On the Spectral properties of Hamiltonians without conservation of the particle number

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Abstract

We consider quantum systems with variable but finite number of particles. For such systems we develop geometric and commutator techniques. We use these techniques to find the location of the spectrum, to prove absence of singular continuous spectrum and identify accumulation points of the discrete spectrum. The fact that the total number of particles is bounded allows us to give relatively elementary proofs of these basic results for an important class of many-body systems with non-conserved number of particles.

I Introduction

This paper is a contribution toward a geometrical theory of quantum systems with variable number of particles. Such systems occur naturally in quantum field theory, condensed matter physics and the theory of chemical reactions. Though often, as in cases involving photons (see e.g. [1]), the number of particles can take arbitrary large values, in other cases such as scattering of spin waves on defects, scattering of massive particles and chemical reactions, there are only few participants at any given time, though their number can change. It is the situations of the second kind that are addressed in this paper. Having the limitation on the total number of particles involved allows us to apply more sophisticated geometrical and positive commutative techniques than is usually the case (see e.g. [1]), while keeping the proof rather simple and obtaining stronger results. In this paper we obtain some basic results for systems described above by developing for them the method of conjugate operators.

The conjugate operator method consists in obtaining diverse spectral properties of a self-adjoint operator H in the Hilbert $\mathcal{H}$ assuming the existence of a self-adjoint operator A such that the commutator $[iH, A]$ has a definite sign when localized in a small spectral interval of H. More precisely, if τ is a closed subset of $\mathbf{R}$ then A is called locally conjugate for H on $\mathbf{R} \setminus \tau$ if and only if for every $\lambda \in \mathbf{R} \setminus \tau$ there is $c > 0$ and a compact operator K such that the Mourre estimate

$$E_{[\lambda-c,\lambda+c]}(H)[iH,A]E_{[\lambda-c,\lambda+c]}(H) \geq cE_{[\lambda-c,\lambda+c]}(H) + K \tag{1.1}$$

holds, where $E_Z(H)$ denotes the spectral projector of H on a Borel set $Z \subset \mathbf{R}$ and the intersection of domains $D(H) \cap D(A)$ is assumed dense in $\mathcal{H}$ allowing to understand the left hand side as a quadratic form.

The method has appeared to be a remarkably fruitful tool in many recent investigations in spectral and scattering theory of many-body Hamiltonians - cf. [2], [3] and [4] see also [5, 6]. If X is a finite-dimensional euclidean space and H_X is the Schrödinger operator in $L^2(X)$ of the form

$$H_X = -\Delta_X + V_X, \tag{1.2}$$

where Δ_X is the Beltrami-Laplace operator on X and V_X has a many-body structure with interaction potentials satisfying some regularity and decay hypotheses, then there exists a closed, countable set $\tau(H_X) \subset \mathbf{R}$ [called the threshold set of H_X] such that the dilation generator

$$A_X = x \cdot D_x + D_x \cdot x \tag{1.3}$$

is locally conjugate for H_X in $\mathbf{R} \backslash \tau(H_X)$. Moreover the eigenvalues of H_X may accumulate (with multiplicities) only at $\tau(H_X)$ and the singular continuous spectrum of H_X is empty. It is also useful to adopt the convention that $H_X = 0$ in $L^2(X) = \mathbf{C}$ in the case $X = \{0\}$.

The aim of this paper is to obtain similar results for a class of Hamiltonians acting in the Hilbert space

$$\mathcal{H} = \bigoplus_{1 \le n \le N} L^2(X(n)), \tag{1.4}$$

where $\mathbf{X} = \{X(n)\}_{1 \le n \le N}$ is a family of configuration spaces which are finite dimensional euclidean spaces. The Hamiltonians we consider,

$$\mathbf{H_X} = \mathbf{H_X}^{diag} + \mathbf{W_X}, \tag{1.5}$$

are perturbations of the diagonal operator formed by many-body Schrödinger operators in $L^2(X(n))$,

$$\mathbf{H_X}^{diag} = \bigoplus_{1 \le n \le N} H_{X(n)} = \bigoplus_{1 \le n \le N} (-\Delta_{X(n)} + V_{X(n)}), \tag{1.6}$$

Such hamiltonians describe a quantum system of at most N particles and the interaction terms can change the number of particles between 1 and N. Hence $\mathbf{H_X}$ acts on the Fock subspace $\mathcal{H}$. In the second-quantized formulation the above interaction can contain any power of creation and annihilation operators leaving the above Fock subspace invariant:

$$\mathbf{W_X} = P_N \mathbf{W_X} P_N$$

where P_N is the projection of the Fock subspace of at most N particles. The locally conjugate operator for $\mathbf{H_X}$ is the diagonal operator

$$\mathbf{A_X} = \bigoplus_{1 \le n \le N} A_{X(n)}, \tag{1.7}$$

where $A_{X(n)}$ are dilation generators in $X(n)$ (defined in (1.3)).

The model is motivated by field theoretical and solid state systems, where the particle number is not conserved. This type of problems has recently attracted a great attention, see e.g. [7, 1, 8, 9, 10]. In [9] the two state atom coupled to a massive free field was studied, spectral and scattering theory was developed using conjugate operators and other N-body techniques.

Previously [7] considered a truncated model of the two state atom where like in this work, the total number of particles is at most $N < \infty$.

The work of [9] was then generalized to a general molecule in a bound state, replacing the two-state atom in [8].

The works of [1, 10] deal with the massless photon/boson field interacting with a bound atom.Papers [1,8,9] use the dilation as conjugate operator;[7, 10] use dilations modified by terms depending on the interaction.

In this work we generalize some of the above models in that we allow the "boson" system to be interacting. We also allow the interaction between the atomic and boson field to be of general nature (In all of the above models the interaction is linear in the boson fields).

Our construction of the conjugate operator is different from the above works as it is more geometric in spirit. Our construction uses the geometry of N-body systems as the guiding principle; in particular we can modify the dilation generator using Graf's construction to allow local singularities in the interactions [11, 12]. Furthermore our interactions are of more general nature, and so we can cover the case of pair annihilation in some cases, e.g. the positronium system, when a bound electron and positron pair are annihilated when coupled to a field.

If X, X' are euclidean spaces then $X' \subseteq X$ denotes the inclusion together with the fact that the euclidean structure of X' is inherited from the structure of X and $X' \subset X$ denotes the strict inclusion, i.e. $X' \subset X \iff (X' \subseteq X$ and $X' \neq X)$.

