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Resonance Theory for Schrödinger Operators

Dedicated to J. L. Lebowitz, on the occasion of his 70th birthday

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Abstract: Resonances which result from perturbation of embedded eigenvalues are studied by time dependent methods. A general theory is developed, with new and weaker conditions, allowing for perturbations of threshold eigenvalues and relaxed Fermi Golden rule. The exponential decay rate of resonances is addressed; its uniqueness in the time dependent picture is shown is certain cases. The relation to the existence of meromorphic continuation of the properly weighted Green's function to time dependent resonance is further elucidated, by giving an equivalent time dependent asymptotic expansion of the solutions of the Schrödinger equation.

Key words. Resonances; Time-dependent Schrödinger equation

1. Introduction and results

1.1. General remarks. Resonances may be defined in different ways, but usually refer to metastable behavior (in time) of the corresponding system. The standard physics definition would be as "bumps" in the scattering cross section, or exponentially decaying states in time, or poles of the analytically continued S matrix (when such an extension exists).

Mathematically, in the last 25 years one uses a definition close to the above, by defining λ to be a resonance (energy) if it is the pole of the meromorphic continuation of the weighted Green's function

$$\chi(H-z)^{-1}\chi$$

with suitable weights χ (usually, in the Schrödinger Theory context, χ will be a C_0^∞ function). Here H is the Hamiltonian of the system. In many cases the equivalence of some of the above definitions has been shown [1–3]. However, the exponential behavior in time, and the correct estimates on the remainder are difficult to produce in general [21]. It is also not clear how to relate the time behavior to a resonance, uniquely, and whether "analytic continuation" plays a fundamental role; see the review [5]. Important progress on such relations has recently been obtained; Orth [6] considered the time dependent behavior of states which can be related to resonances without the assumption of analytic continuation and established some preliminary estimates on the remainder terms. Then, Hunziker [7] was able to develop a quite general relation between resonances defined via poles of analytic continuations in the context of Balslev-Combes theory, to exponential decay in time, governed by the standard Fermi Golden rule. Here the resonances were small perturbations of embedded eigenvalues. In [1] a definition of resonance in a time dependent way is given and it is shown to agree with the one resulting from analytic continuation when it exists, in the Balslev-Combes theory. They also get exponential decay and estimates on the remainder terms.

Exact solutions, including the case of large perturbations, for time dependent potentials have recently been obtained in [8]. Further notable results on the time dependent behavior of the wave equation were proved by Tang and Zworski [9]. The construction of states which resemble resonances, and thus decay approximately exponentially was accomplished e.g. in [10].

For resonance theory based on Balslev-Combes method the reader is referred to the book [21] and its comprehensive bibliography on the subject.

Then, in a time-dependent approach to perturbation of embedded eigenvalues developed in [11] exponential decay and dispersive estimates on the remainder terms were proved in a general context, without the assumption of analytic continuation.

When an embedded eigenvalue is slightly perturbed, we generally get a "resonance". One then expects the solution at time t to be a sum of an exponentially decaying term plus a small term (in the perturbation size) which, however, decays slowly. The lifetime of the resonance is given by Γ^{-1} where Γ, the probability of decay per unit time, enters in the exponential decay rate

$$p(t) \sim e^{-\Gamma t/\hbar}.$$

If an analytic continuation of $\chi(H_0 - z)^{-1}\chi$ exists in a neighborhood of an embedded eigenvalue, then $\Gamma = -2\Im z_0$, and a resonance z_0 is defined as the pole of the analytic continuation of $\chi(H - z)^{-1}\chi$. In this case, Γ has the following expansion in ϵ

$$\Gamma(\lambda_0, \epsilon) = \epsilon^2 \gamma(\lambda_0, \epsilon) + o(\epsilon^2)$$

The expression for $\gamma(\lambda_0, \epsilon)$ is called the Fermi Golden Rule (FGR). A remarkable fact is that this expansion is defined even when analytic continuation does not exist. Previous works on the existence of resonances required that $\gamma(\lambda_0, \epsilon) > 0$ as $\epsilon \to 0$. This condition is sometimes hard to verify, and in the present work we remove this assumption.

1.2. Outline of new results. In this work we improve the theory of perturbation of embedded eigenvalues and resonances in three main directions:

First, the Fermi Golden Rule condition, which originally required as above (sometimes implicitly) that $\Gamma > C\epsilon^2$ as $\epsilon \to 0$ is removed. We show that under (relatively weak) conditions of regularity of the resolvent of the unperturbed Hamiltonian all that is needed is that $\Gamma > 0$. The price one sometimes has to pay is that it may be needed to evaluate Γ at a nearby point of the eigenvalue λ_0 of the unperturbed Hamiltonian (see (3)). In cases of very low regularity of the unperturbed resolvent, we need in general $\Gamma > C\epsilon^m$, with $m > 2$; m becomes larger if more regularity of the resolvent is provided; cf. (1) and (2) below.

The second main improvement relative to known results in resonance theory is that we only require H^η regularity (see §2.1), with $\eta > 0$, of the unperturbed resolvent near the relevant energy. Most works on resonance require analyticity; the recent works [6,11,21] require H^η regularity with $\eta > 1$. This improvement is important to perturbations of embedded eigenvalues at thresholds (e.g., our condition is satisfied by $H_0 = -\Delta$ at $\lambda_0 = 0$ in three or more dimensions, while the previous results only apply to five or more dimensions).

As a third contribution we indicate that under conditions of analytic continuation and with suitable cutoff the term $e^{-\Gamma t}$ can be separated from the solution and the remainder term is given by an asymptotic series in t^{-a}, $a > 0$, times a stretched exponential e^{-t^b}, with $b < 1$, see §5.

Our analyticity assumptions are weaker and thus apply in cases of threshold eigenvalues where standard complex deformation approaches could fail. Furthermore we replace analytic perturbation methods by more general complex theory arguments.

As concrete examples of applications we outline the following two classes of problems;
(1) In many applications $H_0 = -\Delta \oplus H_1$, where H_1 has a discrete spectrum (see e.g. [21]). if $H_1\psi_0 = 0$ has a solution, then H_0 has an embedded eigenvalue at the threshold, since $\sigma(-\Delta) = [0, \infty)$. In this case the known analytic methods do not apply; the methods of [6] apply when $\eta > 1$ which is the case of the Laplacian on $L^2(\mathbb{R}^N)$ if $N \geq 5$. The results of this paper apply down to $N = 3$.
(2) The Hamiltonians one gets by linearizing a nonlinear dispersive completely integrable equation around an exact solution have an embedded eigenvalue corresponding to the soliton/breather etc. Small perturbations of such completely integrable equations then produce a perturbation problem of embedded eigenvalues with self-consistent potential W. In these cases the size of Γ is typically of higher order in ϵ and in certain cases it is even $O(e^{-1/\epsilon^2})$. Hence the previous works are not applicable since they require a lower bound $O(\epsilon^2)$ on Γ.

Our approach follows the setup of the time dependent theory of [11], combined with Laplace transform techniques. It is expected to generalize to the N-body case following [12]. We will follow, in part, the notation of [11]. The analysis in this work utilizes in some ways this framework, but generalizes the results considerably: the required time decay is $O(t^{-1-\eta})$ and we remove here the assumption of lower bound on Γ; it is replaced by

$$\Gamma \geq C\varepsilon^{\frac{2}{1-\eta}} \tag{1}$$

when $\eta < 1$, and

$$\Gamma > 0, \text{ arbitrary} \tag{2}$$

when $\eta > 1$.

