Phase Space Analysis on some Black Hole Manifolds

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October 5, 2005

Abstract

The Schwarzschild and Reissner-Nordstrøm soutions to Einstein's equations describe space-times which contain spherically symmetric black holes. We consider solutions to the liear wave equation in the exterior of a fixed black- hole space- time of this type. We show that for solutions with initial data which decay at infinite, a weighted L^6 norm in space decays like $t^{-\frac{1}{3}}$. This weight vanishes at the event horizon, but not at infinite.

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1 Introduction

Black holes are very important objects in Relativity, but very little is known about their dynamics and interaction. Even the question of stability under small perturbations is a challenging, open problem. Any solution to such questions will require an understanding of the interaction between black holes and gravitational radiation. The structure of the vacuum Einstein equations, which govern these dynamics in the absence of other matter, make it impossible to consider only radial perturbations, so that more general perturbations must be considered immediately. We hope that the study of linear, uncoupled waves outside the black hole will help provide an understanding of these problems.

Even the stability of the empty space was a very difficult problem. It was solved originally by Christodoulou and Klainerman [7], and Lindblad and Rodnianski have since developed a simpler proof [23]. Decay estimates for solutions of non linear wave equations played an important role in both proofs.

In Relativity, the structure of space- time is determined by a Lorentzian pseudo- metric, g, which, in the absence of matter, satisfies the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0. \tag{1.1}$$

The Schwarzschild and Reissner-Nordstrøm solutions are singular solutions to the Einstein equations which describe space- times containing a spherically symmetric black hole. The Schwarzschild solution is a special case of the Reissner-Nordstrøm solutions. We will restrict our attention to the exterior region, outside the black hole. We discuss the geometry of the exterior region further in subsection 2.1. Because the structure of the Einstein equations makes it impossible for these to have spherically symmetric perturbations, it is expected that the study of black- hole stability will require analysis of the more general, non spherically symmetric, rotating, Kerr-Newman black holes. The linearisation of the Einstein equations around the Kerr solution has been shown to have no unstable modes[34].

In 1957, Regge and Wheeler investigated the linear stability of the Schwarzschild black- hole, and were able to reduce the problem to the study of a second order, scalar equation[27]. After appropriate transformations, this equation and the geometrically defined, scalar wave equation differ only by a multiple of the potential term appearing in each. Using these transformations, the exterior region of the black- hole can be decomposed into the product of time and a three dimensional space.

Because of the intrinsic interest in the geometric wave equation and because of its possible applications to the study of black hole stability, we consider the wave equation

$$\Box \tilde{u} = 0 \qquad \tilde{u}(1) = \tilde{u}_0, \qquad \dot{\tilde{u}}(1) = \tilde{u}_1. \tag{1.2}$$

For this equation, we prove the following result.

Theorem 8.21. *If $\tilde{u}$ is a solution to the wave equation (2.21) , and $u = r\tilde{u}$,*

$$\|F^{\frac{1}{2}} \tilde{u}\|_{L^6(\tilde{\mathfrak{M}})} \leq t^{-\frac{1}{3}} C(\|u_0\|_2^2 + E[u_0, u_1] + E_{\mathcal{C}}[u_0, u_1] + \|L^\epsilon u\|^2 + E[L^\epsilon u_0, L^\epsilon u_1])^{\frac{1}{2}}, \tag{1.3}$$

$$\|(\rho_*^2 + 1)^{\frac{-1}{2}} \tilde{u}\|_{L^2(\tilde{\mathfrak{M}})} \leq t^{-\frac{1}{2}} C(\|u_0\|_2^2 + E[u_0, u_1] + E_{\mathcal{C}}[u_0, u_1] + \|L^\epsilon u\|^2 + E[L^\epsilon u_0, L^\epsilon u_1])^{\frac{1}{2}}, \tag{1.4}$$

where $E[v, w]$ and $E_{\mathcal{C}}[v, w]$ are the energy and conformal charge which are defined in section 3, ρ_* is the Regge-Wheeler radial co-ordinate defined in section 2, and L is an angular derivative operator defined in section 7.

We have previously been able to apply our techniques to prove earlier results to both the wave equation on the Schwarzschild manifold and the Regge-Wheeler equation, and expect the same to be true here[4, 5].

