From Gamow States to Resonances for Schrödinger Operators

Y. Strauss[1], **A. Soffer**[2], **I.M. Sigal**[3,4]

[1]Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem 91904, Israel.

[2]Department of Mathematics, Rutgers University, Piscataway, NJ, 08854-8019.

[3]Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556-4618.

[4]Department of Mathematics, University of Toronto, Toronto, ON, M5S 3G3.

Abstract

By applying time dependent methods we show that a given Gamow state solution for a Schrödinger operator with a compactly supported potential can be associated with a corresponding resonance. Our method is to utilize the given Gamow vector for the construction of an appropriate problem of perturbation of an embedded eigenvalue which give rise to the associated resonance.

1. Introduction

In this work we investigate the close relation between the construction of quasimodes (approximate eigenfunctions) and the resonances of certain quantum mechanical potential models. The problem of the construction of approximate eigenfunctions for Schrödinger potential models was considered many times over the years. Many of the diverse methods for producing approximate eigenfunctions came to be known as 'quasimode' constructions. For scattering problems it is possible to construct quasimodes for the scattering Hamiltonian by utilizing a procedure in which a cutoff is applied to a generalized eigenfunction (Gamow vector) corresponding to a scattering resonance. In this way one obtains a Hilbert space state which is an approximate eigenfunction of the Hamiltonian for the scattering. An immediate question which then arises concerns the relation between the quasimode thus obtained and the resonance.

In several recent papers there has been a considerable progress in the understanding of the relationship between resonances near the real axis and the existence of quasimodes for compactly supported perturbations of the Laplacian. Works by Stefanov and Vodev [SV] followed by Tang and Zworski [TZ] showed that the existence of a sequence of real quasimodes $q_j \to \infty$ corresponding to quasimode states supported in a fixed compact set implies the existence of a sequence of resonances (defined as scattering poles) λ_j rapidly converging to the real axis. Stefanov then continued to show [St] the converse statement, i.e., the existence of a sequence of resonances rapidly converging to the real axis implies the existence of quasimodes with quasimode states supported in a compact set whose interior contains the (compactly supported) scatterer.

In this work we introduce a new approach to the problem of the relation between quasimodes and resonances which is suitable for quantum mechanical scattering problems with compactly supported potentials. We make use of recent results obtained by Soffer and Weinstein [SW] and Costin and Soffer [CS] which improve on the theory of resonances and perturbations of embedded eigenvalues and which do not require assumptions on the analytic continuation properties of the resolvent. We show that a construction of a quasimode by truncation of a Gamow state defines a corresponding problem of a perturbation of an embedded eigenvalue. For the case of a Schrödinger operator with compactly supported potential we continue to show that all the premises of the theorem on resonances proved by Soffer and Costin in [CS] hold and, hence, the existence of a resonance is implied.

An interesting implication of the method of proof of Theorem A in this work is that a truncation procedure may be applied to a Gamow vector corresponding to a complex "eigenvalue" in the upper half-plane, with the resulting quasimode again corresponding, via Theorem A, to a unique resonance. As an example we consider a scattering problem with a simple square barrier potential. For this simple example it is easy to find a Gamow state associated with a point in the upper half-plane which, by the cutoff procedure applied in this work, corresponds to a unique resonance. On the other hand, the same Gamow state can be shown to be associated with a zero of the S-matrix in the upper half-plane. The analysis of this example clearly demonstrates the relation between resonances and zeros of the analytic continuation of the S-matrix to the upper half-plane.

The paper is organized as follows: In Section 2 we first introduce the procedure of truncating a Gamow state in order to produce a 'quasimode'. Next we use the quasimode to define a perturbation term and arrive at a decomposition of the Hamiltonian and define a problem of perturbation of embedded eigenvalue. Once this is done we are able to state Theorem A, our main theorem. The proof of Theorem A is given in Section 3. As a first step we state, in Subsection 3.1 the Costin-Soffer theorem for resonances. Proved by time dependent methods, this theorem relates perturbations of embedded eigenvalues to a correspnding time evolution typical of resonance decay. The proof of Theorem A is essentially divided into two parts, corresponding to the two sets of conditions required by the Costin-Soffer theorem, i.e., the conditions imposed on the unperturbed Hamiltonian and on the perturbation respectively. Accordingly, in Subsection 3.2, we prove that the decomposition $H = \tilde{H}_0 + W$ of the Hamiltonian H introduced in Section 2 is such that the perturbation term W satisfies the assumptions of the Costin-Soffer theorem. Some comments regarding the estimates of Subsection 3.2 are given in Subsection 3.3. In Subsection 3.4 we continue to show that the unpertubed Hamiltonian $\tilde{H}_0$ satisfies the corresponding assumptions of the Costin-Soffer theorem. The conclusions of this theorem then follow immediately, and the proof of Theorem A is complete. Section 4 of the paper deals with Gamow states associated with points in the upper-half of the complex plane. As mentioned above, these states also correspond to resonances by an application of the same cutoff procedure described in Section 2 and Theorem A applies again. Next we discuss the example of a square barrier potential problem. Gamow states for a problem of a Schrödinger equation with a simple square barrier potential where calculated, e.g., by M. Gadella and R. de la Madrid in [dMG]. We recall their solution of the problem and note the existence of a Gamow vector, corresponding to a zero of the S-matrix, to which the truncation procedure can be applied and, subsequently, a corresponding resonance can be found. This example exhibit a relation between zeros of the S-matrix in the upper-half plane and resonances.

2. Main results

Our main result in this work states that, for a certain class of quantum mechanical potential problems, quasimode states constructed via the application of an appropriate cutoff procedure to Gamow vectors results in a certain decomposition of the Hamiltonian leading to a corresponding embedded eigenvalue perturbation problem. Analysis of this later perturbation problem then proves the existence of a resonance associated with the original Gamow vector. We first discuss the construction of a quasimode state by the procedure of cutting a Gamow vector solution. Following the construction of the quasimode we can define the corresponding decomposition of the Hamiltonian and then state and prove our main theorem.

Let $H_0 = -1/2\Delta_x$ and let V be a compactly supported potential such that $H = H_0+V$ is self-adjoint on $L^2(R^3)$. We assume furthermore that $\psi^G_{z_0}$ is a Gamow vector 'solution' of the eigenvalue problem for H with complex 'eigenvalue' z_0, $Im[z_0] \neq 0$, i.e., we locally have

$$[H\psi^G_{z_0}](x) = z_0\psi^G_{z_0}(x) \tag{2.1}$$

Of course, $\psi^G_{z_0}$ cannot be normalized and is not an element of $L^2(R^3)$.

Let $R_1 > 0$ be such that $\mathrm{supp}[V] \subset B_1 = \{x \mid |x| < R_1\}$. Take $R_2 > R_1$, denote $B_2 = \{x \mid |x| < R_2\}$ and let $\chi(x) = \chi(|x|)$ be a smooth, non-negative C^∞ cutoff function such that

$$\begin{aligned} \chi(x) &= 1 \quad \text{for} \quad x \in B_1 \\ \chi(x) &= 0 \quad \text{for} \quad x \in R^3 \backslash B_2 \end{aligned} \tag{2.2}$$

Next, we define the procedure of cutting the Gamow vector by defining

$$\tilde{\psi}_{z_0}(x) = \chi(x)\psi^G_{z_0}(x) \tag{2.3}$$

where the resulting state $\tilde{\psi}_{z_0}$ is now an element of $L^2(R^3)$. We call $\tilde{\psi}_{z_0}$ a 'quasimode' associated with the Gamow vector $\psi^G_{z_0}$. Setting notations for the real and imaginary parts of z_0 by writing $z_0 = E_0 - i\Gamma$ and, in addition, denoting $k_0 = \sqrt{z_0}$ (taking the root which makes a smaller angle with R^+), we proceed to define r_0, a parameter with dimension of length (in the natural units we are using here), to be

$$r_0 = \frac{Re[k_0]}{\Gamma} \tag{2.4}$$

It will be seen below that r_0 is the parameter by which we control the estimates needed for proving our main results.

Following the definition of the quasimode $\tilde{\psi}_{z_0}$ we turn to the decomposition of the Hamiltonian. We define W, a rank one interaction term

$$W = \frac{H'|\tilde{\psi}_{z_0}\rangle\langle\tilde{\psi}_{z_0}|H'}{\langle\tilde{\psi}_{z_0}|H'|\tilde{\psi}_{z_0}\rangle} \tag{2.5}$$

where we have denoted $H' = H - E_0$. We define a new Hamiltonian $\tilde{H}_0$ as follows

$$\tilde{H}_0 = H - W \tag{2.6}$$

and so

$$H = \tilde{H}_0 + W \tag{2.7}$$

By construction we have

$$\tilde{H}_0|\tilde{\psi}_{z_0}\rangle = (H - W)|\tilde{\psi}_{z_0}\rangle = H|\tilde{\psi}_{z_0}\rangle - (H - E_0)|\tilde{\psi}_{z_0}\rangle = E_0|\tilde{\psi}_{z_0}\rangle \tag{2.8}$$

hence E_0 is an eigenvalue of $\tilde{H}_0$ with eigenvector $\tilde{\psi}_{z_0}$. Eq. (2.7) provides the decomposition of the Hamiltonian mentioned above where the spectrum of $\tilde{H}_0$ contains the embedded eigenvalue E_0 and W is a rank one perturbation.

Let $\tilde{P}_0$ be the orthogonal projection on the normalized eigenstate $\tilde{\psi}_{z_0}/\|\tilde{\psi}_{z_0}\|$ and let $\tilde{P}_{1b}$ denote the spectral projection on $\mathcal{H}_{pp} \cap (\tilde{P}_0\mathcal{H})^\perp$ (the pure point spectrum of $\tilde{H}_0$ orthogonal to $\tilde{\psi}_{z_0}$). Let Δ be a small interval around E_0 such that there exists a union of intervals Δ_*, disjoint from Δ, which contains a neighborhood of infinity and all of the thresholds of $\tilde{H}_0$ except possibly those in a small neighborhood of Δ. We shall use the notation g_Δ for a smoothed out characteristic function of the interval Δ. We take g_Δ to be a non-negative C^∞ function, identically equal to one on Δ, zero outside a neighborhood of Δ and such that the support of its derivatrives is small compared with the length of Δ. Define an operator $\tilde{P}_1$ as follows

$$\tilde{P}_1 = \tilde{P}_{1b} + g_{\Delta_*}(\tilde{H}_0)$$

where $g_{\Delta_*}(\tilde{H}_0)$ is the smoothed out characteristic function of the subset Δ_* of the spectrum of $\tilde{H}_0$. Furthermore, denote

$$\tilde{P}_c^\sharp = I - \tilde{P}_0 - \tilde{P}_1$$

hence $\tilde{P}_c^\sharp$ is a smoothed out projection on the spectrum of $\tilde{H}_0$ excluding all real eigenvalues, real neighborhoods of thresholds and infinity. Using the notation $\langle x\rangle = (1 + |x|^2)^{1/2}$ we define the following norm for the interaction term W

$$|||W||| = \|\langle x\rangle^{2\sigma} W g_\Delta(\tilde{H}_0)\| + \|\langle x\rangle^\sigma W g_\Delta(\tilde{H}_0)\langle x\rangle^\sigma\| + \|\langle x\rangle^\sigma W(\tilde{H}_0 + c)^{-1}\langle x\rangle^{-\sigma}\| \tag{2.9}$$

With these definitions in place we can state our main result.

