MINIMAL ESCAPE VELOCITIES

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Abstract. We give a new derivation of the minimal velocity estimates [SiSo1] for unitary evolutions. Let H and A be selfadjoint operators on a Hilbert space $\mathcal{H}$. The starting point is Mourre's inequality $i[H,A] \geq \theta > 0$, which is supposed to hold in form sense on the spectral subspace $\mathcal{H}_\Delta$ of H for some interval $\Delta \subset R$. The second assumption is that the multiple commutators $ad_A^{(k)}(H)$ are well-behaved for $k = 1 \ldots n$ $(n \geq 2)$. Then we show that, for a dense set of ψ's in $\mathcal{H}_\Delta$ and all $m < n-1$, $\psi_t = exp(-iHt)$ is contained in the spectral subspace $A \geq \theta t$ as $t \to \infty$, up to an error of order t^{-m} in norm. We apply this general result to the case where H is a Schrödinger operator on R^n and A the dilation generator, proving that $\psi_t(x)$ is asymptotically supported in the set $|x| \geq t\sqrt{\theta}$ up to an error of order t^{-m} in norm.

1. INTRODUCTION

Before posing the problem in abstract form we describe it in its original concrete setting. Consider the Schrödinger equation

$$i\partial_t \psi_t = H\psi_t; \qquad H = \frac{1}{2}p^2 + V(x) \quad \text{on} \quad L^2(R^n) \tag{1.1}$$

for a particle in R^n under the influence of a potential $V(x)$. We are interested in the long–time behavior of orbits $t \to \psi_t$ in the continuous spectral subspace $\mathcal{H}_c$ of H. Under mild conditions on V, H is selfadjoint and

$$\langle p^2 \rangle_\psi \leq \text{const.}\ \langle H + c \rangle_\psi \tag{1.2}$$

for some constant $c \in R$. Here $\langle \Phi \rangle_\psi = (\psi, \Phi\psi)$ denotes the expectation value of an observable Φ in the state ψ. By Ruelle's theorem ([Rue], see also [CFKS], [HuSi1]) any orbit ψ_t in $\mathcal{H}_c$ is *escaping* in a mean ergodic sense :

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt \int_{|x|\leq R} dx\ |\psi_t(x)|^2 = 0 \tag{1.3}$$

for any finite R. The question is : *how fast ?* The answer will of course depend on the initial state ψ. The simplest example is the free particle ($V = 0$): if the Fourier transform $\widehat{\psi}$ of ψ is smooth and supported outside some ball of radius $v > 0$, then a standard asymptotic expansion gives the result

$$\int_{|x|\leq vt} dx\ |\psi_t(x)|^2 \ = \ O(t^{-m}) \quad (t \to \infty) \tag{1.4}$$

for any m. In this sense the orbits ψ_t of the given type are said to have a *minimal escape velocity* v given by the support of $\widehat{\psi}$. To obtain a similar result for $V \neq 0$ we study the long–time behavior of the expectation values $\langle A \rangle_t = (\psi_t\ ,\ A\psi_t)$ for suitable observables A, which evolve according to

$$\partial_t\ \langle A \rangle_t \ = \langle i[H,A] \rangle_t\ . \tag{1.5}$$

Mourre's very fruitful idea [Mou] was to find observables A such that the commutator in (1.5) is *conditionally positive* , in the sense that

$$E_\Delta \, i[H,A] \, E_\Delta \;\geq\; \theta \, E_\Delta \qquad (\theta > 0) \tag{1.6}$$

for some interval $\Delta \subset R$, where E_Δ is the corresponding spectral projection of H. This implies that

$$\langle \, A \, \rangle_t \;\geq\; \theta t \;+\; O(1) \;\rightarrow\; \infty \qquad (t \rightarrow \infty) \tag{1.7}$$

for orbits ψ_t in the spectral subspace $\mathcal{H}_\Delta = \mathrm{Ran}\, E_\Delta$. Evidently A must be unbounded, so that domain questions arise. Also $\mathcal{H}_\Delta$ must be a subspace of $\mathcal{H}_c$ since $\langle A \rangle_t$ is constant and $\langle i[H,A] \rangle_\psi = 0$ for any eigenvector ψ of H. (1.6) is a special case of a more general inequality due to Mourre, which was first proven for Schrödinger operators (including N–body systems), where A was taken as the dilation generator :

$$A \;=\; \frac{1}{2}\,(p \cdot x + x \cdot p) \;; \quad i[H,A] \;=\; p^2 - x \cdot \nabla V(x) \tag{1.8}$$