This decay rate is slower than the rate found in Euclidean space, which is $t^{\frac{-2}{3}}$ [18]. This is consistent with current results for other equations on the Schwarzschild solution which only prove slower rates of decay than those found in $\mathbb{R}^{3+1}$. The L^∞ norm has been shown to decay like $t^{\frac{-5}{6}}$ for both the massive Dirac equation [17] and the massive Klein- Gordon equation [21]. For the Schrödinger equation, the L^∞ has been shown to decay like $t^{\frac{-1}{4}+\epsilon}$[22]. All these results hold only for radial initial data, or data which is the sum of finitely many spherical harmonics.

The rigorous study of scattering for linear waves on black hole backgrounds was begun by Dimock and Kay who proved existence and completeness of wave operators for the linear wave equation [13] and the Klein-Gordon equation[14]. Asymptotic completeness and global existence for the cubic semi linear wave equation have been proven for a wide class of black hole manifolds, including the Kerr- Newman solutions[1, 2, 26]. DeBièvre, Hislop, and Sigal have proven asymptotic completeness on a large class of non compact manifolds, including the Reissner-Nordstrøm manifolds [11]. However, none of these methods give decay estimates even in the linear case. For sufficiently super critical Reissner-Nordstrøm black holes, Strichartz estimates have also been proven[6]. Little is known about fields coupled to the Einstein equations, but for a spherically symmetric, scalar field, Dafermos and Rodnianski have shown the field decays along the event horizon according to an inverse cubic, Price law [9]. They also proved a weaker rate of decay for radial decoupled, radial, semi linear wave equations in the exterior of Reissner-Nordstrøm solutions [10]. All of these results also only hold for radial initial data or for data which is a sum of finitely many spherical harmonics.

We start by introducing an analogue of the conformal charge used by Ginibre and Velo [18]. However, we are prevented from completing the argument which holds for the wave equation in Euclidean space by the presence of a closed geodesic surface in the three dimensional space which describes the exterior region of the black hole. The absence of such geodesics is a non- trapping condition which is commonly imposed in scattering theory. The presence of this geodesic surface and the gravitational lensing they cause is already known[8, 32]. This surface is called the photon sphere.

To over come this obstacle, it is sufficient to prove estimates on the angular derivative of the solution in the region near the closed geodesics. We prove:

Theorem 8.20. *If $\varepsilon > 0$, then if $\tilde{u}$ is a solution to the wave equation (2.21) and $u = r\tilde{u}$*

$$\int_1^\infty \|L^{1-\frac{\varepsilon}{2}}\chi_\alpha u\|^2 dt < C(\|u_0\|_{L^2}^2 + E[u_0, u_1]), \tag{1.5}$$

where L acts like one angular derivative and χ_α is a function with compact support near the closed geodesics.

This is sufficient to prove theorem 8.21. The loss of L^ϵ is responsible for the additional factors of L^ϵ appearing in theorem 8.21.

Estimates on the angular derivative have already been used to prove Strichartz and point wise in time L^p estimates in Euclidean space[31, 24] and on non-trapping manifolds [19].

Our method is to introduce an analogue for the wave equation of the Heisenberg equation for the Schrödinger equation. There is a self-adjoint operator H which determines the time evolution of the solution. We refer to this operator as the Hamiltonian. Other self-adjoint operators are referred to as observables, and the analogue of the Heisenberg identity relates the time derivative of a particular inner product involving the observable to the expectation value of the commutator between the Hamiltonian and the observable. For the Schrödinger equation, the expectation value of the observable is differentiated in time, but for the wave equation, the analogue is a more complicated inner product involving both a solution and its time derivative. For an operator to be a propagation observable, we require a few other conditions.

Our first propagation observable is a radial derivative operator directed away from the geodesic surface. This is used to prove a smoothed Morawetz estimate, which is like theorem 8.20, but with zero powers of the angular derivative and a non compact, but decaying localisation function. A function g of the radial variable, denoted ρ_*, is used to direct this operator away from the geodesics.

Following this, we define the positive operator L by

$$L^2 = 1 - \Delta_{S^2}, \tag{1.6}$$

where Δ_{S^2} is the Laplace- Beltrami operator on the sphere, which acts as the derivative in the angular directions. By rescaling the argument of g by powers of L, we generate a new propagation observable, which we use to prove theorem 8.20 with $\varepsilon = \frac{1}{2}$. We call this rescaling by L angular modulation.