Denote $H_0 = -\frac{1}{2}\Delta_x$ and let n be a natural number, $n \geq 3$. Our assumptions are:

Assumptions on the potential V:

(V1) The support of the potential V is compact.

(V2) The Hamiltonian H satisfies a Mourre estimate on some interval Δ_1 containing the point E_0

$$g_{\Delta_1}(H) i[H, A] g_{\Delta_1}(H) \geq \theta g_{\Delta_1}(H) + K$$

where $\theta > 0$ and K is compact.

(V3) V is H_0 bounded with H_0 bound smaller than 1.

(V4) $(x \cdot \nabla_x)^k V$ is H_0-bounded for all $k \leq n$.

Assumptions on the Gamow vector $\psi^G_{z_0}$:

(G1) The Gamow vector $\psi^G_{z_0}$ is locally a solution of Eq. (2.1) with $z_0 = E_0 - i\Gamma$ and $E_0, \Gamma > 0$.

(G2) In the region outside of the support of the potential the Gamow state solution is of the form

$$\psi^G_{z_0}(x) = F_1 \frac{e^{ik_0 r}}{r}, \qquad r = |x|$$

where F_1 is a constant (with appropriate dimensions such that the norm of $\tilde{\psi}_{z_0}$ is dimensionless).

Before stating the main theorem of this work we remark that the H^η regularity of $\tilde{H}_0$ means that $(\psi, (\tilde{H}_0 - z)^{-1}\phi)$ is in the Sobolev space H^η in the variable z close to the point E_0, where the states ψ and ϕ are taken to be in the dense set $\{\psi \in L^2; \langle x\rangle^\sigma \phi \in L^2\}$.

We can now state our main result:

Theorem A: *Let $H_0 = -\frac{1}{2}\Delta_x$ and assume that V is a potential satisfying conditions (V1)-(V4) ($H = H_0 + V$ is then self-adjoint on $L^2(R^3)$). Assume furthermore that the eigenvalue problem, Eq. (2.1), admits a Gamow state solution $\psi^G_{z_0}$ satisfying conditions (G1)-(G2) and construct a quasimode state $\tilde{\psi}_{z_0}$ as in Eq. (2.3) with the cutoff function $\chi(x)$ satisfying the conditions in Eq. (2.2). Define W and $\tilde{H}_0$ according to Eq. (2.5) and Eq. (2.6) respectively. Then, assuming that r_0 is sufficiently large and $\Gamma^2 r_0$ is sufficiently small we have:*

(α) $\tilde{H}_0$ is H^η regular for some $\eta > 0$ and there exists an open interval Δ with $E_0 \in \Delta$ but containing no other eigenvalue of $\tilde{H}_0$.

(β) The Hamiltonian $H = \tilde{H}_0 + W$ has no eigenvalues in Δ.

(γ) The spectrum of H is purely absolutely continuous in Δ and, for some $\sigma > 0$, we have the following local decay estimate for ϕ_0 with $\langle x\rangle^\sigma \phi_0 \in L^2$:

$$\|\langle x\rangle^{-\sigma} e^{-iHt} g_\Delta(H)\phi_0\|_2 \leq C\langle t\rangle^{-1-\eta} \|\langle x\rangle^\sigma \phi_0\|_2 \tag{2.10}$$

(δ) For $t \geq 0$ we have

$$e^{-iHt} g_\Delta(H)\phi_0 = (I + A_W)(e^{-i\omega_* t} a(0)\tilde{\psi}_{z_0} + e^{-i\tilde{H}_0 t}\phi_d(0)) + R(t) \tag{2.11}$$

where $a(0)$ and $\phi_d(0)$ are given by

$$a(0) = \|\tilde{\psi}_{z_0}\|^{-2}(\tilde{\psi}_{z_0}, g_\Delta(H)\phi_0)$$

and

$$\phi_d(0) = \tilde{P}_c^\sharp(1 - \tilde{P}_0) g_\Delta(H)\phi_0$$

A_W is an operator (the exact form of which will not be specified here; see [CS]) such that

$$\|A_W\| \le C|||W||| \tag{2.12}$$

where $|||W|||$ is the norm defined in Eq. (2.9) and, moreover, for r_0 large enough and $\Gamma^2 r_0$ small enough, and for each $\sigma > 0$, the norm $|||W|||$ can be made as small as desired.

The quantity ω_* is complex and can be written as

$$\omega_* = s_0 - i\gamma \tag{2.13}$$

where s_0 and γ can be found by solving a certain transcendental equation (see [CS]). The remainder $R(t)$ is estimated, for $\eta > 1$, to be $O(t^{-\eta+1})$ and for $\eta < 1$, to be $O(t^{-\eta-1})$.

From Eq. (2.10) we see that the local decay estimates are valid for the full evolution in a suitable interval around E_0. Eq. (2.11) exhibits explicitly the typical resonance behavior with a separation of the exponentially decaying contribution from the background contribution. In addition, the inverse power decay law for times long compared with the resonance lifetime is also included in the results of the theorem.

Theorem A is proved in the next section. The proof is centered on a theory of resonances developed by Soffer and Weinstein [SW] and Costin and Soffer [CS]. The Soffer-Weinstein-Costin theory deals with resonances obtained by a perturbation of embedded eigenvalues and Theorem A is essentially proved by showing that the assumptions of the Costin-Soffer theorem regarding the unperturbed Hamiltonian and the perturbation are satisfied by $\tilde{H}_0$ and W in the decomposition of H in Eq. (2.7).

3. Proof of Theorem A

3.1. The Soffer-Costin-Weinstein resonance theory

For the class of quantum mechanical models defined in Theorem A the construction of a quasimode by cutting a Gamow vector leads, as described in Section 2 above, to the definition of an associated problem of a perturbation of an embedded eigenvalue. The later satisfies the conditions of the theorems of Costin and Soffer [CS] and Soffer and Weinstein [SW] on resonances. We therefore begin the proof of Theorem A in this subsection with a statement of the Soffer-Costin-Weinstein resonance theory. In the subsequent subsections we show that $\tilde{H}_0$ and W satisfy the appropriate conditions of the Costin-Soffer theorem on resonances.

Consider a Hamiltonian H which can be written in the form of an unperturbed Hamiltonian H_0 plus a perturbation W

$$H = H_0 + W = H_0 + \epsilon W^{(\epsilon)}$$

where ϵ is a small parameter, taken to be the size of the perturbation in an appropriate norm (cf. e.g. Eq. (3.6)). Let H_0 be a self-adjoint operator on $\mathcal{H} = L^2(R^n)$, we assume that H_0 has a simple eigenvalue λ_0 with a normalized eigenvector ψ_0 i.e.:

$$H_0\psi_0 = \lambda_0\psi_0, \qquad \|\psi_0\| = 1 \tag{3.1}$$

In a similar way to the definitions in Section 2 above we define P_0 to be the orthogonal projection on ψ_0 and P_{1b} to be the spectral projection on $\mathcal{H}_{pp} \cap (P_0\mathcal{H})^\perp$ (the pure point spectrum of H_0 orthogonal to ψ_0). We take Δ to be a small interval around λ_0 such that there exists a union of intervals Δ_*, disjoint from Δ, which contains a neighborhood of infinity and all of the thresholds of H_0 except possibly those in a small neighborhood of Δ. We again use the notation g_Δ for a smoothed out characteristic function on the interval Δ (with the same properties as listed in Section 2, below Eq. (2.8)). We define P_1 as follows

$$P_1 = P_{1b} + g_{\Delta_*}(H_0) \tag{3.2}$$

and also

$$P_c^\sharp = I - P_0 - P_1 \tag{3.3}$$

hence $P_c^\sharp$ is a smoothed out projection on the spectrum of H_0 excluding all real eigenvalues, real neighborhoods of thresholds and infinity.

The Soffer-Costin-Weinstein theorem on resonances requires the following assumptions on the unperturbed Hamiltonian H_0 and the perturbation W:

Assumptions on H_0:

(I) H_0 a is self-adjoint operator with domain $\mathcal{D}$ dense in $L^2(R^n)$.

(II) λ_0 is a simple embedded eigenvalue of H_0. We denote the normalized eigenfunction corresponding to the eigenvalue λ_0 by ψ_0.

(III) There exists an open interval Δ with $\lambda_0 \in \Delta$ but containing no other eigenvalues of H_0.

(IV) (local decay estimate) Let $r > 1$. There exists $\sigma > 0$ such that if $\langle x\rangle^\sigma f \in L^2$ then

$$\|\langle x\rangle^{-\sigma} e^{-iH_0 t} P_c^\sharp f\|_2 \le C\langle t\rangle^{-r} \|\langle x\rangle^\sigma f\|_2 \tag{3.4}$$

(V) The L^2 norm of the operator $\langle x\rangle^\sigma (H_0 + c)^{-1} \langle x\rangle^{-\sigma}$ (where c is real) can be made sufficiently small by an appropriate choice of the number c.

In addition, we assume that H_0 is H^η regular, i.e., that $(\psi, (H_0 - z)^{-1}\phi)$ is in the Sobolev space H^η, of order η, in the z variable for z close to λ_0. The states ψ and ϕ are taken to be in the dense set $\{\psi \in L^2; \langle x\rangle^\sigma \phi \in L^2\}$.

In order to specify the assumptions on the perturbation W we make use of the definition of the norm $|||W|||$ in Eq. (2.9), with H_0 replacing $\tilde{H}_0$.

Assumptions on W:

(a) W is symmetric and $H = H_0 + W$ is self-adjoint on $\mathcal{D}$. There exists a constant $c \in R$ such that c lies in the resolvent sets of H_0 and H (remark: the constant c mentioned here can be used in assumption (V) above)

(b) For the same value of σ as in (IV) and (V) above we require

$$|||W||| < \infty \tag{3.5}$$

and

$$\|\langle x\rangle^{\sigma} W(H_0+c)^{-1}\langle x\rangle^{\sigma}\| < \infty \tag{3.6}$$

(c) (resonance condition) For a suitable choice of λ we have

$$\Gamma(\lambda,\epsilon) \equiv \Gamma(\lambda) \equiv \pi\epsilon^2 (W^{(\epsilon)}\psi_0, \delta(H_0-\lambda)(I-P_0)W^{(\epsilon)}\psi_0) \neq 0 \tag{3.7}$$

Assumption (c) here is a condition requiring that the Fermi golden rule does not vanish.

Soffer and Costin's improved version of the Soffer-Weinstein theorem on resonances states the following:

Theorem (Soffer-Costin-Weinstein): *Let H_0 satisfy conditions (I)-(V) and W satisfy conditions (a)-(c). We assume that ϵ is sufficiently small and either*

(VI) H_0 is H^η regular with $\eta > 1$

or

(VI') H_0 is H^η regular with $0 < \eta < 1$ and

$$\Gamma > C\epsilon^n, \qquad n \geq 2$$

with $\eta > (n-2)/n$.

