DISSIPATION THROUGH DISPERSION

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Abstract. In this note I describe a time dependent approach to analyze dispersion mediated energy transfer for both linear and nonlinear systems.

Examples, applications and a discussion of some open problems are included.

Section 0. - Introduction.

Dissipative processes are fundamental to our understanding of many physical properties of large systems. They are important in approach to equilibrium, measurement processes, heat transfer ect.

On the other hand, physical theories are expected to conserve energy, so the resulting dynamics is hamiltonian.

To reconcile the two, seemingly contradictory processes, we invoke radiation damping, or dispersion of the energy from one ("small") system to a large, infinite dimensional system.

I'll describe a new, time dependent approach to analyze this dispersion mediated energy transfer, and some examples and applications. This approach turns out to be surprisingly general and applied to diverse problems including resonances in quantum mechanics, asymptotic stability of Nonlinear hamiltonian PDE's, Atoms-laser systems, nonlinear optics and more. Besides describing some of these results, I'll mention a few open problems in different fields, of related nature.

The class of models we are interested here are of the type "small" + infinite systems. [W, Dav, Sew, A-E]

Typically the small system will have finite dimensional phase space, and the infinite part will be dispersive field, or medium. A classical example, due to Lamb [L] is the spring + string system, in which a particle of mass m on a spring, is also coupled to an infinite string.

The system is hamiltonian, yet, if one looks at (finite energy) states, the asymptotic behavior of the mass on the spring is dissipative, due to energy losses to infinity via the string. In some sense all the models considered here are generalizations of this system. This particular model can be solved exactly. Here I'll describe a method to deal with the general case of finite dimensional system coupled to a dispersive (not necessarily linear) medium. [S-W1-6, MSW, K-W]

All the models I'll describe are at zero temperature. See the discussion in sections 4,5 concerning the finite temperature case.

Section 1. Models, Examples.

A beautiful consequence of dissipation phenomena is the Einstein relation. Suppose p, the momentum of a particle coupled to a heat bath satisfies

$$\frac{dp}{dt} = -\alpha p + \eta(t) \tag{1}$$

where $\eta(t)$ is a random noise, with zero average $\langle \eta \rangle = 0$ and

$$\langle \eta(t)\eta(s) \rangle = 2Dg(t-s) \tag{2}$$

as correlation.

By solving for $\langle p^2 \rangle$, after approximating g by a δ-function we get

$$\langle p^2 \rangle = \frac{D}{\alpha} \tag{3}$$

If the white noise term $\eta(t)$ is modeling a bath at temperature T, we have from thermal equilibrium

$$\frac{\langle p^2 \rangle}{2M} = \frac{1}{2}kT \tag{4}$$

and therefore

$$D = Mk\alpha T \tag{5}$$

which is the Einstein relation.

Here M is the mass of the particle.

Deriving this result directly from the hamiltonian of the system is a major problem. See section 5.

Dispersive systems can be cast in the following Schrödinger form:

$$i\frac{\partial}{\partial t}\Psi = K\Psi + NL(\Psi)$$

where Ψ is in general a vector valued function and K is a linear matrix hamiltonian, self-adjoint operator.

$NL(\Psi)$ stands for a general perturbation/coupling which can be linear or non-linear. For integrable systems see [D-Z].

Example 1. Consider a quantum particle coupled to an infinite system, described in some "mean field" approximation; the resulting equation is, for the effective wave function Ψ:

$$i\frac{\partial\Psi}{\partial t} = (-\Delta + V)\Psi + f(\Psi).$$

Here $f(\Psi)$ can be of the general form

$$f(\Psi) = \int R(x-y)|\Psi(y)|^p \Psi(x) dy$$

for some real valued function R.

When $-\Delta + V$ has bound states the large time behavior of this system is incredibly complicated.

Example 2. Nonlinear Wave Equations.

The equation

$$(\Box + V(x,t))u = f(u)$$

with $V(x,t)$ real, the nonlinear, nonhomogeneous wave equation, appears naturally in many applications e.g., in nonlinear optics models or when linearizing around solitary type solutions of homogeneous systems, as well as in the study of blackhole radiation. The original model of Lamb is of the above form with $f(u) = 0$ for linear string, and $f(u) \neq 0$ if the string is nonlinear.