([Mou], [PSS], see also [CFKS], [HuSi1]). In this case the intervals Δ for which (1.6) holds fill the continous spectrum of H, in the sense that the corresponding subspaces $\mathcal{H}_\Delta$ span $\mathcal{H}_c$. Moreover, since

$$A = i\,[H, \frac{1}{2} x^2] \tag{1.9}$$

is itself a commutator, (1.5) can be written as $\partial_t{}^2 \langle x^2 \rangle_t \geq 2\theta$ for orbits ψ_t in $\mathcal{H}_\Delta$, which implies that

$$\langle x^2 \rangle_t \;\geq\; \theta \, t^2 \;+\; O(t) \quad (t \rightarrow \infty) \; . \tag{1.10}$$

This is of course weaker than (1.4) : *it only says that the mean value of x^2 for the probability distribution $|\psi_t(x)|^2$ diverges like $\theta \, t^2$, whereas we want to prove that the support of this distribution is asymptotically contained in $|x| \geq t\,\sqrt{\theta}$ as $t \rightarrow \infty$* . The first step is to derive a corresponding result for the spectral support of ψ_t with respect to A. We state this result in abstract form for a pair $(H \, , \, A)$ of selfdjoint operators on a Hilbert space $\mathcal{H}$.

Theorem 1.1. *Suppose that $ad_A^{(k)}(f(H))$ is bounded for any $f \in C_0^\infty(R)$ and $k = 1 \ldots n$, $n \geq 2$, and that the Mourre inequality (1.6) holds for some open interval $\Delta \subset R$. Let $\chi^\pm$ be the characteristic function of $R^\pm$. Then*

$$||\chi^-(A - a - \vartheta t) e^{-iHt} g(H) \chi^+(A - a)|| \leq \text{const.}\; t^{-m} \tag{1.11}$$

for any $g \in C_0^\infty(\Delta)$, any ϑ in $0 < \vartheta < \theta$ and any $m < n-1$, uniformly in $a \in R$.

Since $g \in C_0^\infty(\Delta)$ and $a \in R$ are arbitrary, the vectors of the form

$$\psi = g(H)\,\chi^+(A - a)\varphi; \qquad \varphi \in \mathcal{H} \tag{1.12}$$

are dense in $\mathcal{H}_\Delta$. (1.11) says that, for any such ψ, $\psi_t = \exp(-iHt)\psi$ has spectral support in $[\,t\,\vartheta\,,\,+\infty)$ with respect to A , up to a remainder of order t^{-m} in norm.

Remarks

Commutators. The hypothesis $ad_A^{(k)}(f(H)) \in L(H)$ may be replaced by conditions on $ad_A^{(k)}(H)$ which are more subtle to formulate since the operators A and H are generally unbounded (see e.g. [ABG], [JMP]). For the special case (1.8) this is further discussed below.

Resolvent smoothness and local decay. We indicate briefly how minimal velocity estimates are related to resolvent smoothness [JMP] and to local decay [PSS]. Let $\rho(A) = (1+A^2)^{1/2}$. Setting $a = -\theta t/2$ and using that

$$\rho(A)^{-\alpha} = \rho(A)^{-\alpha}\,\chi^{\pm}(A \pm ct) + O(t^{-\alpha}) \ ,$$

we obtain from (1.11)

$$||\rho(A)^{-\alpha}e^{-iHt}g(H)\rho(A)^{-\alpha}|| \le \text{const.}\ (1+t)^{-\min(\alpha,m)} \ .$$

For $\alpha,\ m > 1$ this is integrable over $-\infty < t < +\infty$, which (by Fourier transform) leads to the resolvent estimate

$$\sup_{z \notin R} ||\rho(A)^{-\alpha}(z-H)^{-1}g(H)\rho(A)^{-\alpha}|| \le \infty \ . \tag{1.13}$$

Similar estimates for the derivatives with respect to z (higher powers) of the resolvent $(z-H)^{-1}$ are obtained using correspondingly higher values of α, m (resolvent smoothness). Replacing g by g^2 in (1.13) it also follows that the operator $\rho(A)^{-\alpha}$ is H–smooth for $\alpha > 1$ and therefore ([RSIV] Theorem XIII.25 and corollary)

$$\int_{-\infty}^{+\infty} dt\, ||\rho(A)^{-\alpha}e^{-iHt}\psi||^2 \le \text{const.}\ ||\psi||^2 \quad \forall \psi \in \mathcal{H}_{\Delta'} \ , \tag{1.14}$$

where Δ' is any fixed compact subset of Δ (local decay). By an independent argument of Mourre (given in [PSS]) the estimate (1.13) and therefore (1.14) can be improved from $\alpha > 1$ to $\alpha > 1/2$.