Then:

(α') The perturbed Hamiltonian $H = H_0 + W$ has no eigenvalues in Δ.

(β') The spectrum of H is purely absolutely continuous in Δ and we have the following estimate for ϕ_0 with $\langle x\rangle^\sigma \phi_0 \in L^2$

$$\|\langle x\rangle^{-\sigma} e^{-iHt} g_\Delta(H)\phi_0\|_2 \leq C_\epsilon \langle t\rangle^{-1-\eta} \|\langle x\rangle^\sigma \phi_0\|_2 \tag{3.8}$$

(γ') For $t \geq 0$ we have

$$e^{-iHt} g_\Delta(H)\phi_0 = (I + A_W)(e^{-i\omega_* t} a(0)\psi_0 + e^{-iH_0 t}\phi_d(0)) + R(t) \tag{3.9}$$

where $a(0)$ and $\phi_d(0)$ are determined by the initial data, A_W is an operator (the exact form of which will not be specified here; see [CS]) such that

$$\|A_W\| \leq C\epsilon |||W||| \tag{3.10}$$

The remainder $R(t)$ is estimated, for $\eta > 1$, to be $R(t) = O(\epsilon^2 t^{-\eta+1})$ and for $\eta < 1$ (ϵ fixed), to be $R(t) = O(\Gamma^{-1} t^{-\eta-1})$. The quantity ω_* is complex and can be written as

$$\omega_* = s_0 - i\Gamma \tag{3.11}$$

where s_0 and Γ can be found by solving a certain transcendental equation (see [CS]).

Eq. (3.8) shows that the local decay estimates are valid for the evolution generated by H in a suitable interval around λ_0. Eq. (3.9) provides the typical resonance behavior with a separation of the exponentially decaying contribution from the background contribution. The theorem also includes the inverse power decay law for times long compared with the resonance lifetime.

In the next subsection we take up the task of showing that, under the assumptions of Theorem A, the decompostion $H = \tilde{H}_0 + W$ in Eq. (2.7) is such that $\tilde{H}_0$ satisfies conditions (I)-(V) and W satisfies conditions (a)-(c) above. The existence of a corresponding resonance is then a direct result of the Soffer-Costin theorem.

3.2 Conditions on the interaction W

In this subsection we prove that the interaction W satisfies conditions (a)-(c) of the Soffer-Costin theorem. It turns out that the main difficulty in the proof of the theorem is to show that it is possible to control the first term on the r.h.s. of Eq. (2.9). Other difficult intermediate results which are needed essentially follow from the estimate on this term. Hence, we start with an estimate of the operator norm $\|\langle x\rangle^{2\sigma} W g_\Delta(\tilde{H}_0)\|$. Denoting $H' = H - E_0$, we have

$$
\begin{aligned}
\|\langle x\rangle^{\sigma} W g_\Delta(\tilde{H}_0)\| &= \sup_{\|\xi\|\le 1, \|\zeta\|\le 1} |\langle \zeta|\langle x\rangle^{\sigma} W g_\Delta(\tilde{H}_0)|\xi\rangle| = \\
&= \sup_{\|\xi\|\le 1, \|\zeta\|\le 1} \left| \frac{\langle \zeta|\langle x\rangle^{\sigma} H'|\tilde{\psi}_{z_0}\rangle\langle H'\tilde{\psi}_{z_0}|g_\Delta(\tilde{H}_0)\xi\rangle}{\langle\tilde{\psi}_{z_0}|H'|\tilde{\psi}_{z_0}\rangle} \right| = \\
&= \frac{\left[\sup_{\|\zeta\|\le 1} |\langle \zeta|\langle x\rangle^{\sigma} H'|\tilde{\psi}_{z_0}\rangle|\right]\left[\sup_{\|\xi\|\le 1} |\langle H'\tilde{\psi}_{z_0}|g_\Delta(\tilde{H}_0)\xi\rangle|\right]}{|\langle\tilde{\psi}_{z_0}|H'|\tilde{\psi}_{z_0}\rangle|}
\end{aligned}
\tag{3.12}
$$

But

$$
\sup_{\|\zeta\|\le 1} \langle \zeta|\langle x\rangle^{\sigma} H'\tilde{\psi}_{z_0}\rangle| = \|\langle x\rangle^{\sigma}(H - E_0)\tilde{\psi}_{z_0}\| \tag{3.13}
$$

and

$$
\sup_{\|\xi\|\le 1} |\langle H'\tilde{\psi}_{z_0}|g_\Delta(\tilde{H}_0)\xi\rangle| = \sup_{\|\xi\|\le 1} |\langle g_\Delta(\tilde{H}_0)H'\tilde{\psi}_{z_0}|\xi\rangle| = \|g_\Delta(\tilde{H}_0)H'\tilde{\psi}_{z_0}\| \tag{3.14}
$$

and so

$$
\|\langle x\rangle^{\sigma} W g_\Delta(\tilde{H}_0)\| \le \frac{\|\langle x\rangle^{\sigma} H'\tilde{\psi}_{z_0}\| \, \|g_\Delta(\tilde{H}_0)H'\tilde{\psi}_{z_0}\|}{|\langle\tilde{\psi}_{z_0}|H'|\tilde{\psi}_{z_0}\rangle|} \tag{3.15}
$$

We will estimate, in turn, each factor in Eq. (3.15). First we note that, by definition of B_1, the potential V vanishes in $R^3\backslash B_1$ and, furthermore, by assumption (G2), in this region the Gamow vector solution has the form

$$\psi^G_{z_0}(x) = F_1 \frac{e^{ik_0 r}}{r}, \qquad r = |x|, \quad x \in R^3 \backslash B_1 \tag{3.16}$$

Thus we have

$$\begin{aligned} \|\langle x\rangle^\sigma H' \tilde{\psi}_{z_0}\|^2 &= \Gamma^2 \int_{B_1} d^3x\, \langle x\rangle^{2\sigma} |\psi^G_{z_0}(x)|^2 + \int_{B_2\backslash B_1} d^3x\, \langle x\rangle^{2\sigma} |\Delta'_x \tilde{\psi}_{z_0}(x)|^2 \\ &= \Gamma^2 \int_{B_1} d^3x\, \langle x\rangle^{2\sigma} |\psi^G_{z_0}(x)|^2 + 4\pi |F_1|^2 \int_{R_1}^{R_2} dr\, r^2 \langle r\rangle^{2\sigma} \left| \Delta'_r \left(\chi(r) \frac{e^{ik_0 r}}{r} \right) \right|^2 \end{aligned} \tag{3.17}$$

where $\Delta'_x = -\Delta_x - E_0$, $\Delta'_r = -\Delta_r - E_0$ and Δ_r is the radial part of the Laplacian. At this point we make use of the fact that the cutoff function $\chi(x)$ was chosen in such a way that all of its derivatives are continuous and bounded in $B_2\backslash B_1$. We have

$$\begin{aligned} &\int_{R_1}^{R_2} dr\, r^2 \langle r\rangle^{2\sigma} \left| \Delta'_r \left(\chi(r) \frac{e^{ik_0 r}}{r} \right) \right|^2 = \\ &= \int_{R_1}^{R_2} dr\, \langle r\rangle^{2\sigma} \left| \left(\frac{d^2\chi(r)}{dr^2} + 2|\mathrm{Im}\, k_0| \frac{d\chi(r)}{dr} \right) + i \mathrm{Re}\, k_0 \left(2\frac{d\chi(r)}{dr} + \frac{1}{r_0}\chi(r) \right) \right|^2 e^{2|Im k_0| r} \end{aligned}$$

Now, the assumption that r_0 is large implies that $|\mathrm{Im}\, k_0|$ is small. In fact, Eq. (3.45) below shows that $|\mathrm{Im}\, k_0| = o(r_0^{-1})$. Thus we obtain the result that there exist two constants $M_1(\sigma)$, $M_2(\sigma)$ such that

$$\int_{R_1}^{R_2} dr\, r^2 \langle r\rangle^{2\sigma} \left| \Delta'_r \left(\chi(r) \frac{e^{ik_0 r}}{r} \right) \right|^2 \leq M_1(\sigma) + (\mathrm{Re}\, k_0)^2 M_2(\sigma) \tag{3.18}$$

We conclude that

$$\|\langle x\rangle^\sigma H' \tilde{\psi}_{z_0}\|^2 \leq \Gamma^2 \langle R_1\rangle^{2\sigma} \int_{B_1} d^3x\, |\psi^G_{z_0}(x)|^2 + 4\pi |F_1|^2 (M_1(\sigma) + (\mathrm{Re}\, k_0)^2 M_2(\sigma)) \tag{3.19}$$

We turn now to the task of evaluating the second term in the numerator of Eq. (3.15). The starting point is the expression for $g_\Delta(\tilde{H}_0)$

$$g_\Delta(\tilde{H}_0) = \int_0^\infty d\lambda\, \chi_{\Delta,E_0}(\lambda) |\lambda\rangle\langle\lambda| \tag{3.20}$$

where χ_{Δ,E_0} is a smoothed out characteristic function on the interval Δ centered at E_0 and $|\lambda\rangle$ are generalized (continuous spectrum) eigenvectors of $\tilde{H}_0$, i.e. $\langle\lambda|\tilde{H}_0|f\rangle = \lambda\langle\lambda|f\rangle$. We have

$$\|g_\Delta(\tilde{H}_0) H' \tilde{\psi}_{z_0}\|^2 = \langle g_\Delta(\tilde{H}_0) H' \tilde{\psi}_{z_0} | g_\Delta(\tilde{H}_0) H' \tilde{\psi}_{z_0}\rangle = \int_0^\infty d\lambda\, \chi^2_{\Delta,E_0}(\lambda) |\langle \tilde{\psi}_{z_0} | H' | \lambda\rangle|^2 \tag{3.21}$$

We would like to calculate the quantity $\langle\tilde{\psi}_{z_0}|H'|\lambda\rangle$. For this purpose we recall that $\tilde{H}_0 = H - W$ and use wave operators intertwining H and $\tilde{H}_0$

$$\tilde{H}_0\Omega_{\pm} = \Omega_{\pm}H \tag{3.22}$$

Denoting the continuous spectrum generalized eigenvectors of H by $|\lambda\rangle_H$ we have

$$\langle f|\tilde{H}_0|\lambda\rangle = \langle f|\tilde{H}_0\Omega_-|\lambda\rangle_H = \langle f|\Omega_- H|\lambda\rangle_H = \lambda\langle f|\Omega_-|\lambda\rangle_H = \lambda\langle f|\lambda\rangle$$

with $|\lambda\rangle = \Omega_-|\lambda\rangle_H$. Eq. (3.21) can then be written

$$\|g_\Delta(\tilde{H}_0)H'\tilde{\psi}_{z_0}\|^2 = \int_0^\infty d\lambda\, \chi^2_{\Delta,E_0}(\lambda)|\langle\tilde{\psi}_{z_0}|H'\Omega_-|\lambda\rangle_H|^2 \tag{3.23}$$