Whenever a meromorphic continuation of the S-matrix or Green's function exists, the poles give an unambiguous definition of "resonance." A time dependent approach or other definitions are less precise, not necessarily unique, as was observed in [6], but usually apply in more general situations, where analytic continuation is either hard to prove or not available.

We provide some information about defining resonance by time dependent methods and its relation to the existence of "analytic continuation".

In particular, we will show that in general one can find the exponential decay rate up to higher order corrections depending on η and Γ.

In case it is known that analytic continuation exists, our approach provides a definition of a unique resonance corresponding to the perturbed eigenvalue. It is given by the solution of some transcendental equation in the complex plane and it also corresponds to a pole of the weighted Green's function.

2. Main results

We begin with some definitions. Given H_0, a self-adjoint operator on $\mathcal{H} = L^2(\mathbb{R}^n)$, we assume that H_0 has a simple eigenvalue λ_0 with normalized eigenvector ψ_0:

$$H_0\psi_0 = \lambda_0\psi_0, \|\psi_0\| = 1 \tag{3}$$

Our interest is to describe the behavior of solutions of

$$i\frac{\partial \phi}{\partial t} = H\phi, \quad H := H_0 + \epsilon\, W^{(\epsilon)} \tag{4}$$

where ϵ is a small parameter, taken to be the size of the perturbation in an appropriate norm (cf. e.g. (8)), $\phi(0) = E_\Delta\phi_0$, where E_Δ is the spectral projection of H on the interval Δ and Δ is a small interval around λ_0. (Note that $W^{(\epsilon)}$ depends on ϵ in general, and may not even have a limit as $\epsilon \to 0$.) Furthermore, we will describe, in some cases, the analytic structure of $(H-z)^{-1}$ in a neighborhood of λ_0. W is a symmetric perturbation of H_0, such that H is self-adjoint with same domain as H_0.

For an operator A, $\|A\|$ denotes its norm as an operator from L^2 to itself. We interpret functions of a self-adjoint operator as being defined by the spectral theorem. In the special case where the operator is H_0, we omit the argument, i.e., $g(H_0) = g$.

For an open interval Δ, we denote an appropriate smoothed characteristic function of Δ by $g_\Delta(\lambda)$. In particular, we shall take typically $g_\Delta(\lambda)$ to be a nonnegative C^∞ function, which is equal to one on Δ and zero outside a neighborhood of Δ. The support of its derivative is furthermore chosen to be small compared to the size of Δ. We further require that $|g^{(n)}(\lambda)| \le c_n|\Delta|^{-n}, n \ge 1$.

P_0 denotes the projection on ψ_0, i.e., $P_0 f = (\psi_0, f)\psi_0$. P_{1b} denotes the spectral projection on $\mathcal{H}_{pp} \cap \{\psi_0\}^\perp$, the pure point spectral part of H_0 orthogonal to ψ_0. That is, P_{1b} projects onto the subspace of $\mathcal{H}$ spanned by all the eigenstates other than ψ_0. In our treatment, a central role is played by the subset of the spectrum of the operator H_0, $T^\sharp$ on which a sufficiently rapid local decay estimate holds. For a decay estimate to hold for $e^{-iH_0 t}$, one must certainly project out the bound states of H_0, but there may be other obstructions to rapid decay. In scattering theory these are called threshold energies. Examples of thresholds are: (i) points of stationary phase of a constant coefficient principal symbol for two body Hamiltonians and (ii) for N–body Hamiltonians, zero and eigenvalues of subsystems. We will not give a precise definition of thresholds. For us it is sufficient to say that away from thresholds the favorable local decay estimates for H_0 hold.

Let Δ_* be a union of intervals, disjoint from Δ, containing a neighborhood of infinity and all thresholds of H_0 except possibly those in a small neighborhood of λ_0. We then let

$$P_1 = P_{1b} + g_{\Delta_*}$$

where $g_{\Delta_*} = g_{\Delta_*}(H_0)$ is a smoothed characteristic function of the set Δ_*. We also define for $x \in \mathbb{R}^n$

$$\langle x\rangle^2 = 1 + |x|^2, \quad \overline{Q} = I - Q, \text{ and } \quad P_c^\sharp = I - P_0 - P_1 \tag{5}$$

Thus, $P_c^\sharp$ is a smoothed out spectral projection of the set $T^\sharp$ defined as

$$T^\sharp = \sigma(H_0) \setminus \{\text{eigenvalues, real neighborhoods of thresholds and infinity}\} \tag{6}$$

We expect $e^{-iH_0 t}$ to satisfy good local decay estimates on the range of $P_c^\sharp$; (see (**H4**) below).

2.1. Hypotheses on H_0. We assume H^η regularity for H_0. By this we mean that $(\psi, (H_0 - z)^{-1}\phi)$ is in the Sobolev space of order η, H^η, in the z variable for z near the relevant energy. Here ψ, ϕ are in the dense set $\{\phi \in L^2 : \langle x\rangle^\sigma \phi \in L^2\}$.

(**H1**) H_0 is a self-adjoint operator with dense domain $\mathcal{D}$, in $L^2(\mathbb{R}^n)$.

(**H2**) λ_0 is a simple embedded eigenvalue of H_0 with (normalized) eigenfunction ψ_0.

(**H3**) There is an open interval Δ containing λ_0 and no other eigenvalue of H_0.

(**H4**) *Local decay estimate*: Let $r > 1$. There exists $\sigma > 0$ such that if $\langle x\rangle^\sigma f \in L^2$ then

$$\|\langle x\rangle^{-\sigma} e^{-iH_0 t} P_c^\sharp f\|_2 \leq C\langle t\rangle^{-r} \|\langle x\rangle^\sigma f\|_2, \tag{7}$$

(**H5**) By appropriate choice of a real number c, the L^2 operator norm of $\langle x\rangle^\sigma (H_0 + c)^{-1}\langle x\rangle^{-\sigma}$ can be made sufficiently small.

Remarks:

(i) We have assumed that λ_0 is a simple eigenvalue to simplify the presentation. Our methods can be easily adapted to the case of multiple eigenvalues.

(ii) Note that Δ does not have to be small and that Δ_* can be chosen as necessary, depending on H_0.

(iii) In certain cases, the above local decay conditions can be proved even when λ_0 is a threshold; see [13].

(iv) Regarding the verification of the local decay hypothesis, one approach is to use techniques based on the Mourre estimate [14–16]. If Δ contains no threshold values, then quite generally, the bound (7) holds with r arbitrary and positive.
We now specify the conditions we require of the perturbation, W.

Conditions on W.

(**W1**) W is symmetric and $H = H_0 + W$ is self-adjoint on $\mathcal{D}$ and there exists $c \in \mathbb{R}$ (which can be used in (**H5**)), such that c lies in the resolvent sets of H_0 and H.

(**W2**) For the same σ as in (**H4**) and (**H5**) we have :

$$|||W||| := \|\langle x\rangle^{2\sigma} W g_\Delta(H_0)\| \\ + \|\langle x\rangle^{\sigma} W g_\Delta(H_0)\langle x\rangle^{\sigma}\| + \|\langle x\rangle^{\sigma} W (H_0+c)^{-1}\langle x\rangle^{-\sigma}\| < \infty$$

and

$$\|\langle x\rangle^{\sigma} W (H_0+c)^{-1}\langle x\rangle^{\sigma}\| < \infty \tag{8}$$

(**W3**) *Resonance condition–nonvanishing of the Fermi golden rule*:
For a suitable choice of λ (which will be made precise later)

$$\Gamma(\lambda,\epsilon) := \Gamma(\lambda) := \pi\,\epsilon^2 (W^{(\epsilon)}\psi_0, \delta(H_0-\lambda)(I-P_0)W^{(\epsilon)}\psi_0) \neq 0 \tag{9}$$

In most cases $\Gamma = \Gamma(\lambda_0)$. But in the case Γ is very small it turns out that the "correct" Γ will be

$$\Gamma(\lambda_0 + \delta)$$

with δ given in the proof of Proposition 12. See also Section 4.