Our second result is an application of Theorem 1.1 to the Schrödinger equation (1.1). Here A is given by (1.8), and

$$i^k ad_A^{(k)}(H) = 2^{k-1}p^2 + (-x \cdot \nabla)^k\, V(x) \ . \tag{1.15}$$

A simple way to satisfy the hypothesis of Theorem 1.1 in this case is to assume that V has relative bound less than 1 with respect to $\frac{1}{2}p^2$, and that the distributions $(x\cdot\nabla)^k V(x)$ are locally L^2 and (as multiplications operators) bounded relative to p^2 for $k = 1 \ldots n$. Then the operators (1.15) are bounded relative to H and it is straightforward to compute and to estimate the the norms of $ad_A^{(k)}((z-H)^{-1})$ for $\mathrm{Im}(z) \ne 0$. Representing $f(H)$ by the resolvent $(z-H)^{-1}$ (e.g. using the Helffer-Sjöstrand formula ([HeSj], [Dav]) it then follows that $ad_A^{(k)}(f(H))$ is bounded for $k = 1 \ldots n$. The result is that fast decay (large m) in (1.11) must be paid for by high smoothness of $V(x)$ for all x. This is unnatural, and in fact there is a better way to construct A which requires only smoothness of $V(x)$ for arbitray large $|x|$. The idea is to replace x^2 in (1.9) by a smooth, convex function $G(x)$ which is equal to x^2 for large $|x|$ but constant in some abitrary large ball $|x| \le R$. Then A changes to

$$A = \frac{1}{2}(\nabla G(x)\cdot p + p \cdot \nabla G(x)) \ ,$$

and the Mourre inequality can be established as before. Then the operators $ad_A^{(k)}(p^2)$ remain second order in p with bounded coefficients, while

$$i^k\, ad_A^{(k)}(V) \;=\; (-\nabla G(x)\cdot\nabla)^k V(x)$$

requires only derivatives of $V(x)$ in the region $|x| > R$. A more carful construction of $G(x)$ due to Graf [Gra1] is specially adapted to the N-body case, requiring only smoothness of the pair potentials for large separations (see [Skib], [Gri]). The following result can also be proven in this more general setting.

Theorem 1.2. *Let H and A be given by (1.1) and (1.8). Then, under the hypothesis of Theorem 1.1 ,*

$$||\chi^-(x^2 - 2at - \vartheta\, t^2)\, e^{-iHt} g(H)\, \chi^+(A-a)|| \le \text{const. } t^{-m} \tag{1.16}$$

for any ϑ in $0 < \vartheta < \theta$, any $m < n-1$ and any $a \in R$.

We remark that for initial states of the form (1.13) this is equivalent to

$$\int_{|x|\le v\, t} dx\, ||\psi_t(x)||^2 \;\le\; \text{const. } t^{-2m} \tag{1.17}$$

for any $v < \sqrt{\theta}$.

The proofs of these results are given in section 2. The main tool is the method of commutator expansions summarized in section 3.

We conclude the introduction with some (not exhaustive) bibliographical notes. Minimal velocity estimates were first given by Sigal and Soffer [SiSo1] and then extended by Skibsted [Ski] and by Gérard and Sigal [GeSi], with applications to scattering theory ([SiSo2], [Sig], [HeSk]) and to the theory of resonances ([GeSi], [SoWe], [Nier]). Our derivation is similar in spirit to [Ski] and incorporates remarks by Froese and Loss [FrLo]. The related subject of resolvent smoothness and local decay is more fully treated by time–independent methods e.g. in [PSS], [JMP] and [GIS], where further references can be found. The generalization of Mourre's theorem using the construction of Graf [Gra1] first appears in [Ski]. A simpler proof due to Graf [Gra2] is given in [Gri]. For other applications of Mourre's inequality to wave equations and spectral geometry see e.g. [DHS], [DBiPr], [FHP].

