A remark on asymptotic completeness for the critical nonlinear Klein-Gordon equation

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Abstract

We give a short proof of asymptotic completeness and global existence for the cubic Nonlinear Klein-Gordon equation in one dimension. Our approach to dealing with the long range behavior of the asymptotic solution is by reducing it, in hyperbolic coordinates to the study of an ODE. Similar arguments extend to higher dimensions and other long range type nonlinear problems.

1 Introduction

We are interested in the completeness problem for the critical nonlinear Klein-Gordon in one space dimension:

$$\Box v + v = -\beta v^3, \qquad \beta \geq 0 \tag{1.1}$$

where $\Box = \partial_t^2 - \partial_x^2$.

Such problem appear naturally in the study of some nonlinear dynamical problems of mathematical physics, among them radiation theory [F], general relativity [L-R] , the scattering and stability of kinks, vortices and other coherent structures. The long range nature of the scattering, coupled with the nonlinearity poses a challenge to scattering theory. In the Linear case, one can compute in advance the asymptotic corrections to the solution due to the long range nature of the solution. That is not possible in the nonlinear case. It may also break global exsitence [H2]. Since the asymptotic behavior is not free other tools are needed to prove scattering and global exsitence, e.g. normal forms for dealing with u^2 terms [Sh, D1]. Inspired by Delort's ground breaking work [D1], here we see that one can nevertheless get the correct asymptotic behavior from that of an ODE. Recall first that a solution of the linear Klein-Gordon, i.e. $\beta = 0$, is asymptotically given by (as ϱ tends to infinity)

$$u(t,x) \sim \varrho^{-1/2} e^{i\varrho}\, a(x/\varrho) + \varrho^{-1/2} e^{-i\varrho}\, \overline{a(x/\varrho)}, \qquad \text{where} \quad \varrho = (t^2 - |x|^2)^{1/2} \geq 0. \tag{1.2}$$

Here $a(x/\varrho) = (t/\varrho)\widehat{u}_+(-x/\varrho) = \sqrt{1 + x^2/\varrho^2}\,\widehat{u}_+(-x/\varrho)$, where $\widehat{u}_+(\xi) = \int u_+(x)\, e^{-ix\xi}\, dx$ denotes the Fourier transform with respect to x only, $\widehat{u}_+ = (\widehat{u}_0 - i(|\xi|^2+1)^{-1/2}\widehat{u}_1)/2$, where $u_0 = u\big|_{t=0}$ and

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$u_1 = \partial_t u\big|_{t=0}$. Here the right hand side is to be interpreted as 0 outside the light cone, when $|x| > t$. (1.2) can be proven using stationary phase, see e.g. [H1], where a complete asymptotic expansion into negative powers of ϱ was given. One can also get precise estimates by vector-field multipliers [K1]. Recently, Delort[D1] showed that (1.1) with small initial data have a global solution with asymptotics of the form

(1.3)
$$v(t,x) \sim \varrho^{-1/2} e^{i\phi_0(\varrho, x/\varrho)}\, a(x/\varrho) + \varrho^{-1/2} e^{-i\phi_0(\varrho, x/\varrho)}\, \overline{a(x/\varrho)}, \qquad \phi_0(\varrho, x/\varrho) = \varrho + \frac{3}{8}\beta |a(x/\varrho)|^2 \ln \varrho$$

Delort's proof is for more general equations but it is rather involved ; furthermore there is an application of Gronwall's lemma, used to obtain the estimate (3.27) from (3.26) that seem to require an additional argument. This is because the integrand in (3.26) depends explicitly on t. In [L-S] we considered the inverse problem of scattering, we showed that for any given asymptotic expansion of the above form (1.3) there is a solution agreeing with it at infinity. The purpose of this note is to present a simple proof of the completeness for the special case considered here.

For technical reasons we will give initial data of compact support $|x| \le 1$ in the support of u_0 and u_1 when $t = 2$:

$$v(2,x) = \varepsilon u_0(x), \qquad \partial_t v(2,x) = \varepsilon u_1(x).$$

so that solution is supported inside the forward light cone $t - |x| \ge 1$, when $t \ge 2$.