The main results of this paper are summarized in the following theorem.

Theorem 1 *Let H_0 satisfy the conditions* **(H1)**...**(H5)** *and the perturbation satisfy the conditions* **(W1)...(W3)**. *Assume moreover that ϵ is sufficiently small and either:*
(i) H_0 has regularity as in §2.1 with $\eta > 1$
or
(ii) We have lower regularity $0 < \eta < 1$ supplemented by the conditions

$$\Gamma > C\epsilon^n, \quad n \geq 2$$

and $\eta > \frac{n-2}{n}$.

Then
a) $H = H_0 + \epsilon W$ has no eigenvalues in Δ.
b) The spectrum of H is purely absolutely continuous in Δ, and

$$\|\langle x\rangle^{-\sigma}e^{-iHt}g_\Delta(H)\Phi_0\|_2 \le C_\epsilon\langle t\rangle^{-1-\eta}\|\langle x\rangle^{\sigma}\Phi_0\|_2 \tag{10}$$

c) For $t \ge 0$ we have

$$e^{-iHt}g_\Delta(H)\Phi_0 = (I + A_W)\Big(e^{-i\omega_* t}a(0)\psi_0 + e^{-iH_0 t}\phi_d(0)\Big) + R(t) \tag{11}$$

where $A_W := K(I-K)^{-1} - I$ and K is an integral operator defined in (35) and

1. *if $\eta < 1$ and $\epsilon \to 0$ with $t\Gamma$ fixed we have $R(t) = O(\epsilon^2\Gamma^{\eta-1})$ while as $t\Gamma \to \infty$ we have $R(t) = O(\Gamma^{-1}t^{-\eta-1})$*
2. *for $\eta > 1$ we have $R(t) = O(\epsilon^2 t^{-\eta+1})$*
3. $$\|A_W\| \le C\epsilon|||W|||, \tag{12}$$

   $a(0)$ and $\phi_d(0)$ are determined by the initial data. The complex frequency ω_* is given by

   $$-i\omega_* = -is_0 - \Gamma$$

   where s_0 solves the equation

   $$s_0 + \omega + \epsilon^2\Im\{F(\epsilon, is_0)\} = 0 \tag{13}$$

   (see (47) and (49) below) and
4. $$\Gamma = \epsilon^2\Re\{F(\epsilon, is_0)\} \tag{14}$$

Remark: ω_* can be found by solving the transcendental equation (13) by either expansion or iteration if sufficient regularity is present (see also Proposition 12 and note following it and Lemma 18).

2.2. Sketch of the proof of the Theorem 1. The proof of Theorem 1 is given in Secs. 3 and 4. Sec. 3 prepares the ground for the proof, Subsec. 4.1 provides key definitions while Subsecs. 4.2 and 4.3 contain the proof of Theorem 1 (ii) and (i) respectively. As an intuitive guideline, the solution $\phi(t)$ of the time dependent problem is decomposed into the projection $a(t)\psi_0$ on the eigenfunction of H_0 and a remainder (see (18)). The remainder is estimated from the detailed knowledge of $a(t)$ (see (34) and (39).

Thus it is essential to control $a(t)$; once that is done, parts (a) and (b) follow from the Proposition 4; this $a(t)$ satisfies an integral equation, cf. (43). We chiefly use the Tauberian type duality between the large t behavior of $a(t)$ and the regularity properties of its Laplace transform, cf. Proposition 9 and also Eq. (55). Then, an essential ingredient in the proof of the estimate (11) is Proposition 15. When enough regularity is present, no lower bound on $\Gamma > 0$ is imposed; Proposition 16 and Proposition 17 are key ingredients here.

2.3. Further results.

Lemma 2 *Assuming the conditions of Theorem 1 with $\eta > 1$ then*

$$\omega_* = \lambda_0 + \epsilon(\psi_0, W\psi_0) + (\Lambda + i\Gamma) + o(\epsilon^2) \tag{15}$$

where

$$\Lambda = \epsilon^2(W\psi_0, P.V.(H_0 - \lambda_0)^{-1}W\psi_0) \tag{16}$$

$$\Gamma = \pi\epsilon^2(W\psi_0, \delta(H_0 - \lambda_0)(I - P_0)W\psi_0) \tag{17}$$

This follows from the proof of Proposition 12 and the Remarks below it.

3. Decomposition and isolation of resonant terms

We begin with the following decomposition of the solution of (4):

$$e^{-iHt}\phi_0 = \phi(t) = a(t)\psi_0 + \tilde{\phi}(t) \tag{18}$$

$$\Big(\psi_0, \tilde{\phi}(t)\Big) = 0, \quad -\infty < t < \infty \tag{19}$$

Substitution into (4) yields

$$i\partial_t\tilde{\phi} = H_0\phi + \epsilon W\tilde{\phi} - (i\partial_t a - \lambda_0 a)\psi_0 + a\epsilon W\psi_0 \tag{20}$$

Recall now that $I = P_0 + P_1 + P_c^\sharp$. Taking the inner product of (20) with ψ_0 gives the amplitude equation:

$$i\partial_t a = (\lambda_0 + (\psi_0, \epsilon W\psi_0)\,)a + (\psi_0, \epsilon W P_1\tilde{\phi}) + (\psi_0, \epsilon W\phi_d), \tag{21}$$

where

$$\phi_d := P_c^\sharp\tilde{\phi} \tag{22}$$

The following equation for ϕ_d is obtained by applying $P_c^\sharp$ to equation (20):

$$i\partial_t\phi_d = H_0\phi_d + P_c^\sharp\epsilon W(P_1\tilde{\phi} + \phi_d) + aP_c^\sharp\epsilon W\psi_0 \tag{23}$$

To derive a closed system for $\phi_d(t)$ and $a(t)$ we now propose to obtain an expression for $P_1\tilde{\phi}$, to be used in equations (21) and (23). Since $g_\Delta(H)\phi(\cdot,t) = \phi(\cdot,t)$ we find

$$(I - g_\Delta(H))\phi = (I - g_\Delta(H))\Big(a\psi_0 + P_1\tilde{\phi} + P_c^\sharp\tilde{\phi}\Big) = 0 \tag{24}$$

or

$$(I - g_\Delta(H) g_I(H_0)) P_1 \tilde{\phi} = -\overline{g}_\Delta(H)\Big(a\psi_0 + \phi_d\Big) \tag{25}$$

where $g_I(\lambda)$ is a smooth function which is identically equal to one on the support of $P_1(\lambda)$, and which has support disjoint from Δ. Therefore

$$P_1 \tilde{\phi} = -B \overline{g}_\Delta(H)(a\psi_0 + \phi_d), \tag{26}$$

where

$$B = (I - g_\Delta(H) g_I(H_0))^{-1}. \tag{27}$$

This computation is justified in Appendix B of [11]. The following was also shown there:

Proposition 3 ([11]) *For small ϵ, the operator B in (27) is a bounded operator on $\mathcal{H}$.*

From (26) we get

$$\phi(t) = a(t)\psi_0 + \phi_d + P_1\tilde{\phi} = \tilde{g}_\Delta(H)(a(t)\psi_0 + \phi_d(t)), \tag{28}$$

with

$$\tilde{g}_\Delta(H) := I - B\overline{g}_\Delta(H) = B g_\Delta(H)(I - g_I(H_0)). \tag{29}$$

see (5). Although $\tilde{g}_\Delta(H)$ is not really defined as a function of H, we indulge in this mild abuse of notation to emphasize its dependence on H. In fact, in some sense, $\tilde{g}_\Delta(H) \sim g_\Delta(H)$ to higher order in ϵ [11].