2. PROOFS

The following lemma gives the basic estimate for a proof by bootstrap of Theorem 1.1. A smooth function f on R is said to be of order p if for each k

$$|f^{(k)}(x)| \;\le\; \text{const. } |x|^{p-k}\ .$$

Lemma 2.1. *Suppose that A, H and g satisfy the hypothesis of Theorem 1.1. Let f be a positive C^∞-function on R of order < 4 with $f' \le 0$ and $f(x) = 0$ for $x \ge 0$. Let $1 \le s < \infty$; $a \in R$, $A_s = s^{-1}(A-a)$, and $\varepsilon \le 1$. Then*

$$\begin{aligned} g(H)i[H, f(A_s)]g(H) \le\ & s^{-1}\theta g(H)f'(A_s)g(H) \\ & + s^{-(1+\varepsilon)} g(H) f_1(A_s) g(H) \\ & + \text{const. }\ s^{-(2n-1-\varepsilon)} g^2(H) \end{aligned} \tag{2.1}$$

uniformly in $a \in R$, where f_1 is a function on R which again satisfies the hypothesis stated for f.

Proof. In the commutator $i[H, f(A_s)]$ occurring in (2.1) we can replace H by a bounded operator

$$H_b = H\, b(H) \tag{2.2}$$

where $b \in C_0^\infty(R)$; $b \equiv 1$ on $\mathrm{supp}(g)$. Then the commutators

$$B_k = i\; ad_A^{(k)}(H_b)\ , \quad k = 1 \ldots n\ ,$$

are bounded by hypothesis. As a first step we show that

$$i[H_b, f(A_s)] = -s^{-1}(-f'(A_s))^{1/2} B_1 (-f'(A_s))^{1/2} + \text{ remainder}\ . \tag{2.3}$$

Here and in the rest of the proof a *remainder* is defined as a quadratic form $\mathrm{rem}(s)$ with an estimate

$$\pm \mathrm{rem}(s) \le s^{-(1+\varepsilon)} f_1(A_s) + \text{const.}\ \ s^{-(2n-1-\varepsilon)}\ , \tag{2.4}$$

uniformly in a, where f_1 is a function on R satisfying the hypothesis for f. Any such remainder clearly fits into (2.1) and needs no further consideration. To prove (2.3) we factorize $f = F^2$ and then expand the commutator

$$\begin{aligned} i[H_b, f] &= i[H_b, F]F + F i[H_b, F] \\ &= \sum_{k=1}^{n-1} \frac{1}{k!} s^{-k} (F^{(k)} B_k F + F B_k^* F^{(k)}) \\ &\quad + s^{-n}(RF + FR^*) \end{aligned} \tag{2.5}$$

using (3.1). Since $n \ge 2$ and since F is of order < 2 it follows from (3.2) that R is bounded uniformly in s and a. Now we observe that all the terms in the expansion (2.5) except the leading term ($k = 1$) are remainders. In particular

$$|(\psi, F^{(k)} B_k F \psi)| \le ||B_k||\, ||F^{(k)}\psi||\, ||F\psi|| \le \text{const.}\ \ (\psi, f_1 \psi), \tag{2.6}$$

where f_1 is a common upper bound for $F^2 = f$ and $(F^{(k)})^2$ which satisfies the hypothesis for f. The last term in (2.5) is estimated using the operator inequality

$$\pm(P^* Q + Q^* P) \le P^* P + Q^* Q$$

for $Q = s^{-\frac{1}{2}(1+\varepsilon)} F$; $P^* = s^{-n+\frac{1}{2}(1+\varepsilon)} R$, with the result

$$\pm s^{-n}(RF + FR^*) \le s^{-(1+\varepsilon)} f + s^{-(2n-1-\varepsilon)} ||R||^2\ . \tag{2.7}$$

Therefore it remains to consider the leading term ($k = 1$) in (2.5), which is rewritten as

$$s^{-1}(F' B_1 F + F B_1 F') = -s^{-1}(v^2 B_1 u^2 + u^2 B_1 v^2)$$

by factorizing $F = u^2$, $-F' = v^2$. Since u is of order < 1 it follows from (3.2) that $||[B_1, u]|| \le \text{const.}\ \ s^{-1}$ uniformly in a, and similarly for $[B_1, v]$. As in (2.6) this leads to the form estimate

$$\begin{aligned} s^{-1}|v^2 B_1 u^2 - uv B_1 uv| &= s^{-1}|v[B_1, u]uv + v[v, B_1]u^2| \\ &\le \text{const.}\ \ s^{-2}(v^2 + u^2 v^2 + u^4) \le \text{const.}\ \ s^{-2} f_1(A_s)\ , \end{aligned}$$

where f_1 shares the properties of $f = u^4$ (note that v is of lower order than u). Since the same estimate holds with u and v interchanged, we conclude that

$$s^{-1}(F'B_1F + FB_1F') = -2s^{-1}\, uv\, B_1\, uv$$

plus a remainder. (2.3) now follows since $f' = 2FF' = -2(uv)^2$.