To prove the result for all positive ε, we use phase space analysis. We introduce the phase space variables, which are defined in terms of the original radial co-ordinate, ρ_*, and radial derivative, $\frac{\partial}{\partial r_*}$.

$$\mathbf{x}_m = L^m \rho_*, \tag{1.7}$$

$$\boldsymbol{\xi}_n = L^{n-1}\frac{\partial}{\partial r_*}. \tag{1.8}$$

We localise a propagation observable in phase space by multiplying by a compactly supported function of the phase space variable, $X(\mathbf{x}_m)$ and $\Phi(\boldsymbol{\xi}_n)$. We also localise with functions which are not compactly supported but which decay away from a certain region. By both rescaling the argument of g and localising in $\boldsymbol{\xi}_n$, we can control powers of L in various regions of phase space. However, these estimates require control of different powers of L in other regions of phase space. In a process of phase space induction, we are able to combine the estimates across all regions, and use the estimate in one region to control the remainder terms in another. This is sufficient to prove theorems 8.20 and 8.21.

1.1 Structure of the paper

Section 2 introduces the wave equation on the Reissner-Nordstrøm solution, some common notation and transformations to simplify the equation, and discusses important regions of the geometry, including the photon sphere. Section 3 introduces the energy, conformal charge, and a Sobolev estimate. In that section, we prove point wise in time L^p estimates can be reduced to bounding weighted space-time integrals of solutions and their angular derivatives. Section 4 translates some of our conditions into terminology commonly used by Physicists. In particular, we show that finite energy or conformal charge solutions need not vanish at the bifurcation sphere.

The remainder of the paper deals with proving weighted space- time integral bounds. Section 5 provides the basic propagation observable frame work, using a commutator formalism analogous to the Heisenberg equation from quantum mechanics. Section 6 introduces a propagation observable γ which majorates a decaying weight. This is analogous to a smoothed Morawetz estimate. This is sufficient to control the weighted space- time integral needed in the proof of the L^p estimates. The section concludes with an L^p estimate for radial data and with a similar result for non-radial data, which reduces the problem to one of controlling space- time integrals of one angular derivative in L^2. This localisation is in an arbitrarily small neighbourhood of the photon sphere. Section 7 uses a new technique of angular modulation to control $\frac{3}{4}$ angular derivatives in L^2 in the desired region. The method is to rescale the previous propagation observable by fractional powers of the angular derivative operator.

In the final section, section 8, we introduce a family of propagation observables to control $1-\epsilon$ angular derivatives. The propagation observables are like the one from the angular modulation argument, but are also localised in both the radial variable and radial derivative. We refer to this as phase space analysis. Subsection 8.1 introduces these observables and rescaled versions of the radial variable and radial derivative, which we call the phase space variables. Subsection 8.2 provides a commutator theorem which allows us to rearrange localisations in the non commuting phase space variables. It also provides lemmas which cover particularly common cases. Subsection 8.3 begins the argument by using the commutators to bound fractional powers of the angular derivative. These estimates have decaying weights in the rescaled radial variable and derivative. Subsection 8.4 removes the weights in the rescaled radial variable, leaving estimates localised in the radial derivatives only. All these estimates involve remainder terms which include lower powers of the angular derivative, but not localised in the same region. Subsection 8.5 combines the results inductively to control $1-\epsilon$ angular derivatives with out phase space localisation. It concludes with an L^p estimate with ϵ loss of angular derivatives in the energy.

2 The wave equation on the Reissner-Nordstrøm space

2.1 The Reissner-Nordstrøm solution

The Reissner-Nordstrøm solution to the Einstein equations represents the space-time outside a spherically symmetric, charged, massive body. It is the unique spherically symmetric, static[1], asymptotically flat solution to the vacuum Einstein- Maxwell equations, which govern the structure of space-time in the presence of gravity and electro- magnetic fields. This solution can be represented in terms of the co-ordinates (t, r, θ, ϕ)

[1]The static condition is redundant since spherically symmetric solutions are necessarily static.

by the Lorentzian pseudo-metric:

$$ds^2 = F dt^2 - F^{-1} dr^2 - r^2(d\theta^2 + \sin^2(\theta) d\phi^2), \tag{2.1}$$

$$F = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \tag{2.2}$$

where $M \geq 0$ is the central mass and $Q \in \mathbb{R}$ is the central charge, both as measured by observers at infinity.