For a Hamiltonian $H_\alpha = H_0 + \alpha|\psi\rangle\langle\psi|$ with a rank one perturbation we can use the Aronszajn-Krein formulas which can be found, for example, in [AK]: The wave operators are given by

$$\Omega_{\pm}(H_\alpha, H_0) = I - \frac{1}{2\pi i}\int_{R^+} \frac{\alpha}{1+\alpha F_{\mp}(\lambda)}(H_0 - \lambda \pm i\epsilon)^{-1}|\psi\rangle\langle\psi| \\ [(H_0 - \lambda - i\epsilon)^{-1} - (H_0 - \lambda + i\epsilon)^{-1}]d\lambda$$

with

$$F_{\pm}(\lambda) = \langle\psi|(H_0 - \lambda \mp i\epsilon)^{-1}|\psi\rangle$$

Note that, when using these expressions for the wave operators, we are considering here H to be the unperturbed Hamiltonian while H_α, the perturbed Hamiltonian, is taken to be $\tilde{H}_0$. In order to use these expressions we rewrite the perturbation W as

$$W = \frac{(H-E_0)|\tilde{\psi}_{z_0}\rangle\langle\tilde{\psi}_{z_0}|(H-E_0)}{\langle\tilde{\psi}_{z_0}|(H-E_0)|\tilde{\psi}_{z_0}\rangle} = \frac{\langle\tilde{\psi}_{z_0}|{H'}^2|\tilde{\psi}_{z_0}\rangle}{\langle\tilde{\psi}_{z_0}|H'|\tilde{\psi}_{z_0}\rangle}P_{\hat{\psi}_{z_0}} = -\alpha P_{\hat{\psi}_{z_0}} \tag{3.24}$$

where $P_{\hat{\psi}_{z_0}}$ is the projection operator on $|\hat{\psi}_{z_0}\rangle = H'|\tilde{\psi}_{z_0}\rangle$ and

$$\alpha = -\frac{\langle\tilde{\psi}_{z_0}|{H'}^2|\tilde{\psi}_{z_0}\rangle}{\langle\tilde{\psi}_{z_0}|H'|\tilde{\psi}_{z_0}\rangle} \tag{3.25}$$

With Eq. (3.24) and (3.25) we can write $\tilde{H}_0 = H - W = H + \alpha P_{\hat{\psi}_{z_0}}$ and use the expression for the wave operator from above to obtain

$$\begin{aligned}\Omega_-(\tilde{H}_0, H) = I - \frac{1}{2\pi i}\int_{R^+} &\frac{\alpha}{1+\alpha F_+(\lambda)}(H-\lambda-i\epsilon)^{-1}\frac{|\hat{\psi}_{z_0}\rangle\langle\hat{\psi}_{z_0}|}{\langle\hat{\psi}_{z_0}|\hat{\psi}_{z_0}\rangle} \\ &[(H-\lambda-i\epsilon)^{-1} - (H-\lambda+i\epsilon)^{-1}]d\lambda = \\ = I - \frac{1}{2\pi i}\int_{R^+} &\frac{\alpha}{1+\alpha F_+(\lambda)}(H-\lambda-i\epsilon)^{-1}\frac{H'|\tilde{\psi}_{z_0}\rangle\langle\tilde{\psi}_{z_0}|H'}{\langle\tilde{\psi}_{z_0}|{H'}^2|\tilde{\psi}_{z_0}\rangle} \\ &[(H-\lambda-i\epsilon)^{-1} - (H-\lambda+i\epsilon)^{-1}]d\lambda\end{aligned} \tag{3.26}$$

with

$$F_+(\lambda) = \frac{\langle\hat{\psi}_{z_0}|(H-\lambda-i\epsilon)^{-1}|\hat{\psi}_{z_0}\rangle}{\langle\hat{\psi}_{z_0}|\hat{\psi}_{z_0}\rangle} \tag{3.27}$$

applying the wave operator to $|\lambda\rangle_H$ we obtain

$$\Omega_-|\lambda\rangle_H = |\lambda\rangle_H - \frac{\alpha}{1+\alpha F_+(\lambda)}(H-\lambda-i\epsilon)^{-1}\frac{H'|\tilde{\psi}_{z_0}\rangle\langle\tilde{\psi}_{z_0}|\lambda\rangle_H(\lambda-E_0)}{\langle\tilde{\psi}_{z_0}|H'^2|\tilde{\psi}_{z_0}\rangle} \tag{3.28}$$

With the help of Eq. (3.27) and Eq. (3.28) we get

$$\begin{aligned}\langle\tilde{\psi}_{z_0}|H'|\lambda\rangle &= \langle\tilde{\psi}_{z_0}|H'\Omega_-|\lambda\rangle_H = \\ &= (\lambda-E_0)\langle\tilde{\psi}_{z_0}|\lambda\rangle_H\left[1-\frac{\alpha F_+(\lambda)}{1+\alpha F_+(\lambda)}\right] = (\lambda-E_0)\langle\tilde{\psi}_{z_0}|\lambda\rangle_H\left[\frac{1}{1+\alpha F_+(\lambda)}\right]\end{aligned} \tag{3.29}$$

In order to continue we need to obtain an expression for $\alpha F_+(\lambda)$. Writing this quantity explicitly we get

$$\begin{aligned}\alpha F_+(\lambda) &= -\frac{\langle\tilde{\psi}_{z_0}|H'^2|\tilde{\psi}_{z_0}\rangle}{\langle\tilde{\psi}_{z_0}|H'|\tilde{\psi}_{z_0}\rangle}\,\frac{\langle\tilde{\psi}_{z_0}|H'(H-\lambda-i\epsilon)^{-1}H'|\tilde{\psi}_{z_0}\rangle}{\langle\tilde{\psi}_{z_0}|H'^2|\tilde{\psi}_{z_0}\rangle} = \\ &= -\langle\tilde{\psi}_{z_0}|H'|\tilde{\psi}_{z_0}\rangle^{-1}\int_0^\infty d\lambda'\,\frac{(\lambda'-E_0)^2|\langle\tilde{\psi}_{z_0}|\lambda'\rangle_H|^2}{\lambda'-\lambda-i\epsilon}\end{aligned} \tag{3.30}$$

hence

$$\begin{aligned}1+\alpha F_+(\lambda) &= -\langle\tilde{\psi}_{z_0}|H'|\tilde{\psi}_{z_0}\rangle^{-1}\left(\int_0^\infty d\lambda'\,\frac{(\lambda'-E_0)^2|\langle\tilde{\psi}_{z_0}|\lambda'\rangle_H|^2}{\lambda'-\lambda-i\epsilon} - \langle\tilde{\psi}_{z_0}|H'|\tilde{\psi}_{z_0}\rangle\right) = \\ &= -\langle\tilde{\psi}_{z_0}|H'|\tilde{\psi}_{z_0}\rangle^{-1}\int_0^\infty d\lambda'\left(\frac{(\lambda'-E_0)^2}{\lambda'-\lambda-i\epsilon}-(\lambda'-E_0)\right)|\langle\tilde{\psi}_{z_0}|\lambda'\rangle_H|^2 = \\ &= -\langle\tilde{\psi}_{z_0}|H'|\tilde{\psi}_{z_0}\rangle^{-1}(\lambda-E_0)\int_0^\infty d\lambda'\,\frac{\lambda'-E_0}{\lambda'-\lambda-i\epsilon}|\langle\tilde{\psi}_{z_0}|\lambda'\rangle_H|^2\end{aligned} \tag{3.31}$$

Inserting the r.h.s. of Eq. (3.31) into Eq. (3.29) we have

$$\begin{aligned}\langle\tilde{\psi}_{z_0}|H'|\lambda\rangle &= (\lambda-E_0)\langle\tilde{\psi}_{z_0}|\lambda\rangle_H\left[\frac{1}{1+\alpha F_+(\lambda)}\right] = \\ &= -\langle\tilde{\psi}_{z_0}|\lambda\rangle_H\langle\tilde{\psi}_{z_0}|H'|\tilde{\psi}_{z_0}\rangle\left(\int_0^\infty d\lambda'\,\frac{\lambda'-E_0}{\lambda'-\lambda-i\epsilon}|\langle\tilde{\psi}_{z_0}|\lambda'\rangle_H|^2\right)^{-1}\end{aligned} \tag{3.32}$$

Combining Eq. (3.32) with Eq. (3.21) we finally get

$$\|g_\Delta(\tilde H_0)H'\tilde\psi_{z_0}\|^2 = \int_0^\infty d\lambda\,\chi^2_{\Delta,E_0}(\lambda)|\langle\tilde\psi_{z_0}|H'|\lambda\rangle|^2 = \\ = \langle\tilde\psi_{z_0}|H'|\tilde\psi_{z_0}\rangle^2 \int_0^\infty d\lambda\,\chi^2_{\Delta,E_0}(\lambda)|\langle\tilde\psi_{z_0}|\lambda\rangle_H|^2|G(\lambda)|^{-2} \tag{3.33}$$

where

$$G(\lambda) \equiv \int_0^\infty d\lambda'\,\frac{\lambda'-E_0}{\lambda'-\lambda-i\epsilon}|\langle\tilde\psi_{z_0}|\lambda'\rangle_H|^2 \tag{3.34}$$

We prove below that for a properly chosen interval Δ, centered at E_0, and for some constant $C>0$, the following inequality holds for any λ in the support of χ_{Δ,E_0}

$$|G(\lambda)| \geq C\|\tilde\psi_{z_0}\|^2, \qquad \lambda\in\mathrm{supp}[\chi_{\Delta,E_0}] \tag{3.35}$$

We note that for $\lambda=E_0$ we have $G(E_0)=\|\tilde\psi_{z_0}\|^2$ and by continuity in λ there always exists an interval centered at E_0 for which Eq. (3.35) holds. Hence, Eq. (3.35) should hold in the sense that the interval Δ is independent of the parameter r_0 controling the estimates. Assuming for the moment that the inequality in Eq. (3.35) is valid we can use it to simplify Eq. (3.33) with the result

$$\|g_\Delta(\tilde H_0)H'\tilde\psi_{z_0}\| \leq C|\langle\tilde\psi_{z_0}|H'|\tilde\psi_{z_0}\rangle|\|\tilde\psi_{z_0}\|^{-1} \tag{3.36}$$

Eq. (3.15), together with Eq. (3.19) and (3.36) imply that

$$\|\langle x\rangle^\sigma W g_\Delta(\tilde H_0)\| \leq \frac{\|\langle x\rangle^\sigma H'\tilde\psi_{z_0}\|\,\|g_\Delta(\tilde H_0)H'\tilde\psi_{z_0}\|}{|\langle\tilde\psi_{z_0}|H'|\tilde\psi_{z_0}\rangle|} \\ \leq C\|\tilde\psi_{z_0}\|^{-1}\left(\Gamma^2\langle R_1\rangle^{2\sigma}\int_{B_1} d^3x\,|\psi^G_{z_0}(x)|^2 + 4\pi|F_1|^2(M_1(\sigma)+(\mathrm{Re}\,k_0)^2M_2(\sigma))\right)^{1/2} \\ \leq C\left(\frac{\Gamma^2\langle R_1\rangle^{2\sigma}\int_{B_1} d^3x\,|\psi^G_{z_0}(x)|^2 + 4\pi|F_1|^2(M_1(\sigma)+(\mathrm{Re}\,k_0)^2M_2(\sigma))}{\int_{B_1} d^3x\,|\psi^G_{z_0}(x)|^2}\right)^{1/2} \tag{3.37}$$

where in order to obtain the second inequality on the r.h.s. of Eq. (3.37) we have used the relation

$$\|\tilde\psi_{z_0}\|^2 = \int_{B_1} d^3x\,|\psi^G_{z_0}(x)|^2 + \int_{B_2\setminus B_1} d^3x\,|\tilde\psi_{z_0}(x)|^2 > \int_{B_1} d^3x\,|\psi^G_{z_0}(x)|^2 \tag{3.38}$$

We will presently show that

$$\int_{B_1} d^3x\,|\psi^G_{z_0}(x)|^2 = 4\pi|F_1|^2 r_0 e^{\frac{R_1}{r_0}} + o(r_0^{-1}) \tag{3.39}$$

where r_0 is the parameter defined in Eq. (2.4). Assuming that Eq. (3.39) is true we immediately obtain the desired estimate since, recalling that $\Gamma r_0 = Re[k_0]$, we find that for fixed r_0 large enough the r.h.s. of Eq. (3.37) is $o(\Gamma r_0^{1/2})$.