Example 3. Resonances in QM

It may sound surprising that the resonance problem in QM is of the type discussed above. In fact, one can reformulate the resonance problem in many cases as follows

$$i\frac{\partial\Psi}{\partial t} = H_0\Psi + W\Psi$$

with H_0 has embedded eigenvalues in the continuum, and W is a perturbation that couples the eigenvalues to the continuum see [H-S] ans cited ref. In general the eigenvalues disintegrate, exponentially fast into radiation, up to small corrections. For other time dependent analysis of this problem see e.g. [Hu, P-F, G-S, CLR] and cited ref. Again the Lamb's model is exactly of this type, where now H_0 stands for the hamiltonian describing the motion of a mass on a spring and a string decoupled from each other. W is then the coupling term between the mass and the string. The point spectrum of the matrix H_0 comes from the basic frequency of the mass-spring system, and the continuous spectrum from the wave equation of infinite string.

Section 2. Linear Theory.

The discussion of the last example of section 1 is the basis for the approach we utilize here; the dissipative behavior should appear as the exponential decay of a "resonance", and is therefore valid only for a finite time scale. In this way we resolve the apparent impossibility of closed hamiltonian systems to describe dissipative behavior.

Of course, to fulfill the above promise, one needs a general enough resonance theory, which is time dependent, and gives explicit rate of exponential decay and bounds on the correction terms,**uniformly in time**. We first consider linear systems; examples include atom + radiation systems, finite dimensional linear systems of masses and springs coupled to linear wave equations (e.g. a mechanical clock, coupled to sound or water waves if floating..., in the linear approximation). When the environment is in a finite temperature (e.g. as in Einstein relation case) such formulation is still possible in terms of Liuville dynamics on the space of density matrices. [J-P 1,2, Dav, Sew].

Consider then the following system:

The hamiltonian of the system is given by a self-adjoint operator acting on a Hilbert space $\mathcal{H}, H = H_0 + W$ and let Ψ_0 be an embedded eigenvector of H_0, with eigenvalue λ_0

$$H_0\Psi_0 = \lambda_0\Psi_0 \qquad \|\Psi_0\|_{\mathcal{H}} = 1.$$

The main result established in [S-W,4] for this case is

Theorem 1.

Let $H = H_0 + W$ be a self-adjoint operator acting on a Hilbert space $L^2(R^n)$. $H_0\Psi_0 = \lambda_0\Psi_0, \lambda_0 \in \Delta \subset I; I-$ the continuous spectral interval of H_0. Suppose that H_0 satisfies conditions (H) and W satisfies conditions (W) below. Then

(a) $H = H_0 + W$ has no eigenvalues in the interval Δ.

(b) The spectrum of H in Δ is purely absolutely continuous; in particular we have local decay for H;

$$\|\langle x\rangle^{-\sigma} e^{-iHt} g_\Delta(H)\phi\|_{L^2} \le c\langle t\rangle^{-r+1}\|\langle x\rangle^{\sigma}\phi\|_{L^2}$$

for σ as in $(H), (W)$.

Here we use the notation $\langle x\rangle = \sqrt{1+|x|^2}$

$g_\Delta(H)$ is a smooth characteristic function of the interval Δ, with argument the operator H.

(c) For ϕ_0 in the range of $g_\Delta(H)$, we have for $t \ge 0$

$$e^{-iHt}\phi_0 = (1 + O(W))[e^{-\Gamma t} a(0)\Psi_0 e^{-i\theta t} + e^{-iH_0 t}\phi_d(0)] + R(t)$$

$a(0), \psi_d(0)$ are determined by the initial data ϕ_0.

$$\theta = \lambda_0 + (\Psi_0, W\Psi_0) - \Lambda + O(W^3) \equiv \omega - \Lambda + O(W^3)$$

$$\Lambda = (W\Psi_0, P.V.(H_0 - \omega)^{-1} W\Psi_0)$$

$$\Gamma = \pi(W\Psi_0, \delta(H_0 - \omega)(I - P_0)W\Psi_0)$$

P_0 is the Projection on Ψ_0.

$$\|\langle x\rangle^{-\sigma} R(t)\|_{L^2} \le c|||W|||$$
$$\|\langle x\rangle^{-\sigma} R(t)\|_{L^2} \le c|||W|||^{\varepsilon}\langle t\rangle^{-r+1} \qquad t \ge |||W|||^{-2(1+\delta)}.$$

Conditions (H)

(H1) H_0 is a selfadjoint operator with dense domain $\mathcal{D} \subset L^2$.

(H2) λ_0 is a simple eigenvalue embedded in $P_c\mathcal{D}$, and eigenfunction Ψ_0 normalized to 1. P_c is the projection on the continuous spectral part of H_0.