Further on we assume that euclidean spaces $X(n)$ satisfy

$$X(N) \subset ... \subset X(2) \subset X(1). \tag{1.8}$$

The configuration space of the system, $X(1)$ is assumed to be an Euclidean space, corresponds to N particles, of various types or masses; e.g:

$$X(1) = \{x = (x_1, \cdots x_N) | x_i \in \mathbf{R}^m, i = 1, \cdots N\}$$

endowed with the metric

$$x, y \in X(1), x \cdot y = \sum_{i=1}^{N} m_i x_i y_i, \qquad m_i > 0.$$

$X' \subseteq X(1)$ then denotes a subspace of $X(1)$ endowed with the induced metric of $X(1)$.

Then we take

$$X(j) \text{ to denote the subspace of } X(1)$$

with the $j-1$ particles x_N to x_{n-j+1} removed (namely,the corresponding coordinates are set to zero)

$$X(j) = \{x = (x_1, \cdots x_{N-j+1})\}.$$

We then have subspaces $X(j)$ such that

$$X(N) \subset X(N-1) \subset \cdots X(2) \subset X(1).$$

This notation, while nonstandard will allow us to do the inductive proofs in a standard way, and also emphasizes the generality of the N-body type systems involved: the subspaces $X(j)$ can be chosen more generally for X, so we can cover the generalized N-body systems of Agmon[14].

To describe the many-body structure of the considered operators we assume that $\mathbf{Y}_0$ is a finite subset of euclidean spaces contained in $X(1)$ [with inherited euclidean structures] and for every $Y \in \mathbf{Y}_0$ the associated interaction potential $v_Y \in L^2_{loc}(Y)$ is real-valued and satisfies some regularity and decay hypotheses [to be described later]. For every $X \subseteq X(1)$ we define $V_X : H^2(X) \to L^2(X)$ as the operator of multiplication by

$$V_X(x) = \sum_{Y \in \mathbf{Y}_0,\, Y \subseteq X} v_Y(\pi^Y x), \tag{1.9}$$

where $x \in X$, π^Y is the orthogonal projection onto Y and $H^s(X)$ denotes the Sobolev space on X.

To describe the perturbation $\mathbf{W_X}$ we consider also operators $W : H^2(X') \to L^2(X)$ with $X' \subset X$, saying that W is the operator of multiplication by $w \in L^2_{loc}(X)$ if

$$(W\varphi)(x) = w(x)\varphi(\pi^{X'} x). \tag{1.10}$$

Such interaction terms allow quite general non-particle number conserving terms.The decay assumptions on w(x) imply that the created particle by the interaction have localized wave-function.This is natural as local field theories of massive particles satisfy this condition.It may be violated by massless theories with (strong) infrared interactions.

The simplest case is when the interaction creates “one particle” (linear in the creation/annihilation term):

$$w(x) = f(x_\alpha), x_\alpha \in \mathbf{R}^3, f \in L^2(\mathbf{R}^3) \cap L^\infty(\mathbf{R}^3)$$

$$X' = \mathbb{R}^3 \oplus \cdots \mathbb{R}^3, n - times$$

$$X = X' \oplus \mathbb{R}^3.$$

In this case

$$(W\varphi)(x) = f(x_\alpha)\varphi(\Pi^{X'} x)$$

is the operator that creates a particle with wave function $f(x_\alpha)$, acting on the space of n particles into the space of $n+1$ particles.

The construction (1.10) allows much more general type of interactions, for example creating pairs of particles in some subspaces of other particles; this can be achieved by choosing

$$w(x) = f(x_\alpha, x_\beta), x_\alpha, x_\beta \in \mathbf{R}^3,$$

which corresponds to a (sum of) products of two creation operators.

We define $\tilde{\mathbf{X}} = \{\tilde{X}(n)\}_{2 \le n \le N}$ by the relation

$$X(n-1) = X(n) \oplus \tilde{X}(n) \text{ for } 2 \le n \le N. \tag{1.11}$$

We assume that for $2 \leq n \leq N$ and $Y \in \mathbf{Y}_0$ such that $Y \subseteq X(n)$ we have $w_{n-1,Y} \in L^2_{loc}(Y \oplus \tilde{X}(n))$ satisfying some regularity and decay hypotheses [to be described later] and $W_{n-1} : H^2(X(n)) \to L^2(X(n-1))$ is the operator of multiplication by $w_{n-1} \in L^2_{loc}(X(n-1))$ of the form

$$w_{n-1}(x(n), \tilde{x}(n)) = \sum_{Y \in \mathbf{Y}_0, \{0\} \subset Y \subseteq X(n)} w_{n-1,Y}(\pi^Y x(n), \tilde{x}(n)), \tag{1.12}$$

where $(x(n), \tilde{x}(n)) \in X(n) \oplus \tilde{X}(n) = X(n-1)$. We assume that $\mathbf{W_X}$ is the self-adjoint operator in $\mathcal{H}$ defined by the quadratic form

$$\mathbf{W_X}[\varphi, \varphi] = (\mathbf{W^+_X}\varphi, \varphi) + (\varphi, \mathbf{W^+_X}\varphi) \tag{1.13}$$

where $\mathbf{W^+_X}(\varphi) = (W_1\varphi_2, ..., W_{N-1}\varphi_N, 0)$ for $\varphi = (\varphi_1, ..., \varphi_N) \in \mathcal{H}$.
We shall prove

Theorem I.1 *Let $\mathbf{H_X}$ be defined as above, let $\mu > 0$ and $\mu(n) > dim\tilde{X}(n)/2$ for $2 \leq n \leq N$. We assume that for all*

$$Y \in Y_0$$

,

$$\langle y \rangle^{\mu+|\alpha|} \partial^\alpha_y v_Y(y)(I - \Delta_Y)^{-1} \tag{1.14}$$

are compact operators on $L^2(Y)$ for $|\alpha| \leq 1$ and

$$\langle \tilde{x}(n) \rangle^{\mu(n)+|\tilde{\alpha}|} \langle y \rangle^{\mu+|\alpha|} \partial^\alpha_y \partial^{\tilde{\alpha}}_{\tilde{x}(n)} w_{n-1,Y}(y, \tilde{x}(n))(I - \Delta_{Y \oplus \tilde{X}(n)})^{-1} \tag{1.15}$$

are compact operators on $L^2(Y \oplus \tilde{X}(n))$ for $|\alpha| + |\tilde{\alpha}| \leq 1$, where we denote $\langle x \rangle = (1 + |x|^2)^{1/2}$. Then there exists a closed, countable set $\tau(\mathbf{H_X}) \subset \mathbf{R}$

such that $\mathbf{A_X}$ *is locally conjugate for* $\mathbf{H_X}$ *in* $\mathbf{R} \setminus \tau(\mathbf{H_X})$ *and the eigenvalues of* $\mathbf{H_X}$ *may accumulate (with multiplicities) only at* $\tau(\mathbf{H_X})$.