We are left with the task of proving the validity of Eq. (3.39) and Eq. (3.35). We start with Eq. (3.39). Denote $H' = H - E_0$ and consider the quantity $\langle\tilde{\psi}_{z_0}|H'|\tilde{\psi}_{z_0}\rangle$ (identical to the denominator of the interaction W; see Eq. (2.5)). We have

$$\begin{aligned}\langle\tilde{\psi}_{z_0}|H'|\tilde{\psi}_{z_0}\rangle &= -i\Gamma\int_{B_1} d^3x\,|\psi^G_{z_0}(x)|^2 \\ &- 4\pi|F_1|^2\int_{R_1}^{R_2} dr\, e^{-i\bar{k}_0 r}\chi(r)\left[\frac{d^2\chi(r)}{dr^2} + 2ik_0\frac{d\chi(r)}{dr} + i\Gamma\chi(r)\right]e^{ik_0 r} = \\ &= -i\Gamma\int_{B_1} d^3x\,|\psi^G_{z_0}(x)|^2 \\ &- 4\pi|F_1|^2\int_{R_1}^{R_2} dr\,\chi(r)\left[\frac{d^2\chi(r)}{dr^2} + 2ik_0\frac{d\chi(r)}{dr} + i\Gamma\chi(r)\right]e^{2|Imk_0|r}\end{aligned} \tag{3.40}$$

We will need below an estimated value for k_0. Recall that $z_0 = E_0 - i\Gamma$ and assume that $\Gamma/E_0 << 1$. It will be shown presently that this is equivalent to the assumption that the parameter r_0 defined in Eq. (2.4) can be as large as we want. Writing z_0 in polar coordinates as $z_0 = r_{z_0}e^{-i\theta_{z_0}}$ we have

$$z_0 = E_0 - i\Gamma = r_{z_0}(\cos(\theta_{z_0}) - i\sin(\theta_{z_0})) \tag{3.41}$$

and so

$$k_0 = \sqrt{z_0} = r_{z_0}^{\frac{1}{2}}(\cos(\theta_{z_0}/2) - i\sin(\theta_{z_0}/2)) \tag{3.42}$$

The condition $\Gamma/E_0 << 1$ implies that $\theta_{z_0} \approx \Gamma/E_0 << 1$ and $r_{z_0}^{1/2} = \sqrt{E_0} + o(\Gamma^2/E_0^2)$. We have also $\cos(\theta_{z_0}/2) = 1 + o(\theta_{z_0}^2)$ and $\sin(\theta_{z_0}/2) \approx \theta_{z_0}/2 + o(\theta_{z_0}^3)$. Eq. (3.42) then implies that

$$k_0 = \sqrt{E_0}(1 - i\frac{\Gamma}{2E_0} + o(\Gamma^2/E_0^2)) = \sqrt{E_0} - i\frac{\Gamma}{2\sqrt{E_0}} + o(\Gamma^2/E_0^{3/2}) \tag{3.43}$$

With Eq. (3.43), the parameter r_0 defined in Eq. (2.4) can be approximated as follows

$$r_0 = \frac{Re[k_0]}{\Gamma} \approx \frac{\sqrt{E_0}}{\Gamma} \approx \frac{1}{2|Im[k_0]|} \tag{3.44}$$

and we find that

$$k_0 = \Gamma r_0 - i\frac{1}{2r_0} + io(r_0^{-2}) \tag{3.45}$$

where the quantity $o(r_0^{-2})$ is real. Using this expression for k_0 in Eq. (3.40) we get

$$
\begin{aligned}
&(\tilde{\psi}_{z_0}, H'\tilde{\psi}_{z_0}) = -i\Gamma \int_{B_1} d^3x\, |\psi^G_{z_0}(x)|^2 - 4\pi |F_1|^2 \times \\
&\int_{R_1}^{R_2} dr\, \chi(r) \left[\frac{d^2\chi(r)}{dr^2} + 2i \left(\Gamma r_0 - i \frac{1}{2r_0} + io(r_0^{-2}) \right) \frac{d\chi(r)}{dr} + i\Gamma \chi(r) \right] e^{\frac{r}{r_0} + o(r_0^{-2}) r} = \\
&= -i\Gamma \int_{B_1} d^3x\, |\psi^G_{z_0}(x)|^2 \\
&- 4\pi |F_1|^2 \int_{R_1}^{R_2} dr\, \chi(r) \left[\frac{d^2\chi(r)}{dr^2} + \frac{1}{r_0} \frac{d\chi(r)}{dr} + o(r_0^{-2}) \frac{d\chi(r)}{dr} \right] e^{\frac{r}{r_0}} + o(r_0^{-2}) \\
&- i4\pi |F_1|^2 \Gamma \int_{R_1}^{R_2} dr\, \chi(r) \left[2r_0 \frac{d\chi(r)}{dr} + \chi(r) \right] e^{\frac{r}{r_0}} + o(r_0^{-1})
\end{aligned}
\tag{3.46}
$$

Since $Im\langle \tilde{\psi}_{z_0} | H' | \tilde{\psi}_{z_0} \rangle = 0$ we expect that

$$
\int_{B_1} d^3x\, |\psi^G_{z_0}(x)|^2 + 4\pi |F_1|^2 \int_{R_1}^{R_2} dr\, \left[2r_0 \frac{d\chi(r)}{dr} + \chi(r) \right] e^{\frac{r}{r_0}} + o(r_0^{-1}) = 0 \tag{3.47}
$$

but the second integral in Eq. (3.47) can be evaluated and we find

$$
\begin{aligned}
\int_{R_1}^{R_2} dr\, \chi(r) \left[2r_0 \frac{d\chi(r)}{dr} + \chi(r) \right] e^{\frac{r}{r_0}} &= r_0 \int_{R_1}^{R_2} dr\, \frac{d}{dr} \left[\chi^2(r) e^{\frac{r}{r_0}} \right] = \\
&= r_0 [\chi^2(r) e^{\frac{r}{r_0}}] \Big|_{R_1}^{R_2} = -r_0 e^{\frac{R_1}{r_0}}
\end{aligned}
\tag{3.48}
$$

where we assume that $\chi(R_2) = 0$ (see Eq. (2.2)). Taken together, Eq. (3.47) and Eq. (3.48) provides us with Eq. (3.39).

We turn now to the proof of the inequality, Eq. (3.35). We have

$$
\begin{aligned}
G(\lambda) &= \int_0^\infty d\lambda'\, \frac{\lambda' - E_0}{\lambda' - \lambda - i\epsilon} |\langle \tilde{\psi}_{z_0} | \lambda' \rangle_H|^2 = \int_0^\infty d\lambda'\, \frac{(\lambda' - \lambda + \lambda - E_0)}{\lambda' - \lambda - i\epsilon} |\langle \tilde{\psi}_{z_0} | \lambda' \rangle_H|^2 = \\
&= (\lambda - E_0) [\int_0^\infty d\lambda'\, \frac{1}{\lambda' - \lambda - i\epsilon} |\langle \tilde{\psi}_{z_0} | \lambda' \rangle_H|^2] + \|\tilde{\psi}_{z_0}\|^2
\end{aligned}
\tag{3.49}
$$

Eq. (3.35) will hold if there exists a constant $0 < C < 1$ for which

$$
\left| (\lambda - E_0) \left[\int_0^\infty d\lambda'\, \frac{1}{\lambda' - \lambda - i\epsilon} |\langle \tilde{\psi}_{z_0} | \lambda' \rangle_H|^2 \right] \right| \leq C \|\tilde{\psi}_{z_0}\|^2, \qquad \lambda \in \mathrm{supp}[\chi_{\Delta, E_0}] \tag{3.50}
$$

Hence, we need to prove that the interval Δ can be chosen so that the inequality in Eq. (3.50) is satisfied. In order to show this we first start by proving that for some interval Δ_1, centered at E_0, the following inequality holds for some $C_1 > 0$

$$\int_0^\infty dt\, \left|\langle\tilde{\psi}_{z_0}|e^{-iHt}g^2_{\Delta_1}(H)|\tilde{\psi}_{z_0}\rangle\right| < C_1\|\tilde{\psi}_{z_0}\|^2 \tag{3.51}$$

The inequality, Eq. (3.51) can be proved by using a strong low velocity estimate. We have the following theorem (see [SS], [GS] and also [GD]):

Theorem (Strong low velocity estimate): Assume that the Hamiltonian H satisfies the Mourre estimate in condition (V2) above and that for any $\chi \in C_0^\infty(R)$ and $n \geq 2$ we have (for the definition of $ad^k_A(B)$ see Eq. (3.60))

$$\|ad^k_A\chi(H)\| < \infty, \qquad k \leq n$$

Suppose that $\lambda_0 > 0$, $\chi \in C_0^\infty(R)$, and $\text{supp}[\chi] \subset [\lambda_0/2,\infty)\backslash\sigma_{pp}(H)$. Then, for any $n-1 > s \geq 0$,

$$\|1_{[0,\lambda_0]}\left(\frac{x^2}{t^2}\right)\chi(H)e^{-iHt}\phi\| \leq C\langle t\rangle^{-s}\|\langle A\rangle^s_-\phi\|$$

where $\langle r\rangle_- = \langle r\rangle$ for $r < 0$ and $\langle r\rangle_- = 1$ for $r \geq 0$. The corollary below follows from the this theorem

Corollary: *Under the assumptions of the theorem above, for any $n-1 > s \geq 0$, we have*

$$\|1_{[0,\lambda_0]}\left(\frac{x^2}{t^2}\right)\chi(H)e^{-iHt}\phi\| \leq C\langle t\rangle^{-s}\|\langle x\rangle^s\phi\|$$