To complete the proof of Lemma 2.1. we multiply (2.3) from both sides with $g(H) = g(H)G(H)$, where $G \in C_0^\infty(\Delta)$ is real and $G \equiv 1$ on supp(g). We also adjust the function b in (2.2) such that $b \equiv 1$ on supp(G). Multiplying (1.6) by $G(H)$ from both sides we obtain the Mourre inequality :

$$G(H)B_1G(H) = G(H)i[H, A]G(H) \geq \theta\, G^2(H). \tag{2.8}$$

Abbreviating $G(H) = G$ and $(-f'(A_s))^{1/2} = j$ we show that

$$s^{-1}GjBjG - s^{-1}jGBGj = s^{-1}\big(\, jGB[j, G] + [G, j]BGj + [G, j]B[j, G]\,\big) \tag{2.9}$$

is a remainder for any bounded $B = B^*$, using for $[G, j]$ the expansion

$$[G, j] = \sum_{k=1}^{n-1} \frac{1}{k!} s^{-k}\, j^{(k)}(A)\, ad_A^{(k)}(G) + s^{-n} R$$

and its adjoint for $[j, G]$. Since $ad_A^{(k)}(G)$ is bounded for $k \leq n$, the right hand side of (2.9) then becomes a sum of terms of the following types :

$$(a) \qquad s^{-(k+l+1)} \left(j^{(k)}\, C\, j^{(l)} + j^{(l)}\, C^*\, j^{(k)} \right); \quad (0 \leq k, l \leq n-1;\; k + l \geq 1);$$

$$(b) \qquad s^{-(k+n+1)} \left(j^{(k)}\, C + C^*\, j^{(k)} \right); \qquad (0 \leq k \leq n-1);$$

$$(c) \qquad s^{-(2n+1)}\, C,$$

where in each case C stands for some operator which is bounded uniformly in a and s. By the same arguments as in (2.6) and (2.7) these terms have corresponding upper and lower bounds

$$(a) \qquad \pm\, s^{-2}\, 2||C|| \left(j^{(k)2} + j^{(l)2} \right);$$

$$(b) \qquad \pm \left(s^{-2}\, j^{(k)2} + s^{-2n}\, ||C||^2 \right);$$

$$(c) \qquad \pm\, s^{-(2n+1)}\, ||C|| \,.$$

For any bounded $B = B^*$ we therefore obtain

$$s^{-1}GjBjG \cong s^{-1}jGBGj \;, \tag{2.10}$$

meaning that the difference of the two expresssions is a remainder (2.4). For $B = B_1 = i[H_b, A]$ we use Mourre's inequality (2.8) to obtain from (2.3) :

$$\begin{aligned} G\, i[H, f]\, G \;&\cong\; -s^{-1}Gj\, B_1\, jG \;\cong\; -s^{-1}jG\, B_1\, Gj \\ &\leq -s^{-1}\theta jG^2j \;\cong\; -s^{-1}\theta Gj^2G \\ &=\; s^{-1}\theta Gf'G \;, \end{aligned}$$

with remainders arising from (2.3) and twice from (2.10). Multiplying from both sides with $g(H)$ removes $G(H)$ and leads directly to (2.1). $\square$

Proof of Theorem 1.1. We prove a slightly stronger version of (1.11), which will serve later in the proof of Theorem 1.2. Let

$$A_{ts} = s^{-1}(A - a - \theta t) \ ,$$

and suppose that F is a positive C^∞-function of order $\leq 1/2$ on R with $F' \leq 0$ and $F(x) = 0$ for $x \geq 0$. Instead of (1.11) we show under the same hypothesis that

$$||F(A_{ts})e^{iHt}g(H)\chi^+(A-a)|| \leq \text{const.} \ \ s^{-m} \tag{2.11}$$

for $m < n-1$, uniformly in $0 \leq t \leq s$ and in $a \in R$. (1.11) then follows by setting $t = s$ and by observing that, since $\vartheta < \theta$, $\chi^-(s^{-1}(A-a)-\vartheta) \leq F(s^{-1}(A-a)-\theta)$ for some F of the required type. To prove (2.11) we consider the operator

$$\phi_s(t) = g(H)f(A_{ts})g(H) \ ; \ f = F^2 \ ,$$

and the evolution

$$\psi_t = e^{-iHt}\chi^+(A-a)\varphi \ ; \ \varphi \in \mathcal{H} \ . \tag{2.12}$$