Substitution of (26) into (23) gives:

$$i\partial_t \phi_d = H_0 \phi_d + a P_c^\sharp \epsilon W \tilde{g}_\Delta(H)\psi_0 + P_c^\sharp \epsilon W \tilde{g}_\Delta(H)\phi_d \tag{30}$$

and

$$\begin{aligned} i\partial_t a &= \Big(\lambda_0 + (\psi_0, \epsilon W \tilde{g}_\Delta(H)\psi_0)\Big) a + (\psi_0, \epsilon W \tilde{g}_\Delta(H)\phi_d) \\ &= \omega a + (\omega_1 - \omega) a + (\psi_0, \epsilon W \tilde{g}_\Delta(H)\phi_d) \end{aligned} \tag{31}$$

where

$$\omega = \lambda_0 + (\psi_0, \epsilon W \psi_0) \tag{32}$$

$$\omega_1 = \lambda_0 + (\psi_0, \epsilon W \tilde{g}_\Delta(H)\psi_0) \tag{33}$$

We write (30) as an equivalent integral equation. We will later need the integral representation of the solution of (30)

$$\phi_d(t) = e^{-iH_0t}\phi_d(0) - i\int_0^t e^{-iH_0(t-s)}a(s)P_c^\sharp \epsilon W\tilde{g}_\Delta(H)\psi_0 ds \\ - i\int_0^t e^{-iH_0(t-s)}P_c^\sharp \epsilon W\tilde{g}_\Delta(H)\phi_d ds \quad (34)$$

This was also used to prove the following statement.

Proposition 4 ([11]) *Suppose* $|a(t)| \le a_\infty \langle t\rangle^{-1-\alpha}$ *and assume that* $\eta > 0$ *and* $\alpha \ge \eta$. *Then for some* $C > 0$ *we have*

$$\|\langle x\rangle^{-\sigma}\phi_d(t)\|_{L^2} \le C\langle t\rangle^{-1-\eta}\left(\|\langle x\rangle^{\sigma}\phi_d(0)\|_{L^2} + a_\infty|||W|||\right)$$

Note The proposition, as we mentioned, implies parts (a) and (b) of the main theorem, given the properties of $a(t)$ which will be shown in the sequel. The absolute continuity stated in the theorem follows from (10) with $\eta > 0$.

We define K as an operator acting on $C(\mathbb{R}^+, \mathcal{H})$, the space of continuous functions on $\mathbb{R}^+$ with values in $\mathcal{H}$ by

$$\Big(K\, f\Big)(t,x) = \int_0^t e^{-iH_0(t-s)}P_c^\sharp \epsilon W\tilde{g}_\Delta(H)f(s,x)ds \quad (35)$$

We introduce on $C(\mathbb{R}^+, \mathcal{H})$ the norm

$$\|f\|_\beta = \sup_{t\ge 0}\langle t\rangle^\beta \|f(\cdot,t)\|_{\mathcal{H}} \quad (36)$$

and define the operator norm

$$\|A\|_{\beta;\sigma} = \sup_{\|f\|_\beta \le 1}\|\langle x\rangle^{-\sigma}A\langle x\rangle^{\sigma}f\|_\beta \quad (37)$$

The above definitions directly imply the following.

Proposition 5 *If* ϵ *is small,* $0 \le \beta \le r$, $r > 1$ *and for some* $\beta_1 > 0$ *we have* $\|\langle x\rangle^{-\sigma}e^{-iH_0t}P_c^\sharp\langle x\rangle^{-\sigma}\| \le Ct^{-1-\beta_1}$, *then for* $0 \le \beta \le \beta_1$ *we have*

$$\|Kf\|_{\beta;\sigma} \le \epsilon\, C_{\beta;\sigma;r} \quad (38)$$

The proof uses the smallness of ϵ which in turn entails the boundedness of $\langle x\rangle^{-\sigma}\tilde{g}_\Delta(H)\langle x\rangle^{\sigma}$. Using the definition of K given above we see that $K(1-K)^{-1} = \sum_{n=1}^\infty K^n$ is also bounded. We can now rewrite the equations for ϕ_d as

$$\phi_d(t) = e^{-iH_0t}\phi_d(0) + K\big(a(t)\psi_0\big) + K\phi_d \\ = (I-K)^{-1}\left\{K\big(a(t)\psi_0\big) + e^{-iH_0t}\phi_d(0)\right\} \quad (39)$$

(recall that we defined $A_W = -I + (I-K)^{-1}K$) and therefore

$$i\partial_t a = \omega_1 a + \big(\psi_0, \epsilon W \tilde{g}_\Delta(H)(I-K)^{-1}K\big(a\psi_0\big)\big) + \\ \big(\psi_0, \epsilon W \tilde{g}_\Delta(H)(I-K)^{-1}e^{-iH_0 t}\phi_d(0)\big) \quad (40)$$

To complete the proof of Theorem 1 we need to estimate the large time behavior of $a(t)$ solving Eq. (40). Since the inhomogeneous term satisfies the required decay $O(t^{-1-\eta})$ by our assumptions on H_0 it is sufficient to study the associated homogeneous equation. Equivalently, we may choose the embedded eigenfunction as initial condition (that is $\phi_d(0) = 0$).

We now define two operators on L^∞ by

$$\tilde{j}(a) = \Big(v, \langle x\rangle^{-\sigma} K(a\psi_0)\Big); \quad \text{where } v = \langle x\rangle^{\sigma}\epsilon W \tilde{g}_\Delta(H)\psi_0 \quad (41)$$

and

$$j(a) = \Big(v, \langle x\rangle^{-\sigma}(I-K)^{-1}K(a\psi_0)\Big) \quad (42)$$

Proposition 6 *The operators $\tilde{j}$ and j are bounded from L^∞ into itself.*

The proposition follows from Proposition 5 with $\beta = 0$.
Remark. The equation for a can now be written in the equivalent integral form

$$a(t) = a(0)e^{-i\omega t} + e^{-i\omega t}\int_0^t e^{i\omega s} j(a)(s)ds := a(0)e^{-i\omega t} + J(a) \quad (43)$$

Definition 1 *Consider the spaces $L^\infty_{T;\nu}$ and L^∞_ν to be the spaces of functions on $[0,T]$ and $\mathbb{R}^+$ respectively, in the norm*

$$\|a\|_\nu = \sup_s |e^{-\nu s}a(s)| \quad (44)$$

Remark 7 *We note that for $T \in \mathbb{R}^+$, the norm on $L^\infty_{T;\nu}$ is equivalent to the usual norm on $L^\infty[0,T]$.*

Proposition 8 *For some constants c, C and $\tilde{c}$* independent of *T we have $\|ja\|_\nu \le c\nu^{-1}\epsilon^2\|a\|_\nu$, $\|Ja\|_\nu \le C\nu^{-2}\epsilon^2\|a\|_\nu$ and $\|\tilde{j}a\|_\nu \le \tilde{c}\nu^{-1}\epsilon^2\|a\|_\nu$, and thus j, J, and $\tilde{j}$ are defined on $L^\infty_{T;\nu}$ and L^∞_ν and their norms, in these spaces, are estimated by*

$$\|j\|_\nu \le c\nu^{-1}\epsilon^2; \quad \|\tilde{j}\|_\nu \le \tilde{c}\nu^{-1}\epsilon^2; \quad \|J\|_\nu \le C\nu^{-2}\epsilon^2 \quad (45)$$