Remark 1.1. The above restriction on the initial data makes it difficult to give an explicit description of the space in which the asymptotic solutions live. An extension of the proof to more general data, such as the Schwarz class is therefore desirable, and probably can be achieved following arguments similar to[S-Taf]

For the proof we start by introducing the hyperbolic coordinates

$$\varrho^2 = t^2 - x^2, \quad t = \varrho \cosh y, \quad x = \varrho \sinh y,$$

or

$$e^{2|y|} = \frac{t + |x|}{t - |x|}, \qquad \varrho^2 = t^2 - x^2 \tag{1.4}$$

Then

$$\Box + 1 = \partial_\varrho^2 - \varrho^{-2}\partial_y^2 + \varrho^{-1}\partial_\varrho + 1$$

and with

$$v(t,x) = \varrho^{-1/2} V(\varrho, y)$$

we get

$$(\Box + 1) v(t,x) = \varrho^{-1/2}\Big(\partial_\varrho^2 + 1 - \varrho^{-2}(\partial_y^2 - \tfrac14)\Big) V(\varrho, y).$$

Hence in these coordinates (1.1) becomes the following equation for $V = \varrho^{1/2} v$:

$$\Psi(V) \equiv \partial_\varrho^2 V + \Big(1 + \frac{\beta}{\varrho} V^2 + \frac{1}{4\varrho^2}\Big) V - \frac{1}{\varrho^2}\partial_y^2 V = 0 \tag{1.5}$$

We are therefore led to first studying the ODE

$$L(g) \equiv \ddot{g} + \Big(1 + \frac{\beta}{\varrho} g^2 + \frac{1}{4\varrho^2}\Big) g = F \tag{1.6}$$

The proof as usual consists of a decay estimate (L^∞ estimate) of a lower number of derivatives and energy estimate (L^2 estimate) of a higher number of derivatives. The L^2 estimate uses the decay estimate $C\varepsilon\rho^{-1/2}$ and allows for growing energies, like Ct^ε. Since the energies are growing we don't directly get back the L^∞ estimate from the energy estimates but instead we get some weaker decay estimates using some weighted Sobolev lemmas. Using these weaker decay estimates we can get further decay from the asymptotic equation, i.e. the ODE above, using that the term $\rho^{-2}\partial_y^2 V$ has a negative power of ρ so there is some room and we can use the weak decay estimates to estimate this term, c.f. [L1], [L-R] The energies we will use will just be the energies on hyperboloids in the coordinates(1.4)

2 The sharp decay estimate

Lemma 2.1. *Suppose that*

$$\ddot{g} + \Big(1 + \frac{\alpha}{\varrho^{1/2}} g + \frac{\beta}{\varrho} g^2 + \frac{1}{4\varrho^2}\Big) g = F \tag{2.1}$$

Then

$$|\dot{g}(\varrho)| + |g(\varrho)| \le 2\Big(|\dot{g}(1)| + |g(1)| + |g(1)|^2 + \int_1^\varrho |F(\tau)|\, d\tau\Big) \tag{2.2}$$

if $\alpha = 0$ and $\beta \ge 0$, or the right hand side is sufficiently small.

Proof. Multiplying (2.1) by $2\dot{g}$ we see that

$$\frac{d}{d\varrho}\Big(\dot{g}^2 + g^2 + \frac{2\alpha}{3\varrho^{1/2}} g^3 + \frac{\beta}{2\varrho} g^4 + \frac{1}{4\varrho^2} g^2\Big) = 2F\dot{g} - \frac{\alpha}{3\varrho^{3/2}} g^3 - \frac{\beta}{2\varrho^2} g^4 - \frac{1}{2\varrho^3} g^2$$

It therefore follows that if $\alpha = 0$ and $\beta \ge 0$ then $|2M\frac{dM}{d\rho}| \le |2F\dot{g}|$, so

$$\Big|\frac{d}{d\varrho} M(\varrho)\Big| \le F(\varrho), \qquad \text{with} \quad M = \Big(\dot{g}^2 + g^2 + \frac{2\alpha}{3\varrho^{1/2}} g^3 + \frac{\beta}{2\varrho} g^4 + \frac{1}{4\varrho^2} g^2\Big)^{1/2} \tag{2.3}$$