Theorem I.2 *We make the same hypotheses as in Theorem 1.1 and assume moreover that the operators given by (1.14) are compact for* $|\alpha| \le 2$ *and the operators given by (1.15) are compact for* $|\alpha| + |\tilde{\alpha}| \le 2$. *Then the singular continuous spectrum of* $\mathbf{H_X}$ *is empty. Moreover,the point spectrum outside the threshold set consists of discrete eigenvalues of finite multiplicities. The operator* $\mathbf{H_X}$ *satisfies the limiting absorption principle with respect to either the operator* $\mathbf{A_X}$ *or the operator* $< x >$.

II Description of the lattice and associated subhamiltonians

For Banach spaces X_1, X_2 we denote by $B(X_1, X_2)$ and $B_\infty(X_1, X_2)$ the set of bounded and compact operators $X_1 \to X_2$ respectively and $B(X_1) = B(X_1, X_1)$, $B_\infty(X_1) = B_\infty(X_1, X_1)$.
For every $X \subseteq X(1)$ we denote by $X^\perp$ the orthogonal complement of X in $X(1)$, i.e.

$$X(1) = X \oplus X^\perp. \tag{2.1}$$

Without any loss of generality we may replace $\mathbf{Y}_0$ by $\mathbf{Y}$ being a larger finite family of subspaces of $X(1)$ [it suffices to set $v_Y = 0$, $w_{n-1,Y} = 0$ identically for $Y \in \mathbf{Y} \setminus \mathbf{Y}_0$]. Setting

$$\mathbf{Y}_1 = \{Y_1 + ... + Y_n : \, n \in \mathbf{N} \text{ and } Y_1, ..., Y_n \in \mathbf{Y}_0 \cup \mathbf{X} \cup \tilde{\mathbf{X}} \cup \{0\}\},$$

$$\mathbf{Y} = \{\pi^{X(1)}Y_1 + ... + \pi^{X(N)}Y_N : \, Y_1, ..., Y_N \in \mathbf{Y}_1\}, \tag{2.2}$$

we have the following properties

$$X(n) \in \mathbf{Y} \text{ for } 1 \le n \le N \text{ and } \tilde{X}(n) \in \mathbf{Y} \text{ for } 2 \le n \le N, \tag{2.3i}$$

$$Y_1, \, Y_2 \in \mathbf{Y} \Rightarrow Y_1 + Y_2 \in \mathbf{Y}, \tag{2.3ii}$$

$$Y \in \mathbf{Y} \Rightarrow \pi^{X(n)}Y \in \mathbf{Y} \text{ for } 1 \le n \le N. \tag{2.3iii}$$

If $Y \subseteq X \subseteq X(1)$, $Y \in \mathbf{Y}$, then we denote

$$\sharp_X Y = \max\{n \in \mathbf{N} : \, Y \subseteq Y_1 \subset Y_2 \subset ... \subset Y_n \subseteq X \text{ for some } Y_1, ..., Y_n \in \mathbf{Y}\}. \tag{2.4}$$

We denote by $S^m_{hg}(X)$ the set of smooth functions which are homogeneous of degree m outside a bounded region, i.e.

$$S^m_{hg}(X) = \{f \in C^\infty(X) : \text{ there is } R > 0 \text{ such that}$$

$$f(\lambda x) = \lambda^m f(x) \text{ holds for } \lambda > 1 \text{ and } |x| > R\}. \tag{2.5}$$

For $r > 0$, $\tilde{r} > 0$, $X' \in \mathbf{Y}$, $X' \subset X$, we define

$$U^{X'}_X(r, \tilde{r}) = \{x \in X : \, |\pi^{X'}x| < r|x| \text{ and}$$

$$|\pi^Y x| > \tilde{r}|x| \text{ for all } Y \in \mathbf{Y} \text{ satisfying } Y \subseteq X, \, Y \not\subseteq X'\} \tag{2.7}$$

if $X' \neq \{0\}$ and $U^{\{0\}}_X(r, \tilde{r}) = U^{\{0\}}_X(\tilde{r})$ with

$$U^{\{0\}}_X(\tilde{r}) = \{x \in X : |x| < 1\} \cup$$

$$\{x \in X : \, |\pi^Y x| > \tilde{r}|x| \text{ for all } Y \in \mathbf{Y} \text{ satisfying } Y \subset X\}. \tag{2.6$'$}$$

If $X' \in \mathbf{Y}$, $X' \subseteq X$ then we may define the following self-adjoint operators in $L^2(X)$:

$$V_X^{X'}(x) = \sum_{Y \in \mathbf{Y}, Y \subseteq X'} v_Y(\pi^Y x), \tag{2.8}$$

$$H_X^{X'} = -\Delta_X + V_X^{X'}, \tag{2.8'}$$

where in the case $X' = \{0\}$ we adopt the convention that $V_X^{X'} = 0$ and $H_X^{X'} = -\Delta_X$.

Let $J_X^{X'} \in S_{hg}^0(X)$ be such that supp $J_X^{X'} \subset U_X^{X'}(r, \tilde{r})$ for certain $r > 0$, $\tilde{r} > 0$. Then

$$(V_X^X - V_X^{X'})J_X^{X'} \in B_\infty(H^2(X), L^2(X)). \tag{2.9}$$

For $X' \in \mathbf{Y}$, $X' \subseteq X(n)$, $2 \leq n \leq N$, we define $W_{n-1}^{X'} : H^2(X(n)) \to L^2(X(n-1))$ as the operators of multiplication by

$$w_{n-1}^{X'}(x(n), \tilde{x}(n)) = \sum_{Y \in \mathbf{Y}, Y \subseteq X'} w_{n-1,Y}(\pi^Y x(n), \tilde{x}(n)) \tag{2.10}$$

and we adopt the convention that $W_{n-1}^{X'} = 0$ if $X' = \{0\}$.