Similar arguments as above lead to

Proposition 9 *The equation (40) has a unique solution in* $L^1_{loc}(\mathbb{R}^+)$*, and this solution belongs to* L^∞_ν *if* $\nu > \nu_0$ *with* ν_0 *sufficiently large. Thus, in the half-plane* $\Re(p) > \nu_0$ *the Laplace transform of a*

$$\hat{a} := \int_0^\infty e^{-pt} a(t) dt \tag{46}$$

exists and is analytic in p. Furthermore, for $\Re(p) > \nu_0$*, the Laplace transform of a satisfies*

$$ip\hat{a} = \omega\hat{a} + ia(0) - i\epsilon^2 F(\epsilon, p)\hat{a}(p) \tag{47}$$

where $F(\epsilon, p)$ *is defined by*

$$F(\epsilon, p) := \left(\psi_0, W\tilde{g}_\Delta(H)\left[\left(I + \frac{iI}{p + iH_0} P_c^\sharp W\tilde{g}_\Delta(H)\right)^{-1} \frac{-iI}{p + iH_0} P_c^\sharp \epsilon W\tilde{g}_\Delta(H)\right]\psi_0\right) + i(\omega_1 - \omega)\epsilon^{-2} \tag{48}$$

so

$$(ip - \omega + i\epsilon^2 F(\epsilon, p))\hat{a}(p) = ia(0) \tag{49}$$

Eq. (47) follows by taking the Laplace transform of (31).

Proof. By Proposition 8, and since $\|e^{-i\omega t}\|_\nu = 1$, for large ν the equation (43) is contractive in $L^\infty_{T;\nu}$ and has a unique solution there. It thus has a unique solution in L^1_{loc}, by Remark 7. Since by the same argument equation (43) is contractive in $L^\infty_{T;\nu}$ and since $L^\infty_\nu \subset L^1_{loc}$, the unique L^1_{loc} solution of (43) is in L^∞_ν as well. The rest is straightforward.

Remark 10 *Note that by construction (47) and (48) define F as a Laplace transform of a function.*

Our assumptions easily imply that if ϵ is small enough, then:

(a) $F(\epsilon, p)$ is analytic except for a cut along $i\Delta$. $F(\epsilon, p)$ is Hölder continuous of order $\eta > 0$ at the cut, i.e.

$$\lim_{\gamma \downarrow 0} F(\epsilon, i\tau \pm \gamma) \in H^\eta$$

the space of Hölder continuous functions of order η.

(b) $|F(\epsilon, p)| \le C|p|^{-1}$ for some $C > 0$ as $|p| \to \infty$.

To see it we write

$$B = B_1 B_2 \langle x\rangle^{-\sigma}; \quad B_1 := \frac{I}{p + iH_0} P_c^\sharp \langle x\rangle^{-\sigma}; \quad B_2 := \epsilon\langle x\rangle^\sigma W\tilde{g}_\Delta(H)\langle x\rangle^\sigma \tag{50}$$

Noting that $P_c^\sharp$ projects on the interval Δ it is clear by the spectral theorem that $\langle x\rangle^{-\sigma}B$ is analytic in p on $\mathcal{D} := \mathbb{C} \setminus (i\Delta)$. By the assumption on the decay rate and the Laplace transform of eq. (7) we have that

$$B_3(p) := \langle x\rangle^{-\sigma}\frac{I}{p+iH_0}P_c^\sharp\langle x\rangle^{-\sigma} \tag{51}$$

is uniformly Hölder continuous, of order η, as $p \to i\Delta$. For $p_0 \in i\Delta$, the two sided limits $\lim_{a\downarrow 0} B_3(p_0 \pm a) = B_3^\pm$ will of course differ, in general. A natural closed domain of definition of B_3 is $\mathcal{D}$ together with the two sides of the cut, $\overline{\mathcal{D}} := \mathcal{D} \cup \partial\mathcal{D}^+ \cup \partial\mathcal{D}^-$. We then write

$$\|B_3\| \le C_1(p) \tag{52}$$

where we note that C_1 can be chosen so that:

Remark 11 *$C_1(p) > 0$ is uniformly bounded for $p \in \overline{\mathcal{D}}$ and $C_1(p) = O(p^{-1})$ for large p.*

Hence for some C_2 we have uniformly in p (choosing ϵ small enough),

$$\|\langle x\rangle^{-\sigma}(B_1B_2)^n\| \le C_2^n\epsilon^n \tag{53}$$

and therefore the operator

$$\epsilon W\tilde{g}_\Delta(H)\left[\left(I - \frac{I}{p+iH_0}P_c^\sharp\epsilon W\tilde{g}_\Delta(H)\right)^{-1}\frac{I}{p+iH_0}P_c^\sharp\epsilon W\tilde{g}_\Delta(H)\right] \tag{54}$$

is analytic in $\mathcal{D}$ and is in $H^\eta(\overline{\mathcal{D}})$.

4. General Case

4.1. Definition of Γ. We have from Proposition 9, eq. (47) that

$$\hat{a}(p) = \frac{ia(0)}{ip - \omega + i\epsilon^2F(\epsilon,p)} \tag{55}$$

We are most interested in the behavior of $\hat{a}$ for $p = is$, $s \in \mathbb{R}$. Γ will be defined in terms of the approximate zeros of the denominator in (55). Let $F =: F_1 + iF_2$.

Proposition 12 *For ϵ small enough, the equation $s + \omega + \epsilon^2F_2(\epsilon, is) = 0$ has at least one root s_0, and $s_0 = -\omega + O(\epsilon^2)$. If $\eta \ge 1$, then for small enough ϵ the solution is unique. If $\eta < 1$ then two solutions s_1 and s_2 differ by at most $O(\epsilon^{\frac{2}{1-\eta}})$.*

Proof. We write $s = -\omega + \delta$ and get for δ an equation of the form $\delta = \epsilon^2 G(\delta)$ where $G(x) = -F_2(\epsilon, ix - i\omega)$, and $G(x) \in H^\eta$. The existence of a solution for small ϵ is an immediate consequence of continuity and the fact that $\delta - \epsilon^2 G(\delta)$ changes sign in an interval of size $\epsilon^2\|G\|_\infty$. If $\eta \geq 1$ we note that the equation $\delta = \epsilon^2 G(\delta)$ is contractive for small ϵ and thus has a unique root. If instead $0 < \eta < 1$ we have, if δ_1, δ_2 are two roots, then for some $K > 0$ independent of ϵ, $|\delta_1 - \delta_2| = \epsilon^2|G(\delta_1) - G(\delta_2)| \leq \epsilon^2 K|\delta_1 - \delta_2|^\eta$ whence the conclusion. $\square$

Remark 13 *Note that s_0 are not, in general, poles of (55) since we only solve for the real part equal to zero.*

Assumption If $\eta < 1$ then we assume that $\epsilon^2 F_1(\epsilon, -i\omega) \gg \epsilon^{\frac{2}{1-\eta}}$ for small ϵ. When $\eta > 1$ this restriction will not be needed, cf. § 4.3.