□

Applying the above lemma to the equation (1.5)

$$\partial_\varrho^2 V + \Big(1 + \frac{\beta}{\varrho} V^2 + \frac{1}{4\varrho^2}\Big) V = \frac{1}{\varrho^2}\partial_y^2 V \tag{2.4}$$

and noting that $|\dot{V}| + |V| \le C_0\varepsilon$ when $\varrho = 1$ by the support assumptions on initial data, we conclude that

$$|\partial_\varrho V(\rho, y)| + |V(\varrho, y)| \le C\varepsilon + \int_1^\varrho \tau^{-2} |\partial_y^2 V(\tau, y)|\, d\tau \tag{2.5}$$

Which gives the bound

$$|\partial_\varrho V(\rho, y)| + |V(\varrho, y)| \le C_2\varepsilon \tag{2.6}$$

if we can prove that for some $\delta > 0$;

$$|\partial_y^2 V(\varrho, y)| \le C_1\varepsilon(1 + \varrho)^{1-\delta} \tag{2.7}$$

3 Energies on hyperboloids and the weak decay estimate

By section 7.6 in Hörmander [H1], if

$$E(\varrho)^2 = \int_{H_\varrho} u_t^2 + u_x^2 + 2\frac{x}{t}u_t u_x + u^2\, dx$$

where $H_\varrho = \{(t,x);\, t^2 - x^2 = \varrho^2\}$ and $G_{\varrho_1,\varrho_2} = \{(t,x);\, \varrho_2 \ge t^2 - x^2 \ge \varrho_1\}$ then (recall $\frac{|x|}{t} \le 1\ \frac{\varrho}{t} \le 1$)

$$\frac{d}{d\varrho}E(\varrho)^2 = 2\int_{H_\varrho} F\, u_t \frac{\varrho}{t}\, dx \le 2\Big(\int_{H_\varrho} |F|^2\, dx\Big)^{1/2} E(\varrho) \tag{3.1}$$

and hence

$$E(\varrho) \le E(1) + \int_1^\varrho \Big(\int_{H_\tau} |F|^2\, dx\Big)^{1/2}\, d\tau.$$

We have

$$u_t^2 + u_x^2 + 2\frac{x}{t}u_t u_x = \Big(u_x + \frac{x}{t}u_t\Big)^2 + \frac{\varrho^2}{t^2}u_t^2 = \Big(u_t + \frac{x}{t}u_x\Big)^2 + \frac{\varrho^2}{t^2}u_x^2.$$

Since $u_\varrho = (tu_t + xu_x)/\varrho$, $u_y = xu_t + tu_x$ and $t = \rho\cosh y$ we see that

$$u_t^2 + u_x^2 + 2\frac{x}{t}u_t u_x = \frac{1}{2\cosh^2 y}\Big(u_\varrho^2 + \frac{u_y^2}{\varrho^2}\Big) + \frac{\varrho^2}{2t^2}\big(u_t^2 + u_x^2\big) \tag{3.2}$$

In order to change variables in the integral we think of $x = \varrho\sinh y$ as a function of y for ϱ fixed. Then $\partial x/\partial y\big|_{\varrho=const} = \varrho\cosh y = t \sim \varrho e^{|y|}$. Hence with $V = \varrho^{1/2}u$ we have

$$E(\varrho)^2 \ge c\int_{H_\varrho} \big(V_y^2\varrho^{-2} + V_\varrho^2\big)e^{-|y|} + V^2 e^{|y|}\, dy, \qquad \varrho \ge 1. \tag{3.3}$$

Here we also used that $V_\varrho = \varrho^{1/2}u_\varrho + V/2\varrho$ and $V^2\varrho^{-2}\cosh^{-2} y = V^2 t^{-2} \le V^2$ if $t \ge 1$.