Let $J_{X(n)}^{X'} \in S_{hg}^0(X(n))$ be such that supp $J_{X(n)}^{X'} \subset U_{X(n)}^{X'}(r, \tilde{r})$ for certain $r > 0$, $\tilde{r} > 0$. Then

$$(W_{n-1} - W_{n-1}^{X'})J_{X(n)}^{X'} \in B_\infty(H^2(X(n)), L^2(X(n-1))). \tag{2.11}$$

For $X' \in \mathbf{Y}$, $X' \subseteq X(N)$ we denote $\mathbf{X}(X') = \{X'(n)\}_{1 \leq n \leq N}$ taking $X'(N) = X'$ and defining successively $X'(N-1)$, $X'(N-2)$,..., $X'(1) \in \mathbf{Y}$ by the relation

$$X'(n-1) = X'(n) \oplus \tilde{X}(n) \text{ for } 2 \leq n \leq N. \tag{2.12}$$

Assume moreover $X' \neq \{0\}$ and introduce

$$\mathbf{H}_{\mathbf{X}}^{X'\ diag} = \bigoplus_{1\le n\le N} H_{X(n)}^{X'(n)}. \tag{2.13}$$

Still assuming $X' \neq \{0\}$ we define $\mathbf{W}_{\mathbf{X}}^{X'}$ as the self-adjoint operator in $\mathcal{H}$ given by the quadratic form

$$\mathbf{W}_{\mathbf{X}}^{X'}[\varphi,\varphi] = (\mathbf{W}_{\mathbf{X}}^{X'+}\varphi,\varphi) + (\varphi, \mathbf{W}_{\mathbf{X}}^{X'+}\varphi) \tag{2.14}$$

where for $\varphi = (\varphi_1, ..., \varphi_N) \in \mathcal{H}$ we have

$$\mathbf{W}_{\mathbf{X}}^{X'+}(\varphi_1, ..., \varphi_N) = (W_1^{X'(2)}\varphi_2, ..., W_{N-1}^{X'(N)}\varphi_N, 0)$$

and set

$$\mathbf{H}_{\mathbf{X}}^{X'} = \mathbf{H}_{\mathbf{X}}^{X'\ diag} + \mathbf{W}_{\mathbf{X}}^{X'}. \tag{2.15}$$

In the case $X' \neq \{0\}$ we define $\mathbf{H}_{\mathbf{X}}^{\{0\}}$ according to (2.15), where we set

$$\mathbf{H}_{\mathbf{X}}^{\{0\}\ diag} = \left(\bigoplus_{1\le n\le N-1} H_{X(n)}\right) \oplus (-\Delta_{X(N)}) \tag{2.13$'$}$$

and $\mathbf{W}_{\mathbf{X}}^{\{0\}}$ defined by (2.14), where for $\varphi = (\varphi_1, ..., \varphi_N) \in \mathcal{H}$ we take

$$\mathbf{W}_{\mathbf{X}}^{\{0\}+}(\varphi_1, ..., \varphi_N) = (W_1^{X'(2)}\varphi_2, ..., W_{N-2}^{X'(N-1)}\varphi_{N-1}, 0, 0).$$

The next aim is to construct a suitable partition of unity described in

Proposition II.1 *There exists a family* $\{\mathbf{J}_{\mathbf{X}}^{X'}\}_{X'\in\mathbf{Y}, X'\subset X(N)}$ *of operators of the form*

$$\mathbf{J}_{\mathbf{X}}^{X'} = \bigoplus_{1\le n\le N} J_n^{X'}, \tag{2.16}$$

where every $J_n^{X'}$ *is an operator of multiplication by real-valued* $J_n^{X'} \in S^0_{hg}(X(n))$, *satisfying*

$$\sum_{X' \in \mathbf{Y},\, X' \subset X(N)} (\mathbf{J}_\mathbf{X}^{X'})^2 = I \tag{2.17}$$

and such that for every $X', X'' \in \mathbf{Y}$, $X', X'' \subset X(N)$, *the following operators*

$$(i + \mathbf{H_X})^{-1}[\mathbf{J}_\mathbf{X}^{X'}, \mathbf{H}_\mathbf{X}^{X''}], \tag{2.18i}$$

$$(i + \mathbf{H_X})^{-1}(\mathbf{H_X} - \mathbf{H}_\mathbf{X}^{X'})\mathbf{J}_\mathbf{X}^{X'}, \tag{2.18ii}$$

$$(i + \mathbf{H_X})^{-1}[\mathbf{J}_\mathbf{X}^{X'}, [\mathbf{H}_\mathbf{X}^{X''}, \mathbf{A_X}]](i + \mathbf{H_X})^{-1} \tag{2.18iii}$$

$$(i + \mathbf{H_X})^{-1}[\mathbf{H_X} - \mathbf{H}_\mathbf{X}^{X'}, \mathbf{A_X}]\mathbf{J}_\mathbf{X}^{X'}(i + \mathbf{H_X})^{-1} \tag{2.18iv}$$

are compact.

III Proof of Proposition 2.1

If $r > 0$, then $F(\cdot \leq r)$ denotes a smoothed characteristic function of $]-\infty, r]$, i.e. $\lambda \to F(\lambda \leq r)$ is a non-negative, smooth function on $\mathbf{R}$ which equals 1 on $]-\infty, r/2]$ and equals 0 on $[r, \infty[$.

Lemma III.1 *Let* $X' \in \mathbf{Y}$, $\{0\} \subset X' \subset X(N)$ *and* $J_N^{X'} \in S^0_{hg}(X(N))$. *If* $r > 0$ *then the relation*

$$J_{n-1}^{X'}(x(n), \tilde{x}(n)) = J_n^{X'}(x(n)) F\left(\frac{|\tilde{x}(n)|}{|x(n-1)|} \leq r\right) \tag{3.1}$$

allows to define successively $J_{N-1}^{X'} \in S^0_{hg}(X(N-1))$, $J_{N-2}^{X'} \in S^0_{hg}(X(N-2))$,..., $J_1^{X'} \in S^0_{hg}(X(1))$. *If moreover supp* $J_N^{X'} \subset U_{X(N)}^{X'}(r_{X'}, \tilde{r}_{X'})$ *and* $r_{X'} >$

0, $r > 0$ are small enough, then

$$\textit{supp } J_n^{X'} \subset U_{X(N)}^{X'(n)}(r_{X'} + (N-n)r,\ \tilde{r}_{X'}/(1+N-n)) \tag{3.2n}$$

holds for every $1 \leq n \leq N$.