Definition We choose one solution $s_0 = -\omega + \delta$ and let Γ be defined by (14).
Note. In the case $\eta < 1$ the choice of s_0 yields, by the previous assumption a (possible) arbitrariness in the definition of Γ of order $O(\epsilon^{\frac{2}{1-\eta}}) = o(\Gamma)$.
Remarks on the verifiability of condition $\Gamma > 0$. As it is generally difficult to check the positivity of Γ itself but relatively easier to find Γ_0, we will look at various scenarios, which are motivated by concrete examples, in which the condition of positivity reduces to a condition on $F(\epsilon, -i\omega)$.

Let

$$\Gamma_0 = \epsilon^2 F_1(\epsilon, -i\omega); \quad \gamma_0 = \epsilon^2 F_2(\epsilon, -i\omega)$$

where we see that Γ_0 and γ_0 are $O(\epsilon^2)$. The equation for δ reads

$$\delta = -\epsilon^2[F_2(\epsilon, -i\omega + i\delta) - F_2(\epsilon, -i\omega)] - \gamma_0 = \epsilon^2 H(\delta) - \gamma_0$$

where $H(0) = 0$. We write $\delta = -\gamma_0 + \zeta$ and get

$$\zeta = \epsilon^2 H(-\gamma_0 + \zeta)$$

and the definition of Γ becomes

$$\Gamma = \epsilon^2 F_1(\epsilon, -i\omega - i\gamma_0 + i\zeta)$$

Proposition 14 *(i) If H_0 satisfies the conditions of Theorem 1 with $\eta > 1$ and $\gamma_0 = o(\epsilon^{-2}\Gamma_0)$, then as $\epsilon \to 0$,*

$$\Gamma = \Gamma_0 + o(\Gamma_0) \tag{56}$$

and in particular Γ is positive for $\Gamma_0 > 0$.

(ii) Assume that $\eta < 1$, $\gamma_0 = o(\epsilon^{-2}\Gamma_0^{1/\eta})$ and $\Gamma_0 \gg \epsilon^{\frac{2}{1-\eta}}$ as $\epsilon \to 0$. Then again (56) holds.

Proof. (i) Since $\zeta = O(\epsilon^2\gamma_0) + O(\epsilon^2\zeta)$ we get $\zeta = O(\epsilon^2\gamma_0)$, implying that

$$\Gamma = \epsilon^2 F_1\left[\epsilon, -i\omega - i\gamma_0(1 + o(1))\right] = \Gamma_0 + O(\epsilon^2\gamma_0) = \Gamma_0 + o(\Gamma_0)$$

(ii) We have

$$\zeta = O(\epsilon^2 \gamma_0^\eta) + O(\epsilon^2 \zeta^\eta) \tag{57}$$

If $\zeta \le \text{const.}\gamma_0$ as $\epsilon \to 0$, then the proof is as in part (i). If on the contrary, for some large constant C we have $\zeta > C\gamma_0$ then by (57) we have $\zeta < \text{const.}\epsilon^2\zeta^\eta$ so that $\zeta = O(\epsilon^{2/(1-\eta)})$ and $\epsilon^2\zeta^\eta = O(\epsilon^{2/(1-\eta)}) = o(\Gamma_0)$. But then

$$\Gamma = \epsilon^2 F_1(\epsilon, -i\omega) + O(\epsilon^2\gamma_0^\eta) + O(\epsilon^2\zeta^\eta) = \Gamma_0 + o(\Gamma_0)$$

4.2. Exponential decay. We now let $p = is_0 + v$. The intermediate time and long time behavior of $a(t)$ are given by the following Proposition

Proposition 15 *For $t\Gamma = O(1)$ (note that Γ in general depends on ϵ), as $\epsilon \to 0$ we have*
(i)

$$a(t) = e^{-is_0 t} e^{-\Gamma t} + O(\epsilon^2 \Gamma^{\eta-1}) \tag{58}$$

(ii) As $t \to \infty$ we have

$$a(t) = O(\Gamma^{-1} t^{-\eta-1}) \tag{59}$$

Proof. (i) Note first that, taking $\Re(v) > 0$ and writing F as a Laplace transform, cf. Remark 10

$$F(\epsilon, -is_0 + v) = \int_0^\infty e^{-is_0 t - vt} f(t)dt$$

we have by our assumptions that

$$\begin{aligned} F(\epsilon, -is_0 + v) &= \int_0^\infty e^{-vt} \left(\int_0^t e^{-is_0 u} f(u)du \right)' \\ = v\int_0^\infty e^{-vt} \int_0^t e^{-is_0 u} f(u)du &= v\int_0^\infty e^{-vt} \left(\int_0^\infty - \int_t^\infty \right) e^{-is_0 u} f(u)du \\ &= \int_0^\infty e^{-is_0 u} f(u)du - v\int_0^\infty e^{-vt} \int_t^\infty e^{-is_0 u} f(u)du \\ &= F(\epsilon, -is_0) - vL[g](v) \end{aligned} \tag{60}$$

where we denoted $g(v) = \int_t^\infty e^{-is_0 u} f(u)du$ and $L[g]$ is its Laplace transform. Now define

$$h(v) = vL[g](v) \tag{61}$$

We have, by the formula for the inverse Laplace transform

$$2\pi i a(t) = e^{-is_0 t} \int_{-i\infty}^{i\infty} \frac{e^{vt}}{v + \Gamma + \epsilon^2 h(v)} dv \tag{62}$$

where by construction we have $h \in H^\eta$, h is analytic in $\mathbb{C} \setminus i\Delta$ and $h(0) = 0$. We write

$$\begin{aligned}\int_{-i\infty}^{i\infty} \frac{e^{vt}}{v + \Gamma + \epsilon^2 h(v)} dv &= \int_{-i\infty}^{i\infty} \frac{e^{vt}}{(v+\Gamma)\,(1 + \epsilon^2 h(v+\Gamma)^{-1})} dv \\ &= \int_{-i\infty}^{i\infty} \frac{e^{vt} dv}{v + \Gamma} - \epsilon^2 \int_{-i\infty}^{i\infty} \frac{1}{v+\Gamma} \frac{h(v+\Gamma)^{-1}}{1 + \epsilon^2 h(v+\Gamma)^{-1}} e^{vt} dv \end{aligned} \tag{63}$$

We first need to estimate $L^{-1}\left[h(v+\Gamma)^{-1}\right]$ (the transformation is well defined, since the function is just $(v+\Gamma)^{-1}(F(\epsilon, -is_0 + v) - F(\epsilon, -is_0)))$. We need to write

$$vL[g](v) =: (v+\Gamma)L[g_1](v) \quad \text{or} \quad L[g_1] = \left(1 - \frac{\Gamma}{v+\Gamma}\right) L[g] \tag{64}$$

which defines the function g_1:

$$g_1 = g - \Gamma e^{-\Gamma t} \int_0^t e^{\Gamma s} g(s) ds \tag{65}$$

Since $|g(t)| < \text{Const.}t^{-\eta}$ we have

$$|g_1(t)| \le \text{Const.}t^{-\eta} + e^{-\Gamma t} \int_0^{\Gamma t} e^u \left(\frac{u}{\Gamma}\right)^{-\eta} du \le \text{Const.}t^{-\eta} \tag{66}$$

A similar inequality holds for

$$Q := L^{-1}\left[\frac{\frac{h}{v+\Gamma}}{1 + \frac{\epsilon^2 h}{v+\Gamma}}\right] \tag{67}$$

Indeed, we have

$$Q = -L^{-1}\left[\frac{h}{v+\Gamma}\right] + \epsilon^2 L^{-1}\left[\frac{h}{v+\Gamma}\right] * Q \tag{68}$$

It is easy to check that for $t \le r\Gamma^{-1}$ and small enough ϵ this equation is contractive in the norm $\|Q\| = \sup_{s \le t} \langle s \rangle^\eta |Q(s)|$.