Moreover we have proven using (3.2), (3.3):

Lemma 3.1. *Suppose that*

$$\partial_t^2 w - \partial_x^2 w + w = F \tag{3.4}$$

Then

$$\Big(\int_{H_\varrho} V_y(\varrho,y)^2\varrho^{-2}e^{-|y|} + V(\varrho,y)^2 e^{|y|}\, dy\Big)^{1/2} \le CE(1) + \int_1^\varrho \tau^{1/2}\Big(\int_{H_\tau} |F(\tau,y)|^2 e^{|y|}\, dy\Big)^{1/2} d\tau \tag{3.5}$$

Lemma 3.2. *Suppose that*

$$\partial_\varrho^2 W + \big(1 + \frac{1}{4\varrho^2}\big)W - \frac{1}{\varrho^2}\partial_y^2 W = F \tag{3.6}$$

Let

$$\|F(\varrho,\cdot)\|_{L^p(H_\varrho)} = \Big(\int_{H_\varrho} |F(\varrho,y)|^p\, e^{|y|}\, dy\Big)^{1/p} \tag{3.7}$$

Then

$$\|W(\varrho\,,\cdot)\|_{L^2(H_\varrho\,)} \le CE(1) + C\int_1^\varrho \|F(\sigma,\cdot)\|_{L^2(H_\sigma)}\,d\sigma \tag{3.8}$$

In the applications

$$\partial_\varrho^2 V^{(k)} + \big(1+\frac{1}{4\varrho^{\,2}}\big)V^{(k)} - \frac{1}{\varrho^{\,2}}\partial_y^2 V^{(k)} = F^{(k)} \tag{3.9}$$

where

$$F^{(k)} = \beta\varrho^{\,-1}\partial_y^k V^3, \qquad V^{(k)} = \partial_y^k V \tag{3.10}$$

We claim that

$$\|F^{(k)}(\varrho\,,\cdot)\|_{L^2(H_\varrho\,)} \le C\varrho^{\,-1}\|V(\varrho\,,\cdot)\|^2_{L^\infty(H_\varrho\,)}\|V(\varrho\,,\cdot)\|_{L^{2,k}(H_\varrho\,)}, \quad k=0,1,2,3, \tag{3.11}$$

where

$$\|F(\varrho\,,\cdot)\|_{L^{p,k}(H_\varrho\,)} = \Big(\int_{H_\varrho}\sum_{m\le k}|\partial_y^m F(\varrho\,,y)|^p\, e^{|y|}\,dy\Big)^{1/p} \tag{3.12}$$

In fact

$$|F^{(0)}| \le C\varrho^{\,-1}|V|^3 \tag{3.13}$$
$$|F^{(1)}| \le C\varrho^{\,-1}|V|^2|\partial_y V| \tag{3.14}$$
$$|F^{(2)}| \le C\varrho^{\,-1}\big(|V|^2\,|\partial_y^2 V| + |V|\,|\partial_y V|^2\big) \tag{3.15}$$
$$|F^{(3)}| \le C\varrho^{\,-1}\big(|V|^2|\partial_y^3 V| + |V|\,|\partial_y V|\,|\partial_y^2 V| + |\partial_y V|^3\big) \tag{3.16}$$

For $k=0,1$ (3.13),(3.14) are obvious and for $k\ge 2$ (3.15),(3.16) follow from interpolation:

Lemma 3.3. *For $k\ge 2\, j\le k$*

$$\|V\|^{k/j}_{L^{2k/j,j}(H_\varrho\,)} \le C\|V\|^{k/j-1}_{L^\infty(H_\varrho\,)}\|V\|_{L^{2,k}(H_\varrho\,)} \tag{3.17}$$

Proof. A proof of the standard interpolation without weights just uses Hölder's inequality, which holds with weights, and integration by parts, which produces lowers order terms included in the norms. □

For $k=2$ we have $(j=1)$

$$\|V(t,\cdot)\|^2_{L^{4,1}} \le C\|V(t,\cdot)\|_{L^\infty}\|V(t,\cdot)\|_{L^{2,2}}, \tag{3.18}$$