The polynomial $r^2 F$ has two roots at

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{2.3}$$

In the sub critical case, $0 < |Q| < M$, these roots determine the location of two event horizons which permit geodesics and energy to cross only in the inward direction, towards decreasing r, and not in the outward direction. In the critical case, $|Q| = M$, there is a single event horizon, which has the same property. In the super critical case, $|Q| > M$, the roots are complex, there are no event horizons to prevent geodesics which start at the singularity $r = 0$ from being extended to $r = \infty$, and the central singularity is called a naked singularity. We will not study the super critical case. In the non super critical cases, the effects of the event horizon and the structure of the interior region are complicated and have been studied extensively [15, 33]. The interior region is called a black hole since anything, including a light ray, may fall in but can not escape.

We will restrict our attention to the exterior region of the non super critical solution,

$$t \in \mathbb{R}, \qquad r > r_+ = M + \sqrt{M^2 - Q^2}, \qquad (\theta, \phi) \in S^2. \tag{2.4}$$

The famous Schwarzschild solution corresponds to the chargeless case, $Q = 0$,

$$ds^2 = F dt^2 - F^{-1} dr^2 - r^2(d\theta^2 + \sin^2(\theta) d\phi^2), \tag{2.5}$$

$$F = 1 - \frac{2M}{r}, \tag{2.6}$$

$$t \in \mathbb{R}, \qquad r > r_+ = 2M, \qquad (\theta, \phi) \in S^2 \tag{2.7}$$

The Schwarzschild solution, $Q = 0$, has only one event horizon at $r = 2M$, because the second root, r_- coincides with the central singularity $r = 0$. In the exterior region, the Schwarzschild solution is representative of all the sub critical solutions. The critical solution is similar; however, F vanishes quadratically instead of linearly towards the horizon, and this affects the rate of decay of other quantities.

In studying the exterior region of the Reissner-Nordstrøm or Schwarzschild solutions it is common to introduce a new radial co-ordinate, the Regge-Wheeler tortoise co-ordinate, r_*, defined by

$$\frac{dr}{dr_*} = F. \tag{2.8}$$

This co-ordinate extends from $-\infty$ to ∞ and has the effect of "pushing the horizon to negative infinity". The original radial co-ordinate, r, is now treated as a function of r_*. The exterior region of the Reissner-Nordstrøm solution is now represented by

$$ds^2 = F dt^2 - F dr_*^2 - r^2(d\theta^2 + \sin^2(\theta) d\phi^2), \tag{2.9}$$

$$F = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \tag{2.10}$$

$$t \in \mathbb{R}, \qquad r_* \in \mathbb{R}, \qquad (\theta, \phi) \in S^2 \tag{2.11}$$

In the Schwarzschild case, r_* can be expressed simply in terms of r and M and has simple asymptotic behaviour.

$$r_* = r + 2M \log\left(\frac{r - 2M}{M}\right) + C_*, \tag{2.12}$$

$$\lim_{r_* \to \infty} r_*^{-1} r = 1, \tag{2.13}$$

$$\lim_{r_* \to -\infty} r_*^{-1} \log\left(\frac{r - 2M}{M}\right) = \frac{1}{2M}. \tag{2.14}$$

In the literature, C_* is commonly taken, for simplicity, to be $-2M\log(2)$ [33, 25] or $-2M\log(M/2)$ [15]. In equation (2.32),we will use a particular choice of C_*, coming from the geometry of the Reissner-Nordstrøm solution.

