\vec{\phi}_1 + H_1\vec{\phi}_1\right) = 0.$$

Therefore, applying $P_b(H_1)$ to (8.14) yields

$$\sum_{j=0}^{M}\Big[\frac{1}{i}\dot{\vec{f_j}}(t) + H_1\vec{f_j}(t)\Big] = \vec{g}(t) := -P_b(H_1)\tilde{V}_2(\cdot - t\vec{e_1})\vec{\phi}.$$

In view of the explicit expression (7.8) for $P_b(H_1)$ one has $|\vec{g}(t)| \lesssim e^{-\varepsilon t}$. Moreover, decomposing $\vec{g}(t) = \sum_{j=0}^{M}\vec{g}_j(t)$ with $\vec{g}_j(t) \in L_i$, one also has $|\vec{g}_j(t)| \lesssim e^{-\varepsilon t}$ for all $0 \le j \le M$. Hence,

$$\begin{aligned}\frac{1}{i}\dot{\vec{f_0}} + H_1\vec{f_0} &= \vec{g}_0\\ \frac{1}{i}\dot{\vec{f_j}} + \omega_j\vec{f_j} &= \vec{g}_j \quad \text{for} \quad 1 \le j \le M\end{aligned} \tag{8.15}$$

with exponentially decaying right-hand sides. It follows from (8.12) that $|\vec{f_j}(t)| \lesssim e^{-\varepsilon t}$ for $1 \le j \le M$. Moreover, applying H_1 to (8.15) one obtains

$$(H_1\vec{f_0})^{\cdot} = iH_1\vec{g}_0(t).$$

Since the right-hand side decays exponentially, and since $H_1\vec{f_0}(t) \to 0$ as $t \to \infty$ by our assumption of starting from a scattering state, it follows that $H_1\vec{f_0}(t)$ decays exponentially, and therefore also $\vec{f_0}(t)$. □

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