Assumptions (V2)-(V4) on the potential V assures us that the conditions of the strong low velocity estimate above are satisfied. Choose $0 < \lambda_0$ such that $[0,\lambda_0]\cap\Delta_1 = \emptyset$, let $\alpha = \sup[\text{supp}[\tilde{\psi}_{z_0}]]$ and choose t_0 such that $\alpha^2/t_0^2 < \lambda_0$. For such a choice of t_0 and λ_0 we have

$$1_{[0,\lambda_0]}\left(\frac{x^2}{t^2}\right)|\tilde{\psi}_{z_0}\rangle = |\tilde{\psi}_{z_0}\rangle, \qquad t \geq t_0 \tag{3.52}$$

Using this identity we have

$$\begin{aligned}\int_{t_0}^\infty dt\left|\langle\tilde{\psi}_{z_0}|e^{-iHt}g^2_{\Delta_1}(H)|\tilde{\psi}_{z_0}\rangle\right| &= \int_{t_0}^\infty dt\left|\langle\tilde{\psi}_{z_0}|1_{[0,\lambda_0]}\left(\frac{x^2}{t^2}\right)e^{-iHt}g^2_{\Delta_1}(H)|\tilde{\psi}_{z_0}\rangle\right| \\ &\leq \|\tilde{\psi}_{z_0}\|\int_{t_0}^\infty dt\left\|1_{[0,\lambda_0]}\left(\frac{x^2}{t^2}\right)g^2_{\Delta_1}(H)\tilde{\psi}_{z_0}(t)\right\|\end{aligned} \tag{3.53}$$

Since $\tilde{\psi}_{z_0}$ is compactly supported and $\alpha = \sup[\text{supp}[\tilde{\psi}_{z_0}]]$, the corollary above to the strong low velocity estimate implies that, for any $s \geq 0$

$$\left\| 1_{[0,\lambda_0]}\left(\frac{x^2}{t^2}\right) g_{\Delta_1}^2(H)\tilde{\psi}_{z_0}(t)\right\| \leq C\langle \alpha/t\rangle^s \|\tilde{\psi}_{z_0}\|, \qquad t \geq t_0 \tag{3.54}$$

Hence, for any $1 < s$ we have

$$\int_{t_0}^{\infty} \|1_{[0,\lambda_0]}\left(\frac{x^2}{t^2}\right) g_{\Delta_1}^2(H)e^{-iHt}\tilde{\psi}_{z_0}\| \leq C\frac{\alpha}{s-1}\left(\alpha/t_0\right)^{s-1}\|\tilde{\psi}_{z_0}\| = C_2\|\tilde{\psi}_{z_0}\| \tag{3.55}$$

where the constant C_2 depends also on the H-bounds of assumption (V4). Since for all values of t we have

$$\left|\langle\tilde{\psi}_{z_0}|e^{-iHt}g_{\Delta_1}^2(H)|\tilde{\psi}_{z_0}\rangle\right| \leq \|\tilde{\psi}_{z_0}\|^2$$

we conclude that there exists $C_1 > 0$ such that the inequality Eq. (3.51) is satisfied. We note that, since the support of $\tilde{\psi}_{z_0}$ is compact and is determined by the cutoff function $\chi(x)$, the constant C_1 is independent of the parameters Γ and r_0 (or Γ and E_0).

We proceed by observing that Eq. (3.51) implies the following inequality

$$\begin{aligned}\left|\int_0^{\infty} d\lambda'\,\frac{1}{\lambda'-\lambda-i\epsilon}|\langle\tilde{\psi}_{z_0}|g_{\Delta_1}(H)|\lambda'\rangle_H|^2\right| &= \left|\langle\tilde{\psi}_{z_0}|g_{\Delta_1}(H)\frac{1}{H-\lambda-i\epsilon}g_{\Delta_1}(H)|\tilde{\psi}_{z_0}\rangle\right| = \\ = \left|\int_0^{\infty} dt\,\langle\tilde{\psi}_{z_0}|e^{-i(H-\lambda-i\epsilon)t}g_{\Delta_1}^2(H)|\tilde{\psi}_{z_0}\rangle\right| &\leq \int_0^{\infty} dt\,\left|\langle\tilde{\psi}_{z_0}|e^{-iHt}g_{\Delta_1}^2(H)|\tilde{\psi}_{z_0}\rangle\right| \leq C_1\|\tilde{\psi}_{z_0}\|^2\end{aligned} \tag{3.56}$$

Suppose that the length of the interval Δ_1 is l_{Δ_1}. Choose a second interval Δ_2 centered at E_0 and having a length $l_{\Delta_2} = l_{\Delta_1}/2$. In this case there exists a constant $C_3 > 0$ such that, for every $\lambda \in \Delta_2$, and every $\lambda' \in \mathrm{supp}[1 - g_{\Delta_1}(H)]$ we have $|\lambda - \lambda'| > 1/C_3$ and so

$$\left|\int_0^{\infty} d\lambda'\,\frac{1}{\lambda'-\lambda-i\epsilon}|\langle\tilde{\psi}_{z_0}|(1-g_{\Delta_1}(H))^2|\lambda'\rangle_H|^2\right| \leq C_3\|\tilde{\psi}_{z_0}\|^2 \tag{3.57}$$

Combining Eq. (3.56) and Eq. (3.57) we get

$$\left|\int_0^{\infty} d\lambda'\,\frac{1}{\lambda'-\lambda-i\epsilon}|\langle\tilde{\psi}_{z_0}|\lambda'\rangle_H|^2\right| \leq C_4\|\tilde{\psi}_{z_0}\|^2, \qquad \lambda \in \Delta_2 \tag{3.58}$$

Finally, we choose an interval $\Delta_3 \subset \Delta_2$, centered at E_0, with length $l_{\Delta_3} < 1/C_4$. If we now take the interval Δ such that $\mathrm{supp}[\chi_{\Delta,E_0}] \subset \Delta_3$ the inequality Eq. (3.50) is satisfied and hence also the inequality Eq. (3.35). Note that the choice of Δ is independent of r_0 and, hence, the estimate of the term $\|\langle x\rangle^{2\sigma}Wg_{\Delta}(\tilde{H}_0)\|$ is completed.

The rest of the estimates needed in order to show that conditions (a)-(c) hold for W are easier to prove. A bound on the last term on the r.h.s. of Eq. (2.9) and on the l.h.s. of Eq. (3.6) is a consequence of a combination of the observation that, as shown in Subsection 3.4 below, $\tilde{H}_0$ satisfies condition (V) of the Costin-Soffer theorem, the fact that

W is compact and that, as an integral operator, W has a kernel with compact support on $R^3 \times R^3$. Indeed we find that

$$\langle x\rangle^{\sigma} W(\tilde{H}_0 + c)^{-1}\langle x\rangle^{-\sigma} = [\langle x\rangle^{\sigma} W\langle x\rangle^{-\sigma}][\langle x\rangle^{\sigma}(\tilde{H}_0 + c)^{-1}\langle x\rangle^{-\sigma}]$$

and

$$\langle x\rangle^{\sigma} W(\tilde{H}_0 + c)^{-1}\langle x\rangle^{\sigma} = [\langle x\rangle^{\sigma} W\langle x\rangle^{\sigma}][\langle x\rangle^{-\sigma}(\tilde{H}_0 + c)^{-1}\langle x\rangle^{\sigma}]$$

are both a product of a bounded operator and a compact operator.

In order to see that the Fermi golden rule condition in Eq. (3.7) is satisfied we first note that

$$W|\tilde{\psi}_{z_0}\rangle = H'|\tilde{\psi}_{z_0}\rangle$$

and so

$$\begin{aligned}\Gamma(\lambda,\epsilon) = \pi\epsilon^2\langle\tilde{\psi}_{z_0}|W^{(\epsilon)}\delta(\tilde{H}_0-\lambda)(I-P_0)W^{(\epsilon)}|\tilde{\psi}_{z_0}\rangle\|\tilde{\psi}_{z_0}\|^{-2} =\\ = \pi\langle\tilde{\psi}_{z_0}|H'\delta(\tilde{H}_0-\lambda)(I-P_0)H'|\tilde{\psi}_{z_0}\rangle\|\tilde{\psi}_{z_0}\|^{-2} = \pi|\langle\tilde{\psi}_{z_0}|H'|\lambda\rangle|^2\|\tilde{\psi}_{z_0}\|^{-2}\end{aligned}$$

where $|\lambda\rangle$ is a continuous spectrum generalized eigenvector for $\tilde{H}_0$. With the help of Eq. (3.32) we obtain

$$\Gamma(\lambda) = \pi|\langle\tilde{\psi}_{z_0}|\lambda\rangle_H\langle\tilde{\psi}_{z_0}|H'|\tilde{\psi}_{z_0}\rangle(G(\lambda))^{-1}|^2\|\tilde{\psi}_{z_0}\|^{-2}$$

resulting in $\Gamma > 0$, which shows that the Fermi golden rule condition in Eq. (3.7) is satisfied. Note that, in particular, Eq. (3.34) implies the following simple formula for $\lambda = E_0$

$$\Gamma(E_0) = \pi|\langle\tilde{\psi}_{z_0}|E_0\rangle_H\langle\tilde{\psi}_{z_0}|H'|\tilde{\psi}_{z_0}\rangle|^2\|\tilde{\psi}_{z_0}\|^{-6}$$

3.3 Comments on the estimates of $\|\langle x\rangle^{\sigma} W g_{\Delta}(\tilde{H}_0)\|$.

We discuss here briefly the basic mechanism enabling the proof of the estimates above. A first, perhaps surprising, observation is that the r.h.s. of the inequality in Eq. (3.37) is made small by the integration on B_1 where we have $\chi(x) = 1$, hence the exact details of the cutoff function are irrelevant.

A second essential observation is that Eq. (3.39) is the fundamental identity which is needed in order to make the r.h.s. of Eq. (3.37) small. The l.h.s. of this identity appears in both the numerator and denominator of Eq. (3.37). The fact that in the numerator this integral is multiplied by Γ^2, and that Γr_0 is bounded (in fact vanishing) r_0 goes to infinity, is crucial.

The last fundamental observation concerns the origin of the identity Eq. (3.39). This identity results from the fact that, since the quantity $\langle\tilde{\psi}_{z_0}|H'|\tilde{\psi}_{z_0}\rangle$ (which is also the denominator of the interaction W) is real, there must be a mechanism for the cancelation of the first term on the r.h.s. of Eq. (3.40) which is purely imaginary. The mechanism for the cancelation of all imaginary terms on the r.h.s. of Eq. (3.40) provides us with the crucial identity Eq. (3.39). It is readily seen that the mechanism mentioned above depends neither on the exact details of the cutoff function nor on the details of the potential, as long as the support of the potential is compact and the Gamow vector is in the form of Eq. (3.16) in the region outside of the support of the potential (there is another possible form for the Gamow state solution in the region outside of the support of the potential; see Section 5 below).

We note that the quantity $\langle\tilde{\psi}_{z_0}|H'|\tilde{\psi}_{z_0}\rangle$ is well defined and real precisely because the multiplication of the Gamow vector $\psi^G_{z_0}(x)$ by the cutoff function $\chi(x)$ renders the resulting $\tilde{\psi}_{z_0}(x)$ an element of the Hilbert space (an L^2 function). Hence, it is the cutoff procedure applied to the Gamow vector which directly provides us with the desired estimates.