In the general sub critical case , $|Q| < M$, the expression for r_* is slightly more complicated, but the asymptotic are the same.

$$r_* =r + \frac{r_+^2}{r_+ - r_-}\ln(\frac{r-r_+}{M}) - \frac{r_-^2}{r_+ - r_-}\ln(\frac{r-r_-}{M}) + C_*, \tag{2.15}$$

$$\lim_{r_*\to\infty} r_*^{-1} r =1, \tag{2.16}$$

$$\lim_{r_*\to -\infty} r_*^{-1}\ln(\frac{r-r_+}{M}) =\frac{r_+ - r_-}{r_+^2}. \tag{2.17}$$

In the critical case, $|Q| = M$, the expression for r_* and the asymptotics are inverse linear, instead of logarithmic, towards the event horizon.

$$r_* =r + 2M\ln(\frac{r-M}{M}) - \frac{M^2}{r-M} + C_*, \tag{2.18}$$

$$\lim_{r_*\to\infty} r_*^{-1} r =1, \tag{2.19}$$

$$\lim_{r_*\to -\infty} r_*(r-M) =-M^2. \tag{2.20}$$

2.2 The wave equation

We wish to study the linear wave equation,

$$\Box_{\mathrm{RN}}\tilde{u} =0, \qquad \tilde{u}(1) =\tilde{u}_0, \qquad \frac{\partial}{\partial t}\tilde{u}(1) =\tilde{u}_1, \tag{2.21}$$

in the exterior region of the Reissner-Nordstrøm solution. In terms of the tortoise co-ordinate the d'Alembertian is

$$\Box_{\mathrm{RN}} =F^{-1}(\frac{\partial^2}{\partial t^2} - r^{-2}\frac{\partial}{\partial r_*}r^2\frac{\partial}{\partial r_*}) - r^{-2}\Delta_{S^2}, \tag{2.22}$$

where Δ_{S^2} is the Laplace-Beltrami operator on the sphere. The substitution

$$u =r\tilde{u} \tag{2.23}$$

simplifies the wave equation to

$$\frac{\partial^2}{\partial t^2}u + Hu =0, \qquad u(1) =u_0, \qquad \frac{\partial}{\partial t}u(1) = u_1, \tag{2.24}$$

where the operator H is composed of the following terms

$$H =\sum_{i=1}^{3} H_i, \tag{2.25}$$

$$H_1 =-\frac{\partial^2}{\partial r_*^2}, \tag{2.26}$$

$$H_2 =V(r) = \frac{1}{r}F\frac{dF}{dr}, \tag{2.27}$$

$$H_3 =V_l(r)(-\Delta_{S^2}), \tag{2.28}$$

$$V_l(r) =\frac{1}{r^2}F. \tag{2.29}$$

We refer to V as the potential and V_l as the angular potential. In the exterior region, $r > r_+$, the angular potential has a single critical point at (in terms of the r variable) [2]

$$\alpha = \frac{3M + \sqrt{9M^2 - 8Q^2}}{2} \tag{2.30}$$

The critical point, α, is extremely important. Geodesics at $r = \alpha$ and tangent to this surface will remain on the surface forever, and geodesics which approach this surface almost tangentially can approach the surface as $t \to \infty$ or orbit the black hole arbitrarily many times before escaping to $r_* \to \pm\infty$ [8, 32]. This geodesic surface at $r = \alpha$ is sometimes called the photon sphere. The region near $r = \alpha$ will also be the most difficult in which to prove decay of solution to the wave equation. This region orly presents a problem for non- radial data.

The value of r_* corresponding to $r = \alpha$ will be denoted α_*, and we will introduce the new radial coordinate

$$\rho_* = r_* - \alpha_* \tag{2.31}$$

This corresponds to taking $r_* = \rho_*$ with the integration constant C_* in equation (2.12), (2.15), or (2.18) chosen so that

$$\alpha_* = 0. \tag{2.32}$$

For this reason, $\frac{\partial}{\partial r_*} = \frac{\partial}{\partial \rho_*}$, etc.

For simplicity, we typically use the measure

$$d^3\mu = dr_* d^2\mu_{S^2}. \tag{2.33}$$

The space $\mathfrak{M}$ refers to $\mathbb{R} \times S^2$ with the measure $d^3\mu$. This defines $L^2(\mathfrak{M})$ and, more generally, $L^p(\mathfrak{M})$ for any $p \geq 1$. Unless otherwise specified all norms and inner products are with respect to $L^2(\mathfrak{M})$.

The function space $H^1(\mathfrak{M})$ is the collection of functions for which u' and $V_l^{\frac{1}{2}} \nabla_S^2 u$ are in $L^2(\mathfrak{M})$. This space is not particularly useful, and we typically consider functions with finite energy or conformal charge, as defined in section 3.