(H3) These exists an open interval Δ around λ_0, with no other eigenvalues.

(H4) For some $\sigma > 0$ we have local decay with some $r \ge 2+\varepsilon, \varepsilon > 0$:

$$\|\langle x\rangle^{-\sigma} e^{-iH_0 t} P_c^{\#} f\|_{L^2} \le C\langle t\rangle^{-r}\|\langle x\rangle^{\sigma} f\|_{L^2}$$

where $P_c^{\#} = g_{\tilde{\Delta}}(H_0),$ with $\tilde{\Delta} \supset \Delta, \tilde{\Delta}$ avoiding any other threshold or eigenvalue of H_0 (except of course λ_0).

(H5) For some choice of c, $\langle x\rangle^{\sigma}(H_0+c)^{-1}\langle x\rangle^{-\sigma}$ can be made small enough.

Conditions (W)

(W1) W is symmetric, $H = H_0 + W$ is self-adjoint on $\mathcal{D}$.

(W2) for some $\sigma > 0$, as above,

$$|||W||| \equiv \|\langle x\rangle^{2\sigma} g_{\Delta}(H_0)W\| + \|\langle x\rangle^{\sigma} W g_{\Delta}(H_0)\langle x\rangle^{\sigma}\| + \|\langle x\rangle^{\sigma} W (H_0)^{-1}\langle x\rangle^{-\sigma}\| < \infty$$

(W3) for some $\delta_0 > 0$

$$\Gamma \ge \delta_0 |||W|||^2$$

(W4) $|||W||| < \delta_1|\Delta|$ for some $\delta_1 > 0$, sufficiently small, depending only on the properties of H_0, in particular the constants in the local decay estimates, but not on $|\Delta|$.

Corollary.
Let $H_* = H - (Re\omega)I_d$.

Then, for any $T > 0$ there is a constant c_T

$$|(\Psi_0, e^{-iH_* t}\Psi_0) - e^{-\Gamma t}| \le c_T|||W||| \text{ as } |||W||| \to 0.$$

Remarks

1) In applications to open systems, H_0 will generally be the direct sum

$$H_0 = H_s \oplus H_r$$

where H_s has discrete spectrum and H_r with continuous spectrum.

Then, the eigenvalues of H_s generally become embedded and therefore unstable under the perturbation W which is the coupling between the system and reservoir.

This extends to the finite temperature case where we now need the Liuvillian to replace the hamiltonian [J-P 1,2].

2) The corollary gives the time scales on which we see the exponential behavior. Generally it is of order $|||W|||^{-2}$ times a (large) constant.

3) A useful and important generalization of this theorem was recently proved [M-Sig]: it allows degenerate eigenvalue λ_0 and it only requires that

$$\|W\Psi_0\|$$

be small; this is very important in applications, especially for N-body systems (The price to pay is that now local decay is needed for H and not H_0, and this is hard to prove in the presence of infraparticles [BFSS]).

The method of proof of this theorem is time-dependent. As such it applies to nonautonomous hamiltonians and nonlinear dispersive equations. For the linear time dependent potential case see [S-W3,6, MSW, K-W]. Nonlinear cases are discussed in the next section.

Section 3 Nonlinear theory - Asympototic Stability for Dispersive systems.

Finite dimensional hamiltonian systems do not have asymptotic stable points (due to Liuville's theorem). The nonlinear systems describing solitary type solutions have such stability: radiation goes away from the solitons to infinity leaving them to move freely. This is a general phenomena: kinks, vortices, black holes, binary stars coupled to gravity waves, all expected (and observed) to behave like that.

This phenomena is therefore intrinsically infinite dimensional; our aim is to understand this type of phenomena as a time dependent nonlinear resonance scattering, in which a small system (solitons, kinks, ect...) coupled to a big system (radiation = dispersive equations) is stabilized as time approaches infinity. This problem can be studied by the methods described here:

Consider the following model, the nonhomogeneous NLKG equation:

$$(\Box + m^2 + V(x))u(x,t) = \lambda F(u) \quad x \in \mathbb{R}^3$$

$$\Box = \partial_t^2 - \Delta$$

and we assume that $-\Delta + V(x)$ has one bound state (isolated eigenvalue) with

$$B^2 - \Delta + V(x) + m^2 > 0$$

$$B^2\varphi = \Omega^2\varphi\,, \varphi \in L^2$$

so that the linear system ($\lambda = 0$) has the following localized time periodic solution

$$u(x,t) = R\cos(\Omega t + \theta)\varphi(x).$$

It is easy to verify that the above system is hamiltonian (when $F(u)$ is real valued) and that 0 is the state of lowest energy. One would guess that as the nonlinearity is turned on, $\lambda \neq 0$, the lump with energy Ω^2 will disintegrate. In fact this is what is proved in [S-W5] by time dependent methods.