Proof. We assume (3.2.n) for a certain $2 \leq n \leq N$ and we show (3.2.n-1). Due to (3.2n), for $(x(n), \tilde{x}(n)) \in$ supp $J_{n-1}^{X'}$ we have

$$|\tilde{x}(n)| \leq r|x(n)| \text{ and } |\pi^{X'(n)}x(n)| < (r_{X'} + (N-n)r)|x(n)|, \tag{3.3}$$

which implies

$$|\pi^{X'(n-1)}x(n-1)| \leq |\pi^{X'(n)}x(n)| + |\tilde{x}(n)| <$$

$$(r_{X'} + (N-n)r + r)|x(n)| \leq (r_{X'} + (N-n+1)r)|x(n-1)|. \tag{3.4}$$

Now we fix $Y \in \mathbf{Y}$ satisfying $Y \subset X(n-1)$, $Y \not\subseteq X'(n-1)$, and it remains to show that

$$|\pi^Y x(n-1)| > \frac{\tilde{r}_{X'}}{2+N-n}|x(n-1)|. \tag{3.5}$$

We define $Y_1 \in \mathbf{Y}$, $Y_1 \subseteq X(n)$ by

$$Y_1 = \pi^{X(n)}Y + X'(n) \tag{3.6}$$

and $Y_2 \subseteq X(n)$ by

$$X(n) = Y_1 \oplus Y_2. \tag{3.7}$$

We note that $x \in X(n) \cap Y^\perp \iff (0 = (x,y) = (x, \pi^{X(n)}y)$ for all $y \in Y) \iff x \in X(n) \cap (\pi^{X(n)}Y)^\perp$. Therefore

$$Y_2 = Y_1^\perp \cap X(n) = (\pi^{X(n)}Y)^\perp \cap X'(n)^\perp \cap X(n) = Y^\perp \cap X''(n) \tag{3.8}$$

with $X''(n) = X'(n)^{\perp} \cap X(n)$, i.e. with $X''(n)$ satisfying the relation

$$X(n) = X'(n) \oplus X''(n). \tag{3.9}$$

Clearly $Y_2 \subseteq X''(n)$ and we are going to check that the inclusion is strict. Indeed, $Y_2 = X''(n) \Rightarrow Y_1 = X'(n) \Rightarrow \pi^{X(n)}Y \subseteq X'(n) \Rightarrow Y = \pi^{X(n)}Y + \pi^{\tilde{X}(n)}Y \subseteq X'(n) + \tilde{X}(n) \subseteq X'(n-1)$, which is in contradiction with the assumption made on Y.

Thus we have $Y_2 \subset X''(n)$, which implies $X'(n) \subset Y_1$, hence $Y_1 \not\subseteq X'(n)$ and due to (3.2.n),

$$|\pi^{Y_1}x(n)|^2 > \left(\frac{\tilde{r}_{X'}}{1+N-n}\right)^2 |x(n)|^2 \tag{3.10}$$

and since $|\pi^{Y_1}x(n)|^2 + |\pi^{Y_2}x(n)|^2 = |x(n)|^2$, (3.10) is equivalent to

$$|\pi^{Y_2}x(n)|^2 < \left(1 - \left(\frac{\tilde{r}_{X'}}{1+N-n}\right)^2\right) |x(n)|^2. \tag{3.11}$$

Therefore using (3.8), (3.11) and (3.3) we obtain

$$|\pi^{Y^{\perp}}x(n-1)| \leq |\pi^{Y^{\perp}\cap X''(n)}x(n-1)| +$$

$$|\pi^{Y^{\perp}\cap X'(n)}x(n-1)| + |\pi^{Y^{\perp}\cap \tilde{X}(n)}x(n-1)| \leq$$

$$|\pi^{Y_2}x(n)| + |\pi^{X'(n)}x(n)| + |\tilde{x}(n)| <$$

$$\left(\left(1 - \left(\frac{\tilde{r}_{X'}}{1+N-n}\right)^2\right)^{1/2} + r_{X'} + (N-n+1)r\right) |x(n)| \tag{3.12}$$

and taking $r_{X'} > 0$, $r > 0$ small enough, we have

$$\left(1 - \left(\frac{\tilde{r}_{X'}}{1+N-n}\right)^2\right)^{1/2} + r_{X'} + (N-n+1)r \leq \left(1 - \left(\frac{\tilde{r}_{X'}}{2+N-n}\right)^2\right)^{1/2}. \tag{3.13}$$

Now it is clear that (3.12) and (3.13) give

$$|\pi^{Y^\perp} x(n-1)|^2 < \left(1 - \left(\frac{\tilde{r}_{X'}}{2+N-n}\right)^2\right) |x(n-1)|^2, \tag{3.14}$$

which gives (3.5) due to $|\pi^Y x(n-1)|^2 + |\pi^{Y^\perp} x(n-1)|^2 = |x(n-1)|^2$. △

Lemma III.2 *Let $X' \in \mathbf{Y}$, $\{0\} \subset X' \subset X(N)$ and assume that $\{J_n^{X'}\}_{1\le n\le N}$ are as in Lemma 3.1 and (3.2n) holds for $1 \le n \le N$. If $\mathbf{J}_{\mathbf{X}}^{X'}$ is given by (2.16), then the operators (2.18i,ii,iii,iv) are compact.*

Proof. i) Clearly it suffices to consider $\mathbf{H}_{\mathbf{X}}$ instead of $\mathbf{H}_{\mathbf{X}}^{X''}$ in (2.18i). Since the commutators $[J_n^{X'}, \Delta_{X(n)}]$ are first order differential operator with coefficients in $S_{hg}^{-1}(X(n))$, i.e. $\Delta_{X(n)}$-compact operators, it is clear that

$$[\mathbf{J}_{\mathbf{X}}^{X'}, \mathbf{H}_{\mathbf{X}}^{diag}] = \bigoplus_{1\le n\le N} [J_n^{X'}, -\Delta_{X(n)}] \tag{3.15}$$

is $\mathbf{H}_{\mathbf{X}}$-compact. In order to prove that $[\mathbf{J}_{\mathbf{X}}^{X'}, \mathbf{W}_{\mathbf{X}}]$ is $\mathbf{H}_{\mathbf{X}}$-compact, it suffices to show that for $2 \le n \le N$ one has

$$J_{n-1}^{X'} W_{n-1} - W_{n-1} J_n^{X'} \in B_\infty(H^2(X(n)), L^2(X(n-1))). \tag{3.16}$$