When dealing with the original function, $\tilde{u}$, we also use the measure

$$d^3\tilde{\mu} = r^2 dr_* d^2\mu_{S^2}. \tag{2.34}$$

The two L^2 spaces coincide, in the sense that

$$\|u\|_{L^2(\mathfrak{M})} = \|\tilde{u}\|_{L^2(\tilde{\mathfrak{M}})} \tag{2.35}$$

The Schwartz space, $\mathbb{S}$, refers to functions which are infinitely differentiable and for which, for a given $v \in \mathbb{S}$,

$$\forall i, j, k \in \mathbb{Z}^+, \exists C_{i,j,k} |\rho_*^i \frac{\partial}{\partial r_*}^j \nabla_{S^2}^k v| < C_{i,j,k} \tag{2.36}$$

The Fourier transform on $\mathbb{R} \times S^2$ is defined by first making a spherical harmonic decomposition in the angular variable, and then applying the one dimensional Fourier transform on each spherical harmonic.

3 Methods for point wise decay

In Minkowski space, $\mathbb{R}^{3+1}$, there is a conformal charge which is conserved and which dominates $t^2\|r^{-1}\nabla_{S^2} u\|^2$. Using a Sobolev estimate, the decay of the angular component of the $H^1(\mathbb{R}^3)$ norm implies decay of the the $L^6(\mathbb{R}^3)$ norm [18]. We introduce an analogous conformal charge for the Reissner-Nordstrøm solution. The growth of this charge is dominated by the sum of localised space-time integrals of u and its angular derivative. Using a Sobolev estimate and the conformal charge, we reduce the proof of a point wise in time $L^p(\mathfrak{M})$ estimate to bounding localised space-time integrals of a solution and its angular derivative.

[2]For the critical Reissner-Nordstrøm solution, the angular potential has a second critical point at the event horizon $R = M$.

3.1 Densities, energy conservation, and the conformal charge

There are many formalisms for studying wave equations. In this subsection, we will introduce various densities and show that they satisfy differential relations. Integrating these relations will show that the energy is conserved and give an identity for the time derivative of the conformal charge.

Definition 3.1. *Given a pair of functions, $(v, w) \in \mathbb{S} \times \mathbb{S}$, the energy, radial momentum, and angular momentum densities are defined respectively by*

$$e[v,w] = \frac{1}{2}(w^2 + v'^2 + Vv^2 + V_L \nabla_{S^2} v \cdot \nabla_{S^2} v), \tag{3.1}$$

$$p_{\rho_*}[v,w] = wv', \tag{3.2}$$

$$\vec{p}_\omega[v,w] = w\nabla_{S^2} v. \tag{3.3}$$

The energy is defined to be

$$E[v,w] = \int e[v,w] d^3\mu \tag{3.4}$$

These densities satisfy the following differential relations.

Lemma 3.2. *If $u \in \mathbb{S}$ is a solution to the wave equation, $\ddot{u} + Hu = 0$, then the following relations hold*

$$0 = \frac{\partial}{\partial t} e[u(t), \dot{u}(t)] - \partial_{\rho_*} p_{\rho_*}[u(t), \dot{u}(t)] - \nabla_{S^2} \cdot (V_L \vec{p}_\omega[u(t), \dot{u}(t)]) = 0, \tag{3.5}$$

$$0 = \frac{\partial}{\partial t} p_{\rho_*}[u(t), \dot{u}(t)] - \partial_{\rho_*} \frac{1}{2}(\dot{u}\dot{u} + u'u' - uVu - \nabla_{S^2} u \cdot V_L \nabla_{S^2} u) - \nabla_{S^2} \cdot (u' V_L \nabla_{S^2} u) \tag{3.6}$$

$$- \frac{1}{2} u V' u - \frac{1}{2} \nabla_{S^2} u \cdot V_L' \nabla_{S^2} u, \tag{3.7}$$

and the energy is conserved

$$\frac{d}{dt} E[u(t), \dot{u}(t)] = 0. \tag{3.8}$$

Proof. The relations are proven using the method of multipliers, in which both sides of the wave equation are multiplied by a quantity, typically a differential operator acting on u, and then the right hand side is rearranged.