Similarly for $k=3$ we have $j=(1,2)$

$$\|V(t,\cdot)\|^3_{L^{6,1}} \le C\|V(t,\cdot)\|^{1/2}_{L^\infty}\|V(t,\cdot)\|_{L^{2,3}}, \tag{3.19}$$
$$\|V(t,\cdot)\|^{3/2}_{L^{3,2}} \le C\|V(t,\cdot)\|^2_{L^\infty}\|V(t,\cdot)\|_{L^{2,3}}, \tag{3.20}$$

Assuming the bound

$$\|V(\varrho\,,\cdot)\|_{L^\infty(H_\varrho\,)} \le C_0\varepsilon\; \rho \ge 1 \tag{3.21}$$

with a constant independent of ϱ we have hence proven that using (3.6)-(3.8)

$$\|V(\varrho\,,\cdot)\|_{L^{2,k}(H_\varrho\,)} \le CE(1) + \int_1^\varrho C\varepsilon^2\sigma^{-1}\|V(\sigma,\cdot)\|_{L^{2,k}(H_\sigma)}\, d\sigma, \quad k=0,1,2,3 \tag{3.22}$$

from which it follows that

$$\|V^{(k)}(\varrho\,,\cdot)\|_{L^2(H_\varrho\,)} \le C\varepsilon\varrho^{\,C\varepsilon^2}, \quad k=0,1,2,3. \tag{3.23}$$

It remains to deduce from this a weak decay estimate. We have

Lemma 3.4.

$$\|V(\varrho\,,\cdot)\|^2_{L^\infty(H_\varrho\,)} \le \|V(\varrho\,,\cdot)\|_{L^2(H_\varrho\,)}\|\partial_y V(\varrho\,,\cdot)\|_{L^2(H_\varrho\,)}$$

Proof. By Hölder's inequality $V^2 \le 2\int |V||V_y|\, dy \le 2\int |V| e^{|y|/2}\, |V_y| e^{|y|/2}\, dy$. □

It therefore follows that

$$\|V^{(k)}(\varrho\,,\cdot)\|_{L^\infty(H_\varrho\,)} \le C\varrho^{\,C\varepsilon^2}, \quad k=0,1,2. \tag{3.24}$$

4 The completeness

We now want to apply the proof of Lemma 2.1 to

$$\partial_\varrho^2 V + \Big(1 + \frac{\beta}{\varrho}V^2 + \frac{1}{4\varrho^{\,2}}\Big)V = F = \frac{1}{\varrho^{\,2}}\partial_y^2 V$$

where we have proven that $|\partial_y^2 V| \le C\varepsilon\varrho^{\,C\varepsilon^2}$. It then follows from the proof of that lemma that the following limit exists

$$\Big|\big(V_\varrho^2 + V^2\big)^{1/2} - a(y)\Big| \le C\varepsilon\varrho^{\,-1+C\varepsilon^2}, \qquad a(y) = \lim_{\varrho\to\infty}\big(V_\varrho\,(\varrho\,,y)^2 + V(\varrho\,,y)^2\big)^{1/2}$$

Let

$$V_\pm = e^{\mp i\varrho}\,\big(\partial_\varrho\, V \pm iV\big)$$

Then $V = (e^{i\varrho}\, V_+ - e^{-i\varrho}\, V_-)/(2i)$, $V_+V_- = V_\varrho^2 + V^2 = |V_+|^2 = |V_-|^2$, $V_- = \overline{V}_+$, and

$$V^3 = -\frac{1}{8i}\big(e^{3i\varrho}\, V_+^3 - e^{-3i\varrho}\, V_-^3 - 3e^{i\varrho}\, V_+V_-V_+ + 3e^{-i\varrho}\, V_+V_-V_-\big)$$

The equation

$$\partial_\varrho^2 V + V + \Big(\frac{\beta}{\varrho}V^2 + \frac{1}{4\varrho^{\,2}}\Big)V = F$$

become

$$\partial_\varrho\, V_\pm + \Big(\frac{\beta}{\varrho}e^{\mp i\varrho}\, V^3 + \frac{1}{4\varrho^{\,2}}e^{\mp i\varrho}\, V\Big) = e^{\mp i\varrho}\, F$$