However $J_{n-1}^{X'} W_{n-1} - W_{n-1} J_n^{X'}$ is the operator of multiplication by

$$g(x(n), \tilde{x}(n)) = \left(F\left(\frac{|\tilde{x}(n)|}{|x(n-1)|} \le r\right) - 1\right) w_{n-1}(x(n), \tilde{x}(n)) J_n^{X'}(x(n)) \tag{3.17}$$

and we may write $g = g_3 g_2 g_1$ with

$$g_1(x(n), \tilde{x}(n)) = \langle \tilde{x}(n) \rangle^{-\mu(n)+\epsilon} J_n^{X'}(x(n)), \tag{3.18i}$$

$$g_2(x(n),\tilde{x}(n)) = \langle x(n)\rangle^{-\epsilon}\langle\tilde{x}(n)\rangle^{\mu(n)} w_{n-1}(x(n),\tilde{x}(n)), \tag{3.18ii}$$

$$g_3(x(n),\tilde{x}(n)) = \left(F\left(\frac{|\tilde{x}(n)|}{|x(n-1)|}\le r\right)-1\right)\langle x(n)\rangle^{\epsilon}\langle\tilde{x}(n)\rangle^{-\epsilon}, \tag{3.18iii}$$

where $\epsilon > 0$ is chosen such that $\mu(n)-\epsilon > \dim\tilde{X}(n)/2$, which implies that the multiplication by g_1 is a bounded operator $H^2(X(n)) \to H^2(X(n-1))$. Next the compactness hypotheses concerning the operators (1.15) guarantee that the multiplication by g_2 is a compact operator $H^2(X(n-1)) \to L^2(X(n-1))$ and we complete the proof noting that the multiplication by g_3 is a bounded operator in $L^2(X(n-1))$ due to the definition of F.

ii) Clearly (2.9) implies that $(\mathbf{H}_{\mathbf{X}}^{diag} - \mathbf{H}_{\mathbf{X}}^{X'\ diag})\mathbf{J}_{\mathbf{X}}^{X'}$ is $\mathbf{H}_{\mathbf{X}}$-compact. For $1 \le n \le N$ let $\tilde{J}_n^{X'} \in S_{hg}^0(X(n))$ be such that $\tilde{J}_n^{X'} = 1$ on supp $J_n^{X'}$ and supp $\tilde{J}_n^{X'} \subset U_{X(n)}^{X'(n)}(r_n^{X'},\tilde{r}_n^{X'})$ for certain $r_n^{X'} > 0$, $\tilde{r}_n^{X'} > 0$. Setting

$$\tilde{\mathbf{J}}_{\mathbf{X}}^{X'} = \bigoplus_{1\le n\le N} \tilde{J}_n^{X'}, \tag{3.19}$$

it is clear that (2.11) implies the fact that $\tilde{\mathbf{J}}_{\mathbf{X}}^{X'}(\mathbf{W}_{\mathbf{X}} - \mathbf{W}_{\mathbf{X}}^{X'})\mathbf{J}_{\mathbf{X}}^{X'}$ is $\mathbf{H}_{\mathbf{X}}$-compact. Since $(I-\tilde{\mathbf{J}}_{\mathbf{X}}^{X'})\mathbf{J}_{\mathbf{X}}^{X'} = 0$, it suffices to note that the operator

$$(I-\tilde{\mathbf{J}}_{\mathbf{X}}^{X'})(\mathbf{W}_{\mathbf{X}} - \mathbf{W}_{\mathbf{X}}^{X'})\mathbf{J}_{\mathbf{X}}^{X'} = (I-\tilde{\mathbf{J}}_{\mathbf{X}}^{X'})[\mathbf{W}_{\mathbf{X}} - \mathbf{W}_{\mathbf{X}}^{X'}, \mathbf{J}_{\mathbf{X}}^{X'}] \tag{3.20}$$

is $\mathbf{H}_{\mathbf{X}}$-compact due to i).

iii) and iv) We may define $\tilde{V}_X^{X'}$, $\tilde{H}_X^{X'}$, $\tilde{W}_{n-1}^{X'}$ according to (2.8), (2.10), where v_Y, $w_{n-1,Y}$ are replaced by

$$\tilde{v}_Y(y) = -y\cdot\nabla_y v_Y(y), \tag{3.21}$$

$$\tilde{w}_{n-1,Y}(y,\tilde{x}(n)) = -\left(y\cdot\nabla_y + \tilde{x}(n)\cdot\nabla_{\tilde{x}(n)} + \frac{\dim\tilde{X}(n)}{2}\right) w_{n-1,Y}(y,\tilde{x}(n)). \tag{3.21$'$}$$

It is clear we still have

$$(\tilde{V}_X^X - \tilde{V}_X^{X'})J_X^{X'} \in B_\infty(H^2(X), L^2(X)), \tag{3.22}$$

$$(\tilde{W}_{n-1} - \tilde{W}_{n-1}^{X'})J_{X(n)}^{X'} \in B_\infty(H^2(X(n)), L^2(X(n-1)), \tag{3.23}$$

where $\tilde{V}_X = \tilde{V}_X^X$, $\tilde{W}_{n-1} = \tilde{W}_{n-1}^X$.