The relation for the time derivative of the energy comes from the multiplier $\frac{\partial}{\partial t} u$.

$$0 = (\frac{\partial}{\partial t} u)(\ddot{u} - u'' + Vu - V_L(\Delta_{S^2})u) \tag{3.9}$$

$$= \dot{u}\ddot{u} + \dot{u}'u' + \partial_{\rho_*}(\dot{u}u') + \dot{u}Vu + \nabla_{S^2} u \cdot (V_L \nabla_{S^2} u) + \nabla_{S^2} \cdot (\dot{u} V_L \nabla_{S^2} u) \tag{3.10}$$

$$= \frac{1}{2}\frac{\partial}{\partial t}(\dot{u}\dot{u} + u'u' + uVu + \nabla_{S^2} u \cdot V_L \nabla_{S^2} u) - \partial_{\rho_*}(\dot{u}u') - \nabla_{S^2} \cdot (\dot{u} V_L \nabla_{S^2} u) \tag{3.11}$$

$$= \frac{\partial}{\partial t} e[u, \dot{u}] - \partial_{\rho_*} p_{\rho_*}[u, \dot{u}] - \nabla_{S^2} \cdot (V_L \vec{p}_\omega[u, \dot{u}]) \tag{3.12}$$

Since the integral of a pure spatial derivative is identically zero, integrating this result gives that $\frac{d}{dt} E[u(t), \dot{(t)}]$ is zero, and that the energy is conserved.

The relation for the time derivative of the radial momentum comes from multiplier $\partial_{\rho_*} u$.

$$0 = (\partial_{\rho_*} u)(\ddot{u} - u'' + Vu + V_L(-\Delta_{S^2})u) \tag{3.13}$$

$$= u'\ddot{u} - u'u'' + u'Vu + u'V_L(-\Delta_{S^2})u \tag{3.14}$$

$$= \frac{\partial}{\partial t}(u'\dot{u}) - \dot{u}'\dot{u} - u'u'' + u'Vu + \nabla_{S^2} u' \cdot V_L \nabla_{S^2} u - \nabla_{S^2} \cdot (u' V_L \nabla_{S^2} u) \tag{3.15}$$

$$= \frac{\partial}{\partial t} p_{\rho_*} - \partial_{\rho_*} \frac{1}{2}(\dot{u}\dot{u} + u'u' - uVu - \nabla_{S^2} u \cdot V_L \nabla_{S^2} u) \tag{3.16}$$

$$- \frac{1}{2} u V' u - \frac{1}{2} \nabla_{S^2} u \cdot V_L' \nabla_{S^2} u \tag{3.17}$$

$$- \nabla_{S^2} \cdot (u' V_L \nabla_{S^2} u) \tag{3.18}$$

□

We now define the conformal charge in terms of the energy and momentum densities. In Minkowski space, the conformal multiplier is found by conjugating the time derivative with the discrete inversion symmetry of Minkowski space. The conformal multiplier is then used to define the conformal charge and its density by the same process which defines the energy and energy density from the time derivative. The Reissner-Nordstrøm solution does not have a discrete inversion symmetry, so we define our conformal multiplier by formally taking the Minkowski conformal multiplier and replacing the Minkowski radial variable by ρ_*, the Reissner-Nordstrøm radial variable.

Definition 3.3. *The conformal multiplier, the conformal charge density for a pair of functions* $(v, w) \in \mathbb{S}\times\mathbb{S}$*, and the conformal charge for the same pair are defined respectively by*

$$\mathcal{C} =(t^2 + \rho_*^2)\frac{\partial}{\partial t} + 2t\rho_*\partial_{\rho_*}, \tag{3.19}$$

$$e_{\mathcal{C}}[v, w] =(t^2 + \rho_*^2)e[v, w] + 2t\rho_* p_{\rho_*}, \tag{3.20}$$

$$E_{\mathcal{C}}[v, w] = \int e_{\mathcal{C}}[v, w]d^3\mu. \tag{3.21}$$

The conformal charge density can be rewritten as a manifestly positive quantity. This form is more useful for making estimates.