In other words

$$\partial_\varrho V_\pm \mp igV_\pm = F_\pm + e^{\mp i\varrho} F,$$

where

$$g = \Big(\frac{\beta}{\varrho}\frac{3}{8}V_+V_- + \frac{1}{\varrho^2}\frac{1}{8}\Big)$$

$$F_\pm = \pm\frac{\beta}{8i\varrho}\big(e^{\pm 2i\varrho} V_\pm^3 - e^{\mp 4i\varrho} V_\mp^3 + 3e^{\mp 2i\varrho} V_+V_-V_\mp\big) \mp \frac{ie^{\mp 2i\varrho}}{8\varrho^2}V_\mp$$

Multiplying by the integrating factor $e^{\mp iG}$, where $G(\varrho, y) = \int g(\varrho, y)\, d\varrho$ we get

$$\frac{d}{d\varrho}\big(V_\pm e^{\mp iG}\big) = e^{\mp iG} F_\pm + e^{\mp iG \mp i\varrho} F$$

Note that

$$e^{\mp iG} F_\pm = \frac{d}{d\varrho}\Big(\frac{e^{\mp iG}}{\varrho} H_\pm\Big) + \frac{1}{\varrho^2} K_\pm + \frac{1}{\varrho} L_\pm F$$

where

$$H_\pm = \pm\frac{\beta}{8i}\Big(\frac{e^{\pm 2i\varrho}}{\pm 2i} V_\pm^3 - \frac{e^{\mp 4i\varrho}}{\mp 4i} V_\mp^3 + 3\frac{e^{\mp 2i\varrho}}{\mp 2i} V_+V_-V_\mp\Big)$$

and

$$|K_\pm| \lesssim |V_\pm|\,(1+|V_\pm|^4), \qquad |L_\pm| \lesssim |V_\pm|^2$$

Hence

$$\frac{d}{d\varrho}\Big(V_\pm e^{\mp iG} \mp \frac{e^{\mp iG}}{\varrho} H_\pm\Big) = \frac{1}{\varrho^2} K_\pm + \big(e^{\mp iG \mp i\varrho} + \frac{1}{\varrho} L_\pm\big) F$$

Since we have already shown that $|V_+| = |V_-| \leq C\varepsilon$ and that $|F| \leq \varrho^{-2} |\partial_y^2 V| \leq \varepsilon\varrho^{-2+C\varepsilon^2}$ it follows that the right hand side is integrable and that

$$\big|V_\pm e^{\mp iG} - a_\pm(y)\big| \leq C\varepsilon\varrho^{-1+C\varepsilon^2}, \qquad \text{where} \quad a_\pm(y) = \lim_{\varrho\to\infty} V_\pm(\varrho, y)\, e^{\mp iG(\varrho, y)}$$

Moreover since $V_+V_- \sim a(y)^2$ it follows from what we have already shown that

$$\Big|G(\varrho, y) - \beta \ln|\varrho|\, \frac{3}{8} a(y)^2\Big| \leq C\varrho^{-1+C\varepsilon^2}$$

and furthermore we must have that $|a_\pm(y)| = a(y)$. The exact phase is hence determined by looking at $a_\pm(y)/a(y)$.

Alternatively we can use:

Lemma 4.1. *Suppose that g is real valued and*

$$i\partial_\varrho V - gV = F$$

Then with $G(\varrho, y) = \int_{s0}^{\varrho} g(\tau, y)\, d\tau$

$$\big|\partial_\varrho |V|\big| + \big|\partial_\varrho (Ve^{iG})\big| \leq 2|F|$$

Proof. Multiplying with $\overline{V}$ gives

$$i\partial_\varrho |V|^2 = \Im F\overline{V}$$

and it follows that $|\partial_\varrho |V|| \leq |F|$. Multiplying with the integrating factor $e^{iG(s)}$, where $G = \int g\, ds$ gives $\partial_\varrho \left(V e^{iG}\right) = -i\, F\, e^{iG}$ and the lemma follows. $\square$

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