But now, for constants independent of ϵ,

$$\epsilon^2 L^{-1}\left[\frac{1}{v+\Gamma}\right] * Q \leq \text{Const.} e^{-\Gamma s}\int_0^t e^{\Gamma s} s^{-\eta} ds$$
$$= \epsilon^2 \text{Const.} e^{-\Gamma s}\Gamma^{-1}\int_0^{\Gamma t} e^u \left(\frac{u}{\Gamma}\right)^{-\eta} du \leq \text{Const.}\frac{\epsilon^2}{\Gamma^{1-\eta}} \quad (69)$$

(ii) We now use (60) and (61) to write

$$\frac{h}{v+\Gamma} = \frac{F(\epsilon, -is_0+v)}{v+\Gamma} - \frac{F(\epsilon, -is_0)}{v+\Gamma}$$

and get

$$H_1 := L^{-1}\left[\frac{h}{v+\Gamma}\right] = e^{-\Gamma t}\int_0^t e^{\Gamma s} f(s) ds + const e^{-\Gamma t}$$

and thus, proceeding as in the proof of (i) we get for some $C > 0$ $|H_1| \leq C\Gamma^{-1}\langle t\rangle^{-\eta-1}$. To evaluate $a(t)$ for large t we resort again to Q as defined in (67) which satisfies (68). This time we note that the equation is contractive in the norm $\sup_{s\geq 0}|\langle s\rangle^{1+\eta}\cdot|$ when ϵ is small enough. □

Using (59), Proposition 4 and (28) imply local decay and therefore χ cannot be an eigenfunction which implies (i). Since the local decay rate is integrable (ii) follows [24]. Part c) follows from (58), (39) and (28) while (12) follows from (39) and the smallness of K.

4.3. Proof of Theorem 1 in case (i) of regularity $\eta > 1$. In this case we obtain better estimates. We write

$$G(v) = L^{-1}[g](v) \quad (70)$$

and (62) becomes

$$a(t) = e^{-is_0 t}\int_{-i\infty}^{i\infty}\frac{e^{vt}}{v+\Gamma+\epsilon^2 vG(v)}dv \quad (71)$$

Now

$$L^{-1}\left[(v+\Gamma+\epsilon^2 vG(v))^{-1}\right]$$
$$= L^{-1}\left[\frac{1}{v+\Gamma}\right] - \epsilon^2 L^{-1}\left[\frac{1}{v+\Gamma}\right] * L^{-1}\left[\frac{\frac{v}{v+\Gamma}G(v)}{1+\epsilon^2\frac{v}{v+\Gamma}G(v)}\right] \quad (72)$$

Proposition 16 *Let*

$$H_2(t) := L^{-1}\left[\frac{\frac{v}{v+\Gamma}G(v)}{1+\epsilon^2\frac{v}{v+\Gamma}G(v)}\right]$$

We have

$$|H_2| \le Const.\langle t\rangle^{-\eta}; \quad \int_0^\infty H_2(t)dt = 0 \tag{73}$$

Proof. Consider first the function

$$h_3 := v(v+\Gamma)^{-1}G(v) = G(v) - \Gamma(v+\Gamma)^{-1}G(v)$$

we see that (cf. (70) and (60))

$$H_3 := L^{-1}h_3 = \int_t^\infty e^{-is_0u}f(u)du - \Gamma\, e^{-\Gamma t}\int_0^t e^{\Gamma s}\int_s^\infty e^{-is_0u}f(u)duds \tag{74}$$

and thus, for some positive constants C_i,

$$|H_3| \le \text{Const.}t^{-\eta} + \text{Const.}e^{-\Gamma t}\int_0^{\Gamma t} e^v\Gamma^{-\eta}\langle v\rangle^{-\eta}dv \tag{75}$$

and thus, since $h_3(0) = 0$ we have

$$|H_3| \le \text{Const.}\langle t\rangle^{-\eta}; \quad \int_0^\infty H_3(t)dt = 0$$

Note now that the function

$$\frac{v}{v+\Gamma}G(v)\left(1+\epsilon^2\frac{v}{v+\Gamma}G(v)\right)^{-1}$$

vanishes for $v = 0$. Note furthermore that

$$H_2 = H_3 - \epsilon^2 H_3 * H_2$$

It is easy to check that this integral equation is contractive in the norm $\|H\| = \sup_{s\le t}|\langle s\rangle^\eta H(s)|$ for small enough ϵ; the proof of the proposition is complete.

Proposition 17

$$L^{-1}\left[(v+\Gamma+\epsilon^2G(v))^{-1}\right] = e^{-\Gamma t} + \Delta(t)$$

where for some constant C independent of ϵ, t, Γ we have

$$|\Delta| \le C\epsilon^2\langle t\rangle^{-\eta+1}$$

Proof. We have, by (72)

$$\Delta(t) = \epsilon^2 e^{-\Gamma t} \int_0^t e^{\Gamma s} \left(\int_s^\infty H_2(u)du \right)' ds$$
$$= \epsilon^2 \int_t^\infty H_2(s)ds - \Gamma e^{-\Gamma t} \int_0^t e^{\Gamma s} \int_s^\infty H_2(u)du \quad (76)$$

The estimate of the last term is done as in (75). □

Theorem 1 part (c) in case (i) follows.

5. Analytic case

Suppose that the function $F(p, \epsilon)$ has analytic continuation in a neighborhood of the relevant energy $-i\omega \neq 0$; in this case we can prove stronger results. In many cases one can show the analyticity of F if the resolvent, properly weighted, has analytic continuation.

Lemma 18 *Assume that for some ω and some neighborhood $\mathcal{D}$ of ω, $E(\epsilon, p)$ is a function with the following properties:*

(i) $E \in H^\eta(\overline{\mathcal{D}})$ and E is analytic in $\mathcal{D}$ (this allows for branch-points on the boundary of the domain, a more general setting that meromorphicity).

(ii) $|E(\epsilon, p)| \leq C\epsilon^2$ for some C.

(iii) $\lim_{a\downarrow 0} \Re E(\epsilon, -i\omega - a) = -\Gamma_0 < 0$.

If (a) $\eta > 1$, $E(\epsilon, -i\omega) = o(\Gamma_0/\epsilon^2)$ or (b) $\eta < 1$ and $E(\epsilon, -i\omega) = O(\Gamma_0)$ and ϵ is small enough, then the function

$$G_1(\epsilon, p) = p + i\omega + E(\epsilon, p)$$

has a unique zero $p = p_z$ in $\overline{\mathcal{D}}$ and furthermore $\Re(p_z) < 0$. In fact,

$$\Re(p_z) + \Gamma_0 = o(\Gamma_0) \quad (77)$$

Remark If the condition that for $\eta > 1$, $E(\epsilon, -i\omega) = o(\epsilon^{-2}\Gamma_0)$ is not satisfied, then we can replace $-i\omega$ by $-i\omega - is_0$ and the uniqueness of the complex zero will still be true.