Then the estimate (2.11) to be proved reads

$$\langle \phi_s(t) \rangle_t = (\psi_t, \phi_s(t)\psi_t) \leq \text{const.} \ ||\varphi||^2 s^{-2m} \ , \tag{2.13}$$

uniformly in $1 < s$, $0 \leq t \leq s$ and in $a \in R$. We compute

$$\partial_t \langle \phi_s(t) \rangle_t = (\psi_t, D_t\phi_s(t)\psi_t) \ ; \tag{2.14}$$

$$\begin{aligned} D_t\phi_s(t) &= i[H, \phi_s(t)] + \partial_t\phi_s(t) \\ &= g(H)i[H, f(A_{ts})]g(H) - s^{-1}\theta g(H)f'(A_{ts})g(H) \ . \end{aligned} \tag{2.15}$$

First we conclude that

$$||D_t\phi_s(t)|| \leq \text{const.} \tag{2.16}$$

uniformly in s, t, a, since f is of order ≤ 1. (cf. the remark after (3.4)). Secondly, by (3.8),

$$\begin{aligned} \langle \phi_s(0) \rangle_0 &\leq ||\varphi||^2 ||F(s^{-1}(A-a))g(H)\chi^+(A-a)||^2 \\ &\leq \text{const.} \ \ s^{-2n}||\varphi||^2 \ . \end{aligned} \tag{2.17}$$

Integrating (2.14) over t and using (2.16) and (2.17) we find the crude estimate

$$\langle \phi_s(t) \rangle_t \leq \text{const.} \ ||\varphi||^2 (s^{-2n} + s)$$

for $0 \leq t \leq s$, which proves (2.13) for $m = -1/2$. Now we bootstrap this estimate. First we note that by (2.15) and Lemma 2.1

$$D_t\phi_s(t) \leq s^{-1-\varepsilon} g(H)f_1(A_{st})g(H) \ + \ \text{const.} \ s^{-(2n-1-\varepsilon)} g^2(H) \ . \tag{2.18}$$

As an induction assumption, suppose that (2.13) holds for some $m < n-1$. Since f_1 also satisfies the hypothesis for f it then follows from (2.18) that

$$|\langle D_t\phi_s(t)\rangle_t| \leq \text{const. } ||\varphi||^2 \cdot s^{-1}\big(s^{-(2m+\varepsilon)} + s^{-(2(n-1)-\varepsilon}\big) \ ,$$

and again by integrating over t :

$$\langle\phi_s(t)\rangle_t \leq \text{const. } ||\varphi||^2\big(s^{-2n} + s^{-(2m+\varepsilon)} + s^{-(2(n-1)-\varepsilon)}\big)$$

uniformly in $0 \leq t \leq s$ and in $a \in R$. Recalling that $\varepsilon \leq 1$, the best decay for $s \to \infty$ is obtained by setting

$$\varepsilon \ = \ \min\big(1, \ (n-1) - m\big) \ ,$$

which boosts the exponent m in (2.13) to $m' = m + \varepsilon/2$. Therefore (2.13) holds for any $m < n-1$. □

Proof of Theorem 1.2. It suffices to prove that

$$||\chi(t^{-2}x^2 - 2t^{-1}a - \vartheta)e^{-iHt}g(H)\chi^+(A-a)|| \leq \text{const. } t^{-m} \tag{2.19}$$

if χ is a smoothed characteristic function of $(-\infty, -\varepsilon)$ for some $\varepsilon > 0$: $\chi(x) = 1$ for large negative x, $\chi' \leq 0$, and $\chi(x) = 0$ for $x \geq -\varepsilon$. Following the line of the previous proof we consider the operators

$$\phi_s(t) = f(x_{ts}^2); \ f = \chi^2; \ x_{ts}^2 = \frac{1}{s^2}(x^2 - 2at - \vartheta t^2) \tag{2.20}$$

and the evolution

$$\psi_t = e^{-iHt}g(H)\chi^+(A-a)\varphi \ , \ \varphi \in \mathcal{H} \ , \tag{2.21}$$

for $0 \leq t \leq s$. The desired inequality (2.19) then reads

$$\langle\phi_s(s)\rangle_s \leq \text{const. } s^{-2m} \ . \tag{2.22}$$

Writing $x^2 - 2at - \vartheta^2 = (x-a)^2 - (a^2+\vartheta)t^2$ we see that

$$\phi_s(t) = 0 \quad \text{ for } \quad t \leq s\sqrt{\frac{\varepsilon}{a^2+\vartheta}} \equiv \alpha s \ ,$$

and therefore

$$\langle\phi_s(s)\rangle_s = \int\limits_{\alpha s}^{s} dt \langle D_t\phi_s(t)\rangle_s \ . \tag{2.23}$$