3.4 Conditions on $\tilde{H}_0$

In this section we prove that $\tilde{H}_0$ satisfies conditions (I)-(IV) of the Soffer-Costin theorem. Here the main task is to prove the local decay estimate Eq. (3.4) for the Hamiltonian $\tilde{H}_0$. This is done by first proving that $\tilde{H}_0$ satisfies a Mourre estimate [Mo]. Let the generator of dilations A be given by

$$A = \frac{1}{2}(D \cdot x + x \cdot D) \tag{3.59}$$

where $D = -i\nabla_x$. Denote

$$ad^n_A(H) = [\cdots [H, A], A], \ldots A] \tag{3.60}$$

In order to obtain the local decay estimate for $\tilde{H}_0$ we need to show that it satisfies two conditions. The first is a boundedness condition

$$g_\Delta(\tilde{H}_0) ad^n_A(\tilde{H}_0) g_\Delta(\tilde{H}_0) \leq C \tag{3.61}$$

for $n = 1, 2$. The second is the Mourre estimate

$$g_\Delta(\tilde{H}_0) i[\tilde{H}_0, A] g_\Delta(\tilde{H}_0) \geq \theta (g_\Delta(\tilde{H}_0))^2 \tag{3.62}$$

for some $\theta > 0$. Validity of the local decay estimate, Eq. (3.4) for $\tilde{H}_0$ is a consequence of the validity of the two conditions Eq. (3.61), (3.62) (see for example [HSS]). We observe also that the Mourre estimate for $\tilde{H}_0$ in the interval Δ ensures that the points in $\sigma_{pp}(\tilde{H}_0) \cap \Delta$ are seperated from each other and hence there must exist an interval Δ containing E_0 but no other eigenvalue of $\tilde{H}_0$. In addition, the H^η regularity of $\tilde{H}_0$ follows from the H^η regularity of H which in turn is a result of assumptions (V1)-(V4) on the potential V.

We prove first the Mourre estimate Eq. (3.62). Noting that $\tilde{H}_0 = H - W$, we have

$$\begin{aligned} g_\Delta(\tilde{H}_0) i[\tilde{H}_0, A] g_\Delta(\tilde{H}_0) &= g_\Delta(\tilde{H}_0) i[H - W, A] g_\Delta(\tilde{H}_0) = \\ &= g_\Delta(\tilde{H}_0) i[H, A] g_\Delta(\tilde{H}_0) - g_\Delta(\tilde{H}_0) i[W, A] g_\Delta(\tilde{H}_0) \end{aligned} \tag{3.63}$$

Denote $H_0 = -\Delta_x$. In order to estimate the first term on the r.h.s. of Eq. (3.63) we calculate

$$i[H,A] = 2H_0 - x\nabla_x V = 2H - 2V - x\nabla_x V = 2\tilde{H}_0 + 2(W-V) - x\nabla_x V \tag{3.64}$$

Hence, we need to obtain an estimate of the operator $g_\Delta(\tilde{H}_0)(2(W-V) - x\nabla_x V)g_\Delta(\tilde{H}_0)$. Since W is compact we conclude that $g_\Delta(\tilde{H}_0)Wg_\Delta(\tilde{H}_0)$ is compact. Furthermore, we have

$$g_\Delta(\tilde{H}_0)(2V + x\nabla_x V)g_\Delta(\tilde{H}_0) = \left[g_\Delta(\tilde{H}_0)\langle D\rangle^2\right]\langle D\rangle^{-2}(2V + x\nabla_x V)\langle D\rangle^{-2}\left[\langle D\rangle^2 g_\Delta(\tilde{H}_0)\right] \tag{3.65}$$

The operator

$$\langle D\rangle^{-2}(2V + x\nabla_x V)\langle D\rangle^{-2} = (1-\Delta_x)^{-1}(2V + x\nabla_x V)(1-\Delta_x)^{-1} \tag{3.66}$$

is compact. Moreover, the operator

$$\langle D\rangle^2 g_\Delta(\tilde{H}_0) = (1-\Delta_x)g_\Delta(\tilde{H}_0) = (1 + \tilde{H}_0 + W - V)g_\Delta(\tilde{H}_0) \tag{3.67}$$

is bounded. Denoting $K_1 = g_\Delta(\tilde{H}_0)(2(W-V) - x\nabla_x V)g_\Delta(\tilde{H}_0)$ we conclude that

$$g_\Delta(\tilde{H}_0)i[H,A]g_\Delta(\tilde{H}_0) = 2\tilde{H}_0 g_\Delta^2(\tilde{H}_0) + K_1 \tag{3.68}$$

where K_1 is compact.

For the second term on the r.h.s. of Eq. (3.63) we have

$$g_\Delta(\tilde{H}_0)WAg_\Delta(\tilde{H}_0) = g_\Delta(\tilde{H}_0)W\langle x\rangle^2\langle x\rangle^{-2}A\langle D\rangle^{-2}\langle D\rangle^2 g_\Delta(\tilde{H}_0) \tag{3.69}$$

but the operator $\langle x\rangle^{-2}A\langle D\rangle^{-2}$ is compact and $\langle D\rangle^2 g_\Delta(\tilde{H}_0)$ is bounded (see Eq. (3.67)). Furthermore, the estimates in Subsection 3.2 assures us that operators of the form $\langle x\rangle^\sigma Wg_\Delta(\tilde{H}_0)$, $\sigma \geq 0$ are bounded (with the norm going to zero when the parameter r_0 goes to infinity). We conclude that $g_\Delta(\tilde{H}_0)i[W,A]g_\Delta(\tilde{H}_0)$ is compact. This result, together with Eq. (3.68) and Eq. (3.63) give the Mourre estimate

$$g_\Delta(\tilde{H}_0)i[\tilde{H}_0,A]g_\Delta(\tilde{H}_0) = 2\tilde{H}_0 g_\Delta^2(\tilde{H}_0) + K \tag{3.70}$$

where K is compact. Eq. (3.62) is an immediate consequence of Eq. (3.70). For $n=1$ Eq. (3.61) is also a direct consequence of Eq. (3.70). Furthermore, Following the general steps of the analysis above with $i(ad_A(\tilde{H}_0)) = i[\tilde{H}_0, A]$ replaced by $i(ad_A^2(\tilde{H}_0)) = i[[\tilde{H}_0, A], A] = 4D^2 + (x\nabla_x)^2 V - i[[W,A],A]$ results in the estimate Eq. (3.61) for $n=2$. Thus, the Hamiltonian $\tilde{H}_0$ satisfies both the condition Eq. (3.61) (with $n=1,2$) and the condition Eq. (3.62). With the Mourre estimate proved for the Hamiltonian $\tilde{H}_0$, a local decay estimate in the form of Eq. (3.4) can be proved for $\tilde{H}_0$ by using techniques based on the Mourre estimate, see for example [JMP], [SS], [HSS] or Appendix D in [SW]. This proves that condition (IV) is satisfied by $\tilde{H}_0$.

It is easier to show that $\tilde{H}_0$ satisfies the other conditions of the Costin-Soffer theorem. Condition (I) follows from the compactness of W and the self-adjointness of H. Condition (II) is trivial and we assume that condition (III) holds. Condition (V) follows from the compactness of W and the fact that H is a self-adjoint Schrödinger operator with a spectrum bounded from below. We use the second resolvent formula to obtain

$$\begin{aligned}\langle x\rangle^{\sigma}(\tilde{H}_0+c)^{-1}\langle x\rangle^{-\sigma} &= \langle x\rangle^{\sigma}(H+c)^{-1}\langle x\rangle^{-\sigma} \\ &\quad - [\langle x\rangle^{\sigma}(H+c)^{-1}W\langle x\rangle^{-\sigma}][\langle x\rangle^{\sigma}(\tilde{H}_0+c)\langle x\rangle^{-\sigma}]\end{aligned} \tag{3.71}$$

Thus

$$\begin{aligned}\|\langle x\rangle^{\sigma}(\tilde{H}_0+c)^{-1}\langle x\rangle^{-\sigma}\| &\le \|\langle x\rangle^{\sigma}(H+c)^{-1}\langle x\rangle^{-\sigma}\| \\ &\quad + \|\langle x\rangle^{\sigma}(H+c)^{-1}W\langle x\rangle^{-\sigma}\|\|\langle x\rangle^{\sigma}(\tilde{H}_0+c)\langle x\rangle^{-\sigma}\|\end{aligned} \tag{3.72}$$

and

$$\left(1-\|\langle x\rangle^{\sigma}(H+c)^{-1}W\langle x\rangle^{-\sigma}\|\right)\|\langle x\rangle^{\sigma}(\tilde{H}_0+c)\langle x\rangle^{-\sigma}\| \le \|\langle x\rangle^{\sigma}(H+c)^{-1}\langle x\rangle^{-\sigma}\| \tag{3.73}$$

Now, the properties of H mentioned above imply that a c can be chosen such that $\|\langle x\rangle^{\sigma}(H+c)^{-1}\langle x\rangle^{-\sigma}\|$ is as small as necessary. Hence we can choose c such that

$$\|\langle x\rangle^{\sigma}(H+c)^{-1}W\langle x\rangle^{-\sigma}\| = [\|\langle x\rangle^{\sigma}(H+c)^{-1}\langle x\rangle^{-\sigma}][\langle x\rangle^{\sigma}W\langle x\rangle^{-\sigma}\|] < 1/2$$

This can be done since W is compact with compact support in $R^3 \times R^3$, therefore $\langle x\rangle^{\sigma}W\langle x\rangle^{-\sigma}$ is compact. In this case we have

$$\|\langle x\rangle^{\sigma}(\tilde{H}_0+c)\langle x\rangle^{-\sigma}\| \le 2\|\langle x\rangle^{\sigma}(H+c)^{-1}\langle x\rangle^{-\sigma}\| \tag{3.74}$$

and condition (V) holds for $\tilde{H}_0$ since $\langle x\rangle^{\sigma}(H+c)^{-1}\langle x\rangle^{-\sigma}$ can be made as small as needed by an appropriate choice of c. Thus conditions (I)-(V) hold for $\tilde{H}_0$ and the proof of Theorem A is complete.