Proof. We have

$$G_1(\epsilon, p_z) = 0 = p_z + i\omega + E(\epsilon, -i\omega) + [E(\epsilon, p_z) - E(\epsilon, -i\omega)]$$

or, letting $p = -i\omega + \zeta$, $\zeta_z := p_z + i\omega$, $\epsilon^2\phi(\epsilon, \zeta) := E(\epsilon, p) - E(\epsilon, -i\omega)$,

$$\zeta_z = -E(\epsilon, -i\omega) - \epsilon^2\phi(\epsilon, \zeta_z)$$

Consider a square centered at $E(\epsilon, -i\omega)$ with side $2|\Re(E(\epsilon, -i\omega))| = 2\Gamma_0$. For both cases (a) and (b) for η considered in part (iii) of the lemma, note that in our assumptions and by the choice of the square we have

$$\left|\frac{\epsilon^2\phi(\zeta,\epsilon)}{\zeta+E(\epsilon,-i\omega)}\right| \to 0 \quad (\text{as } \epsilon \to 0) \tag{78}$$

(on all sides of the square). In case (a) on the boundary of the rectangle we have by construction of the rectangle, $|\zeta + E(\epsilon, -i\omega)| \geq \Gamma_0$. Also by construction, on the sides of the rectangle we have $|\zeta| \leq \Gamma_0$. Still by assumption, $\phi(\epsilon, \zeta) \leq C\zeta = o(\epsilon^{-2}\Gamma_0)$ and the ratio in (78) is $o(1)$. In case (b), we have

$$\epsilon^2\phi(\epsilon,\zeta) = O(\epsilon^2\zeta^\eta) = O(\epsilon^2\Gamma_0^\eta) = o(\Gamma_0)$$

Thus, on the boundary of the square, the variation of the argument of the functions $\zeta + E(\epsilon, -i\omega) + \epsilon^2\phi(\zeta)$ and that of $\zeta + E(\epsilon, -i\omega)$ differ by at most $o(1)$ and thus have to agree exactly (being integer multiples of $2\pi i$); thus $\zeta + E(\epsilon, -i\omega) + \epsilon^2\phi(\zeta)$ has exactly one root in the square. The same argument shows that $p + i\omega + E(\epsilon, p)$ has no root in any other region in its analyticity domain except in the square constructed in the beginning of the proof. □

Theorem 19 *Assume the conditions (H) and (W) as before, and furthermore that the function $F(\epsilon, p)$ has analytic continuation in a neighborhood of $-i\omega$; with an appropriate choice of the cutoff function $E_\Delta(H_0)$, we have that $\chi(H-z)^{-1}\chi$ has a unique pole away from the real axis, near $-i\omega$, corresponding to a resonance with imaginary part near Γ, with appropriate choice of weights χ.*

Proof. First we note that by taking the Laplace transform of (28) and (34) and solving for the resolvent of H we get that

$$\chi(H-z)^{-1}\chi = A(z)\hat{a}(z)\psi_0 + B(z)$$

with $A(z)$ and $B(z)$ analytic in $\mathcal{D}$ by our assumptions (H) and (W), and the assumed analyticity of $F(\epsilon, p)$, $ip := z$. Hence the existence and uniqueness of the pole of $\chi(H-z)^{-1}\chi$ follows from Lemma 18, with $\epsilon^2 F(\epsilon, p) = E(\epsilon, p)$.

As a consequence we obtain the following result.

Proposition 20 *With an appropriate exponential cutoff function, the remainder term decays as a stretched exponential times an asymptotic series.*

Sketch of proof. We need the large t behavior of $a(t)$ which is the Inverse Laplace transform of $G(p) := (p + i\omega + i\epsilon^2 F(\epsilon, p))^{-1}$ and to this end we write

$$G(p) = (p + i\omega_*)^{-1} - i\epsilon^2(p + i\omega_*)^{-1}F_*(\epsilon, p)G(p) \tag{79}$$

where $F_*(\epsilon, p) := F(\epsilon, p) - (\omega_* - \omega)/\epsilon^2$ and ω_* is the unique pole of $G(p)$ found in the previous theorem. Taking inverse Laplace transform of (79) we get an integral equation for $G(t)$, and direct calculations show that $\tilde{F} \sim e^{-\sqrt{t}+i\theta t}\sum a_k t^{-k/4}$ implies $G(t) \sim e^{-i\omega_* t} + O(\epsilon^2)e^{-\sqrt{t}+i\theta t}\sum b_k t^{-k/4}$. To find the asymptotic behavior of $\tilde{F}(t)$ we derive an integral equation by taking the inverse Laplace transform of (48) and the same integral equation arguments as above reduce the asymptotic study of $\tilde{F}$ to that of the following expression for any $u \in L^2$:

$$(u, Be^{-iH_0t}P_c^\sharp B\psi_0) = \int \widetilde{Bu^*}e^{-i\lambda t}g_\Delta\widetilde{B\psi_0}d\mu_{a.c.}(\lambda) := \int \xi(\lambda)e^{-i\lambda t}g_\Delta(\lambda)d\lambda$$

where $B = W\tilde{g}_\Delta(H)$ and $\tilde{\phi}$ is the spectral representation of ϕ associated to H_0. By assumption $B(H_0 - z)^{-1}B$ is analytic in $z \in \mathcal{D}$ hence $\int(\widetilde{Bu^*})(\lambda)(\lambda - z)^{-1}(\widetilde{Bv})(\lambda)f(\lambda)d\lambda$ is analytic for any $v \in L^2$, where $f(\lambda) = d\mu_{a.c.}/d\lambda$; therefore so is its Hilbert transform $\widetilde{Bu^*}\widetilde{Bv}f$ and thus ξ is also analytic. Choosing $g_\Delta(\lambda) = \exp(-(\lambda - a)^{-1} + (\lambda - b)^{-1})$ the asymptotic expansion of $\tilde{F}$ follows from that of the integral $\int_a^b e^{-\frac{1}{\lambda-a}+\frac{1}{\lambda-b}-it\lambda}\xi(\lambda)d\lambda$. $\square$

5.1. Example. Suppose

$$H_0 = \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta + x^2 \end{pmatrix} := -\Delta \oplus (-\Delta + x^2)$$

on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. Assume

$$W = \begin{pmatrix} 0 & \tilde{W} \\ \tilde{W} & 0 \end{pmatrix}$$

with $\tilde{W} = \tilde{W}(x)$ sufficiently regular and exponentially localized. Then, the spectrum of H_0 has embedded eigenvalues corresponding to the spectrum of $-\Delta+x^2$, with Gaussian localized and smooth eigenfunctions. Since the projection $I - P_0$ in the definition of $P_c^\sharp$ eliminates the $-\Delta + x^2$ part in any interval Δ containing an eigenvalue of $-\Delta + x^2$, it is left to verify the conditions of the theorem for H_0 replaced by $-\Delta$. Since

$$e^{-\alpha\langle x\rangle}(-\Delta - z)^{-1}e^{-\alpha\langle x\rangle} \tag{80}$$

has analytic continuation through the cut $(0, \infty)$ and is an analytic function away from $z = 0$, we can now choose an interval $\Delta = [a, b]$ around each eigenvalue E_n of $-\Delta + x^2$, avoiding zero, and let

$$E_\Delta(\lambda) = e^{-(\lambda-a)^{-1}}e^{(\lambda-b)^{-1}}$$

a function analytic in $\mathbb{C}$ except $z = a$ and b.

5.2. Remarks on applications. The examples covered by the above approach include those discussed in [11] as well as the many cases where analytic continuation has been established, see e.g. [21]. Furthermore, following results of [21] it follows that under favorable assumptions on $V(x)$, $-\Delta + V(x)$ has no zero energy bound states in three or more dimensions extending the results of [11], where it was proved for 5 or more dimensions.

It is worth mentioning that the possible presence of thresholds inside Δ makes it necessary to allow for $\eta < \infty$, and that in the case where there are finitely many thresholds inside Δ of known structure, sharper results may be obtained.

Other applications of our methods involve numerical reconstruction of resonances from time dependent solutions data, in cases Borel summability is ensured. This and other implications will be discussed elsewhere.

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