It turns out that as the nonlinearity is turned on (and small)

$$\begin{aligned}
u(x,t) &\sim a(t)\cos(\Omega t + \rho(t))\varphi(x) + \eta(x,t)\\
a(t) &\sim c(1 + \frac{3\lambda^2}{4\Omega}\Gamma t)^{-1/4}\\
\rho(t) &\sim k_1 t^{1/2} + k_2 lnt + O(t^{-1/2})\\
\|\eta(\cdot,t)\|_{L^8(\mathbb{R}^3)} &\leq O(t^{-3/4})\\
\Gamma &\equiv \frac{\pi}{3\Omega}(P_c(B)\varphi^3, \delta(B{-}3\Omega)P_c(B)\varphi^3).
\end{aligned}$$

Here $f(u) = \lambda u^3 + O(u^4)$.

Some ideas of the proof

Begin with the ansatz

$$u(x,t) = a(t)\varphi(x) + \eta$$

and orthogonality $(\varphi, \eta) = 0\,\forall t$.

$$\begin{aligned}
a(0) = (\varphi, u(0)) &\qquad a'(0) = (\varphi, u_t'(0))\\
\eta(0,x) = P_c(B)u(0) &\quad \partial_t\eta(0,x) = P_c(B)u_t'(0).
\end{aligned}$$

Using the equations of motion one can then derive a system of equations for $a(t)$ and η; a simplified version of these equations look like this

$$\begin{aligned}
a'' + \Omega^2 a &= \lambda k a^3 + 3\lambda a^2(\varphi^3, \eta) + \cdots\\
\partial_t^2\eta + (-\Delta + V)\eta + m^2\eta &= \lambda a^3\varphi^3 + \cdots .
\end{aligned}$$

Ignoring the $\cdots$ terms it has the following hamiltonian structure:

$$\mathcal{E}_{\text{appx.}} = \frac{1}{2}\int \left[(\partial_t u)^2 + |\nabla u|^2 + Vu^2\right] dx + \frac{1}{2}(a^{12} + \Omega^2 a^2 - \frac{1}{2}\lambda k a^4)$$
$$\lambda a^3 \int \chi(x)\eta(x,t)dx.$$

It is therefore an anharmonic oscillator coupled to the linear dispersive equation given by the nonhomogeneous KG equation, and coupled linearly in η and nonlinearly in a.

Then, we solve the η equation in terms of the propagator of the linear KG equation (with potential) and treat the $\lambda a^3 \varphi^3$ (and $\cdots$) as source term. Then plug it into the a equation.

Then, by a series of decompositions and integration by parts, we isolate the leading a^m terms in the resulting a equation. The integration by parts leads to evaluation of Green's functions as energies belonging the the continuum of B, which through ε - prescription determines the sign of the dissipation term, according to the direction of time! In particular, as $t \to +\infty$ we get

$$a'' + (\Omega^2 + O(|a|^2))a = -\Gamma|a|^4 a + \sum_{j\geq 3} O(a^i) \quad \Gamma > 0,$$

and $-\Gamma$ is replaced by $+\Gamma$ for $t \to -\infty$.

We then prove by a method based on repeated integration by parts that it is possible to change variables $a \to A$ in such a way that the $\sum_{j\geq 3} O(a^i)$ become higher than the leading $|a|^4 a$ term. We therefore conclude that, the dissipative term $-\Gamma|a|^4 a$ dominates the large time behavior, which then gives the decay of $a(t) \sim t^{-1/4}$, and the rest of the results.

Remarks

1) For Γ small and t not too large, the decay will "look" like exponential.

2) Some of the mathematical consequences of this analysis include global existence for small data in H^2 for this equation, and asymptotic stability of 0 (since the above estimates hold for **all** initial conditions around zero).

The first results based on the time dependent discussed in this article are for the nonlinear Schrödinger equation with one bound state [S-W 1, 2]. This was later developed for other nonlinear equations [Pe-W, B- Per , S-W5, Kev-W]. A different way of estimating the large time behavior and the decoupling between the system and radiation is employed in [K-S], where the Huygens principle for wave equation allows the decoupling to be achieved in essentially finite time.