Thus it suffices to follow the proof of i) and ii) noting that

$$\frac{i}{2}[\mathbf{H}_\mathbf{X}^{X'}, \mathbf{A}_\mathbf{X}] = \tilde{\mathbf{H}}_\mathbf{X}^{X'} = \tilde{\mathbf{H}}_\mathbf{X}^{X'\ diag} + \tilde{\mathbf{W}}_\mathbf{X}^{X'}, \tag{3.24}$$

where $\tilde{\mathbf{H}}_\mathbf{X}^{X'\ diag}$ and $\tilde{\mathbf{W}}_\mathbf{X}^{X'}$ are defined according to (2.13), (2.14) with $H_{X(n)}^{X'(n)}$, $W_{n-1}^{X'(n)}$ replaced by $\tilde{H}_{X(n)}^{X'(n)}$, $\tilde{W}_{n-1}^{X'(n)}$. $\triangle$

Proof of Proposition 2.1. Fix $0 < \tilde{r}_2 < 1$, set $\tilde{r}_{X'} = \tilde{r}_2$ for every X' such that $\sharp_{X(N)}X' = 2$ and choose $r_2 = r_2(\tilde{r}_2) > 0$ small enough to guarantee (3.2n) for $1 \le n \le N$ with $r_{X'} = r_2$. Assuming that we have chosen $r_{X'} = r_{\sharp_{X(N)}X'} > 0$, $\tilde{r}_{X'} = \tilde{r}_{\sharp_{X(N)}X'} > 0$ for every X' such that $\sharp_{X(N)}X' \le k$, we choose sufficiently small $\tilde{r}_{k+1} = \tilde{r}_{k+1}(\tilde{r}_2, r_2, ..., \tilde{r}_k, r_k) > 0$ and $r_{k+1} = r_{k+1}(\tilde{r}_2, r_2, ..., \tilde{r}_k, r_k, \tilde{r}_{k+1}) > 0$ small enough to guarantee (3.2n) for $1 \le n \le N$ with $\tilde{r}_{X'} = \tilde{r}_{k+1}$, $r_{X'} = r_{k+1}$ for every X' such that $\sharp_{X(N)}X' = k+1$. A simple geometric reasoning based on the fact that for $X' \subset X$ and $r > 0$ small, $\{x \in X : |\pi^{X'}x| < r|x|\}$ is a small conical neighbourhood of $X'^{\perp} \cap (X \setminus \{0\})$ in $X \setminus \{0\}$, allows to find $\tilde{r}_{\{0\}} = r_{\{0\}} > 0$

such that $\{U^{X'}_{X(N)}(r_{X'}, \tilde{r}_{X'})\}_{X'\in\mathbf{Y},\, X'\subset X(N)}$, is a covering of $X(N)$. Therefore there exists a partition of unity

$$\sum_{X'\in\mathbf{Y},\, X'\subset X(N)} \tilde{J}^{X'}_N = 1 \tag{3.25}$$

composed of $\tilde{J}^{X'}_N \in S^0_{hg}(X(N))$ such that supp $\tilde{J}^{X'}_N \subset U^{X'}_{X(N)}(r_{X'}, \tilde{r}_{X'})$ and $\tilde{J}^{X'}_N \geq 0$.

For $X' \neq \{0\}$ we define successively $\tilde{J}^{X'}_{N-1} \in S^0_{hg}(X(N{-}1))$, $\tilde{J}^{X'}_{N-2} \in S^0_{hg}(X(N{-}2))$,...,$\tilde{J}^{X'}_1 \in S^0_{hg}(X(1))$, using the relation (3.1) with $\tilde{J}^{X'}_n$ instead of $J^{X'}_n$ and let $\tilde{\mathbf{J}}^{X'}_{\mathbf{X}}$ be defined as in (3.19). Due to the assertion of Lemma 2.1 we may assume that (3.2n) holds for $1 \leq n \leq N$ with $\tilde{J}^{X'}_n$ instead of $J^{X'}_n$ and due to Lemma 3.2 the commutator $[\tilde{\mathbf{J}}^{X'}_{\mathbf{X}}, \mathbf{H}_{\mathbf{X}}]$ is $\mathbf{H}_{\mathbf{X}}$-compact.

Defining $\tilde{\mathbf{J}}^{\{0\}}_{\mathbf{X}}$ by the relation

$$\sum_{X'\in\mathbf{Y},\, \{0\}\subset X'\subset X(N)} \tilde{\mathbf{J}}^{X'}_{\mathbf{X}} = I - \tilde{\mathbf{J}}^{\{0\}}_{\mathbf{X}}, \tag{3.26}$$

it is clear that $[\tilde{\mathbf{J}}^{\{0\}}_{\mathbf{X}}, \mathbf{H}_{\mathbf{X}}]$ is still $\mathbf{H}_{\mathbf{X}}$-compact and

$$\tilde{\mathbf{J}}^{\{0\}}_{\mathbf{X}} = \bigoplus_{1\leq n\leq N} \tilde{J}^{\{0\}}_n, \tag{3.27}$$

with $\tilde{J}^{\{0\}}_n \in S^0_{hg}(X(n))$ for $1 \leq n \leq N$ and supp $\tilde{J}^{\{0\}}_N \subset U^{\{0\}}_{X(N)}(\tilde{r}_{\{0\}})$, hence the operators (2.18ii), (2.18iv) are compact for $X' = \{0\}$ as well.

It is clear that (3.26) implies existence of a constant $c_0 > 0$ such that

$$\mathbf{S}_{\mathbf{X}} = \sum_{X'\in\mathbf{Y},\, X'\subset X(N)} \left(\tilde{\mathbf{J}}^{X'}_{\mathbf{X}}\right)^2 \geq c_0 I \tag{3.28}$$

and then clearly $[\mathbf{S_X}, \mathbf{H_X}]$ is $\mathbf{H_X}$-compact. Hence

$$[f(\mathbf{S_X}), \mathbf{H_X}](i + \mathbf{H_X})^{-1} \in B_\infty(\mathcal{H}) \tag{3.29}$$

if $f(\lambda) = (\lambda \pm i)^{-1}$ and Stone-Weierstrass theorem (cf. e.g. [4]) allows to affirm that (3.26) still holds for every $f \in C(\mathbf{R})$ such that $f(\lambda) \to 0$ when $\lambda \to \infty$. We complete the proof noting that all assertions of Proposition 2.1 are satisfied if we set $\mathbf{J}_\mathbf{X}^{X'} = \tilde{\mathbf{J}}_\mathbf{X}^{X'} \mathbf{S}_\mathbf{X}^{-1/2}$. $\triangle$

IV Proofs of Theorems 1.1 and 1.2

Let $\mathbf{X} = \{X(n)\}_{1 \le n \le N}$ be as before a finite family of euclidean spaces satisfying (1.8) and let $\mathbf{X}' = \{X'(n)\}_{1 \le n \le N'}$ be another finite family of euclidean spaces satisfying

$$X'(N') \subset ... \subset X'(2) \subset X'(1). \tag{4.1}$$

We shall write $\mathbf{X}' \le \mathbf{X}$ if and only if the following three conditions hold

i) $N' \le N$,

ii) $X'(1) \subseteq X(1)$,

iii) $\tilde{X}'(n) = \tilde{X}(n)$ for $2 \le n \le N'$,

where $\tilde{X}(n)$ is given by (1.11) and $\tilde{X}'(n)$ by

$$X'(n-1) = X'(n) \oplus \tilde{X}'(n) \text{ for } 2 \le n \le N'. \tag{4.2}$$

Moreover we shall write $\mathbf{X}' < \mathbf{X} \iff (\mathbf{X}' \le \mathbf{X} \text{ and } \mathbf{X}' \ne \mathbf{X})$.