To find $D_t\,\phi_s(t)$ we first compute

$$\begin{aligned} i[H, \phi_s(t)] &= \frac{i}{2}[p^2, \phi_s(t)] = s^{-2}\big(Af'(x_{ts}^2) + f'(x_{ts}^2)A\big) \\ &= -2s^{-2}u(x_{ts}^2)Au(x_{ts}^2) \ , \end{aligned}$$

where we have factorized $f' = -u^2$ and used that $\big[[A,u]u\big] = 0$. Adding the term

$$\partial_t\phi_s(t) = 2s^{-2}(a + \vartheta t)u^2(x_{ts}^2)$$

we arrive at

$$D_t\phi_s(t) = -2ts^{-2}u(x_{ts}^2)(t^{-1}(A-a)-\vartheta)u(x_{ts}^2) \ . \tag{2.24}$$

Now we use $\vartheta < \theta$ to estimate

$$\begin{aligned} -(t^{-1}(A-a)-\vartheta) &\leq -(t^{-1}(A-a)-\vartheta)\chi^-(t^{-1}(A-a)-\vartheta) \\ &\leq \left(F(t^{-1}(A-a)-\theta)\right)^2 \end{aligned}$$

by some smooth function F of order $1/2$ supported in R^- and with $F' \leq 0$. Setting $A_t = t^{-1}(A-a)-\theta$ we find

$$\langle D_t\phi_s(t)\rangle_t \leq 2ts^{-2}||F(A_t)u(x_{ts}^2)e^{iHt}g(H)\chi^+(A-a)\varphi||^2 \ . \tag{2.25}$$

On the other hand, it follows from (2.11) by setting $t = s$ that

$$||F(A_t)e^{iHt}g(X)\chi^+(A-a)|| \leq \text{const. } t^{-m} \ . \tag{2.26}$$

Before we can use this estimate in (2.25) we must commute the factor $u(x_{ts}^2)$ to the left. The required commutator can be expanded to any order n :

$$[u(x_{ts}^2), F(A_t)] = \sum_{k=1}^{n-1} \frac{1}{k!} t^{-k} ad_A^{(k)}(u) \ F^{(k)}(A_t) \ + \ t^{-n}R \ ,$$

where $||R|| \leq$ const. $||ad_A^{(n)}(u)||$. Since $u \in C_0^\infty(R)$, the commutators $ad_A^{(k)}(u)$ are easily bounded :

$$\begin{aligned} -iad_A^{(1)}(u) &= x \cdot \nabla u(x_{ts}^2) = 2s^{-2}x^2u'(x_{ts}^2) \\ &= 2x_{ts}^2u'(x_{ts}^2) + 2s^{-2}(2at+\vartheta t^2)u'(x_{ts}^2) \end{aligned}$$

and so forth, with the result that $||ad_A^{(k)}(u)|| \leq$ const. uniformly in $1 \leq s < \infty$ and $0 \leq t \leq s$. Since (2.26) also holds if F is replaced by a derivative $F^{(k)}$, we find the estimate

$$\begin{aligned} \langle D_t\phi_s(t)\rangle_s &\leq \text{const. } ts^{-2}(t^{-m} + \sum_{k=1}^{n-1} t^{-k-m} + t^{-n})^2 \\ &\leq \text{const. } s^{-2}t^{-2m+1} \end{aligned}$$

for $1 \leq s < \infty$, $0 \leq t \leq s$, uniformly in a. Therefore, by (2.23),

$$\langle \phi_s(s)\rangle_s \leq \text{const. } s^{-2}\int_{\alpha s}^{s} dt \ t^{-2m+1} \leq \text{const. } s^{-2m} \ . \qquad \square$$