4. Pairs of quasimodes corresponding to a resonance

4.1 A symmetry in the estimates

We notice an interesting symmetry in the estimates of the quantity $\|\langle x\rangle^{\sigma}Wg_{\Delta}(\tilde{H}_0)\|$. Turning back to Eq. (3.40) we consider the complex conjugate of this equation. We find that

$$
\begin{aligned}
\langle\tilde\psi_{z_0}|H'|\tilde\psi_{z_0}\rangle &= i\Gamma\int_{B_1} d^3x\,|\psi^G_{z_0}(x)|^2\\
&- 4\pi|F_1|^2\int_{R_1}^{R_2} dr\, e^{ik_0 r}\chi(r)\left[\frac{d^2\chi(r)}{dr^2}-2i\bar{k}_0\frac{d\chi(r)}{dr}-i\Gamma\chi(r)\right]e^{-i\bar{k}_0 r} =\\
&= i\Gamma\int_{B_1} d^3x\,|\psi^G_{z_0}(x)|^2\\
&- 4\pi|F_1|^2\int_{R_1}^{R_2} dr\,\chi(r)\left[\frac{d^2\chi(r)}{dr^2}-2i\bar{k}_0\frac{d\chi(r)}{dr}-i\Gamma\chi(r)\right]e^{2|Imk_0|r}
\end{aligned}
\tag{4.1}
$$

We readily observe that, starting with Eq. (4.1) instead of Eq. (3.40) the analysis leading to the important identity Eq. (3.39) does not change and we again arrive at the same identity. However, the right hand side of Eq. (4.1) can be obtained if we assume that in the region outside of the support of the potential, and in particular in $B_2\backslash B_1$ the Gamow state solution has the form

$$
\psi^G_{\bar{z}_0}(x) = F_2\frac{e^{-i\bar{k}_0 r}}{r}, \qquad x\in R\backslash B_1, r=|x| \tag{4.2}
$$

Moreover, a replacement of $\psi^G_{z_0}(x)$ in Eq. (3.16) by $\psi^G_{\bar{z}_0}(x)$ does not effect any of the following estimates and again we arrive at Eq. (3.37). An immediate conclusion resulting from these considerations is that if we find a Gamow vector solution of the eigenvalue problem for H which has the form of Eq. (4.2) in the region outside the potential we can apply to it the cutoff procedure and produce a quasimode satisfying all of the estimates above. By our main theorem such a quasimode will also correspond to a resonance. Since $\bar{z}_0 = \bar{k}_0^2$ is a point in the upper half-plane we see that there exist points above the real axis which may be associated with a resonance via the quasimode construction procedure.

In the next subsection we introduce a simple example illustrating the current discussion. In this example, a scattering problem with a square barrier potential, the nature of the points in the upper half-plane which are associated with a resonance by the quasimode construction is clearly demonstrated.

4.2 An example - Gamow vectors, quasimodes and resonances for a square barrier potential problem

The problem of the construction of Gamow state solutions for various quantum mechanical models was addressed many times over the years. In this subsection we illustrate the discussion of the previous subsection by considering a simple example. This relatively simple model, a scattering problem with a square barrier potential, was analyzed, e.g., by M. Gadella and R. de la Madrid in [dMG]. The two authors originally consider an eigenvalue equation in three dimensions for a Schrödinger operator with a spherically symmetric square barrier potential. The Hamiltonian is

$$
H = -\Delta_x + V(x) \tag{4.3}
$$

with

$$V(x) = V(r) = \begin{cases} 0 & ,0 < r < a \\ V_0 & , a < r < b \\ 0 & , b < r < \infty \end{cases} , \qquad r = |x| \tag{4.3'}$$

the problem is reduced, for zero angular momenta (s-wave) to a one dimensional one with a potential which is infinite for $r < 0$. In this case we can consider only the radial part $H_r = -\Delta_r + V(r)$. The eignvalue equation then becomes

$$\left(-\frac{1}{r}\frac{d^2 r}{dr^2} r + V(r)\right)\phi_\lambda(r) = \lambda\phi_\lambda(r) \tag{4.4}$$

A generalized eigenvector corresponding to a generalized eigenvalue λ belonging to the continuous spectrum of H is given by $\phi_\lambda(r) = \psi_\lambda(r)/r$ where

$$\psi_\lambda(r) = \begin{cases} \alpha(k)\sin(kr), & 0 \le r < a \\ \alpha_2(k)e^{iQr} + \beta_2(k)e^{-iQr}, & a \le r < b \\ F_1(k)e^{ikr} + F_2(k)e^{-ikr}, & b \le r \end{cases} , \qquad k = \sqrt{\lambda} \tag{4.5}$$

with

$$Q = \sqrt{k^2 - V_0} \tag{4.6}$$

The coefficients α, α_2, β_2, F_1, F_2 satisfy the following conditions at the various boundary points

$$\begin{aligned} \alpha_2 e^{iQa} + \beta_2 e^{-iQa} &= \alpha\sin(ka) \\ iQ(\alpha_2 e^{iQa} - \beta_2 e^{-iQa}) &= \alpha\, k\cos(ka) \\ F_1 e^{ikb} + F_2 e^{-ikb} &= \alpha_2 e^{iQb} + \beta_2 e^{-iQb} \\ ik(F_1 e^{ikb} - F_2 e^{-ikb}) &= iQ(\alpha_2 e^{iQb} - \beta_2 e^{-iQb}) \end{aligned} \tag{4.7}$$

From these conditions we get

$$\begin{aligned} \alpha_2(k) &= \frac{1}{2}e^{-iQa}(\sin(ka) + \frac{k}{iQ}\cos(ka))\alpha(k) \\ \beta_2(k) &= \frac{1}{2}e^{iQa}(\sin(ka) - \frac{k}{iQ}\cos(ka))\alpha(k) \\ F_1(k) &= \frac{1}{4}e^{-ikb}\Big(e^{iQ(b-a)}(1 + \frac{Q}{k})(\sin(ka) + \frac{k}{iQ}\cos(ka)) \\ &\quad + e^{-iQ(b-a)}(1 - \frac{Q}{k})(\sin(ka) - \frac{k}{iQ}\cos(ka)\Big)\alpha(k) \\ F_2(k) &= \frac{e^{ikb}}{4}\Big(e^{iQ(b-a)}(1 - \frac{Q}{k})(\sin(ka) + \frac{k}{iQ}\cos(ka)) \\ &\quad + e^{-iQ(b-a)}(1 + \frac{Q}{k})(\sin(ka) - \frac{k}{iQ}\cos(ka))\Big)\alpha(k) \end{aligned} \tag{4.8}$$

The S-matrix for the problem is given by

$$S(\lambda) = -\frac{F_1(k)}{F_2(k)} \tag{4.9}$$

with $k = \sqrt{\lambda}$. The condition for finding a resonance pole at the complex point $z_0 = E_0 - i\Gamma$ is the vanishing of the coefficient $F_2(k)$ at $k_0 = \sqrt{z_0}$. A Gamow vector solution for the problem is then found by analytically continuing the generalized state ψ_λ in Eq. (4.5) to the point $k_0 = \sqrt{z_0}$. At this point the S-matrix has a pole and, since $F_2(k_0) = 0$, in the region outside of the support of the potential the resulting Gamow vector $\psi^G_{z_0}$ has the form $\psi^G_{z_0}(r) = F_1(k_0)\exp(ik_0 r)/r$. Multiplying $\psi^G_{z_0}$ by a cutoff function $\chi(r)$ satisfying the conditions in Eq. (2.2) we obtain the quasimode $\tilde{\psi}_{z_0}$ for which, if r_0 is large enough, we can apply Theorem A.

Instead of analytically continuing ψ_λ in the lower half-plane to the point $k_0 = \sqrt{z_0}$ consider the analytic continuation of ψ_λ in the upper half-plane to the point $\overline{k}_0 = \sqrt{\overline{z}_0}$. Denote the resulting state by $\psi^G_{\overline{z}_0}$. At the point $\overline{k}_0$ we have $F_1(\overline{k}_0) = 0$ and we see from Eq. (4.9) that this corresponds to a zero of the S-matrix in the upper half-plane. Furthermore, from Eq. (4.5) we see that, in the region beyond the support of the potential, $\psi^G_{\overline{z}_0}(r) = F_2(\overline{k}_0)\exp(-i\overline{k}_0 r)/r$, which has exactly the form of the r.h.s. of Eq. (4.2). Hence, by applying the cutoff procedure we produce the cut state $\tilde{\psi}_{\overline{z}_0}$ which corresponds to the same resonance.

Recapitulating, it is possible to construct a quasimode by cutting a Gamow vector solution corresponding to a pole of the S-matrix in the lower half-plane, then decomposing the Hamiltonian and proving the estimates as shown above. Theorem A, together with the Soffer-Costin-Weinstein theorem, then states that such a quasimode is associated with a particular resonance of the scattering problem. On the other hand, we can obtain a different quasimode by cutting a solution corresponding to a zero (paired with the pole) of the S-matrix in the upper half-plane. The quasimode we obtain in this way can be used for a decomposition of the Hamiltonian that satisfies the same estimates as the one corresponding to the pole. This quasimode is also associated with a particular resonance, in fact to the same resonance as the other quasimode.

Acknowledgments

The research work for this paper was initiated during the first author's stay at Rutgers university as a postdoctoral fellow at the department of mathematics. In addition, the research of the first author was also sponsored by the Israel Science Foundation (grant no. 188/02) and the Edmund Landau Center for Research in Mathematical Analysis and Related areas, sponsored by the Minerva Foundation (Germany). The work of the second and third authors was partially supported by the NSF.

References

[AK] Albeverio, S., Kurasov, P., Singular Perturbations of Differential Operators Solvable Schrödinger type operators, London Mathematical Society Lecture Notes Series, 271, Cambridge University Press, Cambridge (2000).

[CS] Costin, O., Soffer, A., Resonance theory for Schrödinger operators, Dedicated to Joel L. Lebowitz, Comm. Math. Phys. **224**, No. 1 (2001) p. 133-152.

[dMG] de la Madrid, R., Gadella, M., A pedestrian introduction to Gamow vectors, Amer. J. Phys. **70**, No. 6 (2002) p. 626-638.

[GD] Gerard, C., Derezinski, J., Scattering Theory of Classical and Quantum N-Particle Systems, Texts and monographs in Physics, Springer (1997), Chapter 4, p. 196-198

[GS] Gerard, C., Sigal, I.M., Space-time picture of semiclassical resonances, Comm. Math. Phys. **145** (1992) p. 281-328.

[HSS] Hunziker, H., Sigal, I.M., Soffer, A., Minimal escape velocities, Comm. PDE **24** (11 & 12) (2000) p. 2279-2295.

[JMP] Jensen, A., Mourre, E., Perry, P., Multiple commutator estimates and resolvent smoothness in quantum scattering theory, Ann. Inst. Poincare-Phys. Theor. **41** (1984) p. 207-225.

[Mo] Mourre, E., Absence of singular continuous spectrum for certain self-adjoint operators, Comm. Math. Phys. **78** (1981) p. 391-408.

[SS] Sigal, I.M., Soffer, A., Local decay and velocity bounds for quantum propagation, (1988), Preprint, ftp: //www.math.rutgers.edu/pub/soffer.

[St] Stefanov, P.: Resonances near the real axis imply existence of quasimodes, C.R. Acad. Sci. Paris, t. **330**, Serie 1 (2000) p. 105-108.

[SW] Soffer, A., Weinstein, M.I.: Time dependent resonance theory, Geom. Func. Anal. **8**, No. 6 (1998) p. 1086-1128.

[SV] Stefanov, P.D., Vodev, G., Distribution of resonances for the Neumann problem in Linear elasticity outside a strictly convex body, Duke Math. J. **78**, No. 3 (1995) p. 677-714.

[TZ] Tang, Siu-Hung, Zworski, M., From quasimodes to resonances, Math. Res. Lett. **5**, No. 3 (1998) p. 261-272.