The idea of the proof of Theorem 1.1 is to show the assertion for $\mathbf{H_X}$ with

$$\tau(\mathbf{H_X}) = \bigcup_{\mathbf{X'}<\mathbf{X}} \overline{\sigma_{pp}(\mathbf{H_{X'}})} \tag{4.3}$$

[where $\sigma_{pp}(H)$ denotes the set of eigenvalues of H], assuming that the analogical statement holds for every $\mathbf{H_{X'}}$ with $\mathbf{X'} < \mathbf{X}$. We note first that Proposition 2.1 has the following

Corollary 4.1. Assume that $\mathbf{J}_{\mathbf{X}}^{X'}$ satisfies the assertions of Proposition 2.1. If $f \in C_0^\infty(\mathbf{R})$ then for every $X', X'' \in \mathbf{Y}$, $X', X'' \subset X(N)$, the operators

$$(i + \mathbf{H_X})[\mathbf{J}_{\mathbf{X}}^{X'}, f(\mathbf{H}_{\mathbf{X}}^{X''})], \tag{4.4}$$

$$(i + \mathbf{H_X})(f(\mathbf{H_X}) - f(\mathbf{H}_{\mathbf{X}}^{X'}))\mathbf{J}_{\mathbf{X}}^{X'}, \tag{4.4$'$}$$

are compact. Moreover there exist compact operators K_1, K_2, such that

$$f(\mathbf{H_X}) = \sum_{X'\in\mathbf{Y},\, X'\subset X(N)} \mathbf{J}_{\mathbf{X}}^{X'} f(\mathbf{H}_{\mathbf{X}}^{X'})\mathbf{J}_{\mathbf{X}}^{X'} + K_1, \tag{4.5}$$

$$f(\mathbf{H_X})[i\mathbf{H_X}, \mathbf{A_X}]f(\mathbf{H_X}) =$$

$$\sum_{X'\in\mathbf{Y},\, X'\subset X(N)} \mathbf{J}_{\mathbf{X}}^{X'} f(\mathbf{H}_{\mathbf{X}}^{X'})[i\mathbf{H}_{\mathbf{X}}^{X'}, \mathbf{A_X}]f(\mathbf{H}_{\mathbf{X}}^{X'})\mathbf{J}_{\mathbf{X}}^{X'} + K_2, \tag{4.6}$$

Indeed, Proposition 2.1 allows to get the compactness of operators (4.4-4') for $f(\lambda) = (\lambda \pm i)^{-1}$ and the general case follows as before via Stone-Weierstrass theorem (cf. [4]). Using (2.18) and (4.4-4') we get (4.5-6) as in [4]. We note first that the assertion of Theorem 1.1 holds in the case N=1 corresponding to the case of a standard scalar many-body operator treated in [3] or [4], where

$$\tau(H_X) = \bigcup_{X'\subset X} \overline{\sigma_{pp}(H_{X'})} \tag{4.7}$$

Consider now the case $N \geq 2$ and assume that the assertion of Theorem 1.1 with (4.3) holds for all $\mathbf{X}'$ such that $\mathbf{X}' < \mathbf{X}$. Clearly we may exclude the case $X(N) = \{0\}$, because the corresponding operator may be replaced by $\mathbf{H}_{\mathbf{X}(N-1)}$ with

$$\mathbf{X}(N-1) = \{X(n)\}_{1 \leq n \leq N-1} < \mathbf{X} \tag{4.8}$$

for which the assertion of Theorem 1.1 holds due to our induction hypothesis. It remains to consider the case $X(N) \neq \{0\}$ applying Corollary 4.1 similarly as in [4]. Then for every $X' \in \mathbf{Y}$, $\{0\} \subset X' \subset X(N)$, we use the direct decomposition

$$\mathbf{H}_{\mathbf{X}}^{X'} = \int_{\xi \in X''}^{\oplus} \mathbf{H}_{\mathbf{X}(X')}(\xi)\, d\xi, \tag{4.9}$$

where X'' is such that $X(N) = X' \oplus X''$ and

$$\mathbf{H}_{\mathbf{X}(X')}(\xi) = \xi^2 + \mathbf{H}_{\mathbf{X}(X')}, \tag{4.9$'$}$$

where $\mathbf{H}_{\mathbf{X}(X')}$ is the Hamiltonian associated with the family of euclidean subspaces $\mathbf{X}(X')$ defined by (2.12) with $X'(N) = X'$. Thus for $X' \in \mathbf{Y}$, $\{0\} \subset X' \subset X(N)$ we have $\mathbf{X}(X') < \mathbf{X}$ and the assertion of Theorem 1.1 holds for $\mathbf{H}_{\mathbf{X}(X')}$ due to our induction hypothesis. It remains to note that in the case $X' = \{0\}$,

$$\mathbf{H}_{\mathbf{X}}^{\{0\}} = \mathbf{H}_{\mathbf{X}(N-1)} \oplus (-\Delta_{X(N)}), \tag{4.10}$$

with $\mathbf{X}(N-1)$ given by (4.8), hence satisfies the assertion of Theorem 1.1. Therefore we may complete the proof similarly as in the reasoning described in [4]. Here we point out that compactness is(in general) used in the Fock space with differing number of particles.So compactness of the relevant con-rtibution from the

$$W$$

terms is between two spaces.Smallness of the remainder terms,after fibration,follows as in the usual case. The assertion of Theorem 1.2 follows from Theorem 1 and from the boundedness of $[[\mathbf{H_X}, \mathbf{A_X}], \mathbf{A_X}](i + \mathbf{H_X})^{-1}$ (cf. [2], [13]).

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