3. COMMUTATOR EXPANSIONS

Let H and A be self-adjoint operators on a Hilbert space $\mathcal{H}$ and suppose that H is bounded. To say that the commutor $i[H, A]$ is bounded means that the quadratic form

$$i[(H\psi, A\psi) - (A\psi, H\psi)]$$

on $D(A)$ is bounded and thus defines a bounded, symmetric operator called $i[H,A]$. In the same sense we assume that the higher commutators

$$ad_A^{(k)}(H) = [ad_A^{(k-1)}(H)\ ,\ A]$$

are bounded for $k = 2 \ldots n$. Let f be a real C^∞–function on R. Then, under a further condition given below, the commutator $[H, f(A)]$ has an expansion

$$[H, f(A)] = \sum_{k=1}^{n-1} \frac{1}{k!} f^{(k)}(A) ad_A^{(k)}(H) \ + \ R_n \tag{3.1}$$

with a remainder estimate

$$||R_n|| \leq c_n ||ad_A^{(n)}(H)|| \sum_{k=0}^{n+2} \int dx (1 + |x|)^{k-n-1} |f^{(k)}(x)| \ . \tag{3.2}$$

The further condition on f is that the integrals (3.2) exist. The number c_n is a numerical constant depending on n but not on f, A or H. In particular, the expansion (3.1) holds if

$$f^{(k)}(x) = O(|x|^{n-\varepsilon-k}) \qquad (x \to \pm\infty) \tag{3.3}$$

for $k = 1 \cdots n+2$, i.e. if the function $f(x)$ grows not faster than $|x|^{n-\varepsilon}$, with corresponding slower growth of the successive derivatives. We will refer to (3.3) by saying that f is of order $n - \varepsilon$. In that case (3.1) is defined in form sense on the domain of $f^{(1)}(A)$. Taking the adjoint of (3.1) and noting that

$$ad_A^{(k)}(H)^* = (-1)^k ad_A^{(k)}(H) \ ,$$

we also obtain

$$[H, f(A)] = \sum_{k=1}^{n-1} \frac{1}{k!} \cdot (-1)^{k-1} ad_A^{(k)}(H) f^{(k)}(A) \ - \ R_n^* \ . \tag{3.4}$$

This defines $[H, f(A)]$ as an operator on the domain of $f^{(1)}(A) = f'(A)$. In particular, if f is of order ≤ 1 and $n \geq 2$, then $[H, f(A)]$ is bounded. In the general case where H is not bounded, we will work with operators $g(H)$, $g \in C_0^\infty(R)$, assuming that

$$ad_A^{(k)}(g(H)) \text{ is bounded for } k = 1 \cdots n \ . \tag{3.5}$$

Then, if f is of order $< n$,

$$\begin{aligned} [g(H), f(A)] &= \sum_{k=1}^{n-1} \frac{1}{k!} f^{(k)}(A) ad_A^{(k)}((g(H)) \ + \ R_n \ ; \\ ||R_n|| &\leq \text{const. } ||ad_A^{(n)}((g(H))|| \ , \end{aligned} \tag{3.6}$$

with a constant depending on f and n, and similarly for the adjoint expansion. All these formulas are particularly useful if the role of A is played by a scaled operator, say $s^{-1}A$, $0 < s < \infty$. Then the commutor expansions are expansions in powers of s^{-1}, e.g.

$$[g(H), f(s^{-1}A)] = \sum_{k=1}^{n-1} \frac{1}{k!} s^{-k} f^{(k)}(A) ad_A^{(k)}(g(H)) \ + \ s^{-n} R_n \ . \tag{3.7}$$

A simple but useful observation is the following. Suppose that $f(x) = 0$ for $x \in R^+$, and let χ^+ be the characteristic function of R^+. Then

$$\|\chi^+(A)g(H)f(s^{-1}A)\| \leq \text{const. } s^{-n} \ . \tag{3.8}$$

Proof. Since $\chi^+(A)f(s^{-1}A) = 0$ we have

$$\chi^+(A)g(H)f(s^{-1}A) = \chi^+(A)[g(H), f(s^{-1}A)] \ .$$

Inserting the expansion (3.7) we notice that only the remainder $s^{-n}R_n$ contributes, since

$$\chi^+(A)f^{(k)}(A) = 0. \qquad \square$$

Commutator expansions of this type were introduced in [SiSo1] and have since become an important tool of operator analysis. There are several versions ([SiSo1], [Ski], [IvSi], [ABG]) which differ in the form of the remainder estimate. The results above are derived in [HuSi2] and are based on the Helffer-Sjöstrand functional calculus ([HeSj], [Dav]).

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