3) Time independent resonance theory was used for the study of stability in nonlinear equations in [C-H, Sig].

Section 4 A Comment on Irreversibility, Entropy and More.

Strictly speaking, proving exponential decay does not necessarily determines the direction of time, or show irreversibility. Since the system is hamiltonian, changing the direction of time, will return the system to its original form [Gu, Leb]. However, we do not see such things in nature: Cuckoo clocks do not absorb sound waves and move exponentially faster, solitons and kinks do not swallow radiation to become more wobbly or energetic, and boats do not rock faster by absorbing water waves.

For systems with finite dimensional phase-space, as particle systems, the way it is understood is that the measure of the set of initial conditions which lead to exponential growth is absurdly small, compared with the typical initial conditions. This can be quantified by entropy considerations. Therefore, changing the direction of time and making some arbitrary small change in the state, will lead to a state very different from the initial one [Leb]. I expect a similar behavior for the systems considered here. However the above argument, based on entropy ect... does not seem to apply, since we now deal with systems in infinite dimensional phase-space (that of radiation). The results discussed in the previous sections show that the degrees of freedom of the small observed system move exponentially fast away from the initial state, and that is believed to be a critical aspect in the proofs of irreversibility through "loss of memory" of the initial state [Gu].

Section 5 Some open problems.
Linear resonance theory in $\mathcal{QM}$

a) Consider $H = -\Delta + V(x)$, with $V(x)$ a function with a maximum at some point x_0. $V(x)$ is smooth and vanishing at infinity. Prove, by time dependent methods the appearance of a resonance corresponding to this critical point. Such results were proved, by complex distortion techniques, but to apply the time dependent approach requires new ideas, since there is no obvious way of rewriting the problem as $H_0 + W$, with H_0 having an embedded eigenvalue.

b) Prove the appearance of a resonance due to the existence of quasimodes: solutions of the eigenvalue equation $H\Psi = z\Psi$, with z complex, and Ψ not in L^2. [G-S]

Remark. Problems a) and b) are in fact related. In [G-S] a) is used to construct quasimodes.

In this respect it may be interesting to improve [G-S] to show the decay in time of the remainder terms.

Linear Resonances in Field theory

α) Apply the theorem of time dependent resonance theory for the case of non-relativistic QED hamiltonians [BFS].

Derive the detailed equations of multilevel state molecule, and transition rates. What is the effect of infra-red contributions to these rates?

β) Apply the theorems of time dependent resonance theory to Liuville operator describing an atom coupled to a field at finite temperature T. See [J- P1,2].

γ) Consider the cases α) or β), even without massless particles, when the small molecule is replaced by a "chain" of N-molecules coupled to each other, and N large. [H-L, A - E, Sew]

In this case, for N-sufficiently large the distance between embedded eigenvalues is smaller than $|||W|||$, and hence the standard analysis fails.

δ) Understand the Einstein relation: in this case the small system is replaced by a free particle. Therefore there is no embedded eigenvalue corresponding to the small system (the spectrum of the small system is now continuous $[0, \infty)$). Nevertheless, the methods may be versatile enough to treat such cases [in the study of time dependent perturbations of Quantum hamiltonians, embedded eigenvalues play no role [S - W3,6, MSW, K-W]].

Nonlinear Scattering Problems

1) Extend the results of [B-Per, P-W, S-W 1,2] to multisoliton systems. The only results in this direction is the case of two-solutions in 1-dim NLS in [Per].

2) Asymptotic stability of Kinks. The main difficulty is that now we have to deal with one dimensional NLKG equation, with long range nonlinear scattering and linear part with nongeneric spectrum (zero energy resonance)

3) Schrödinger and KG vortices: in this case the nonlinear long range part is much more serious than in 1).

4) Nonlinear scattering on nonflat domains, e.g. around a black hole. The metric in this case results in bound states and resonances. See [La-S], and cited ref.

Statistical problems

1) Introduce, in a useful way, the analog of entropy and entropy production to analyze radiation mediated dissipation. Is there an analog of the H theorem for radiation phenomena?

2) Suppose we let one of the systems considered in this article move in forward time, for an interval T, change the direction of time and make a small perturbation of the state. Will the small system's energy grow exponentially in time? Will a small change result in a big difference for the small system? The expressions we have for the remainder terms to exponential decay are pretty explicit. Can it be used? Can it be demonstrated numerically?

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