Lemma 3.4. *For any pair* (v, w)*,*

$$(t^2 + \rho_*^2)e[v, w] + 2t\rho_* p[v, w] =\frac{1}{4}(t - \rho_*)^2(w - v')^2 + \frac{1}{4}(t + \rho_*)^2(w + v')^2 \tag{3.22}$$

$$+ \frac{1}{2}(t^2 + \rho_*^2)V v^2 + \frac{1}{2}(t^2 + \rho_*^2)V_L(\nabla_{S^2} v \cdot \nabla_{S^2} v) \tag{3.23}$$

Proof. Only the time and radial derivative terms need to be rearranged.

$$(t - \rho_*)^2(w - v')^2 =t^2w^2 - 2t^2wv' + t^2v'^2 \tag{3.24}$$

$$- 2t\rho_*w^2 + 4t\rho_*wv' - 2t\rho_*v'^2 \tag{3.25}$$

$$+ \rho_*^2w^2 - 2\rho_*^2wv' + \rho^2v'^2 \tag{3.26}$$

From this,

$$(t - \rho_*)^2(w - v')^2 + (t + \rho_*)^2(w + v')^2 =2(t^2 + \rho_*^2)(w^2 + v'^2) + 8t\rho_*wv' \tag{3.27}$$

□

In Minkowski space, since the time derivative is the generator of the time translation symmetry, the conformal multiplier is a composition of symmetries, and hence a symmetry itself. From Noether's theorem, the conformal charge it generates is conserved. Our conformal multiplier is not constructed from symmetries and does not generate a conserved quantity. Heuristically, the change of the Reissner-Nordstrøm conformal charge should only involve the potentials, since, formally, the conformal multiplier is the same as in $\mathbb{R}^{3+1}$, and the wave equation differs only by the presence of potentials. The potential appear in expressions quantities of the form $2V+\rho_*V'$. We call them the trapping terms. In $\mathbb{R}^3$, the analogue of V_l is r^{-2}, and the corresponding trapping term, $2V + rV'$, vanishes.

Lemma 3.5. *If* $u \in \mathbb{S}$ *is a solution to the wave equation,* $\ddot{u} + Hu = 0$*, then*

$$\frac{d}{dt}E_{\mathcal{C}}[u(t), \dot{u}(t)] = \int t(2V + \rho_*V')u^2d^3\mu + \int t(2V_L + \rho_*V_L')(\nabla_{S^2}u \cdot \nabla_{S^2}u)d^3\mu \tag{3.28}$$

(3.29)

Proof. We multiply the wave equation by the conformal multiplier, $\mathcal{C}u$, and then apply the relations from lemma 3.2.

$$0 =(\mathcal{C}u)(\ddot{u} - u'' + Vu + V_L(-\Delta_{S^2})u) \tag{3.30}$$
$$=(t^2+\rho_*^2)\dot{u}(\ddot{u} - u'' + Vu + V_L(-\Delta_{S^2})u) \tag{3.31}$$
$$+ 2t\rho_* u'(\ddot{u} - u'' + Vu + V_L(-\Delta_{S^2})u) \tag{3.32}$$
$$=(t^2+\rho_*^2)\left(\frac{\partial}{\partial t}e[u(t),\dot{u}(t)] - \partial_{\rho_*}p_{\rho_*} - \nabla_{S^2}\cdot(V_L p_\omega)\right) \tag{3.33}$$
$$+ 2t\rho_*\left(\frac{\partial}{\partial t}p_{\rho_*} - \partial_{\rho_*}\frac{1}{2}(\dot{u}\dot{u} + u'u' - uVu - \nabla_{S^2}u\cdot V_L\nabla_{S^2}u) - \nabla_{S^2}\cdot(u'V_L\nabla_{S^2}u)\right) \tag{3.34}$$
$$- 2t\rho_*\left(\frac{1}{2}uV'u - \frac{1}{2}\nabla_{S^2}u\cdot V_L'\nabla_{S^2}u\right) \tag{3.35}$$

Integrating these terms and then integrating by parts in the angular derivatives eliminates the angular gradients.

$$0 = \int (t^2+\rho_*^2)\left(\frac{\partial}{\partial t}e[u(t),\dot{u}(t)] - \partial_{\rho_*}p_{\rho_*} - \nabla_{S^2}\cdot(V_L p_\omega)\right)d^3\mu \tag{3.36}$$
$$+ \int 2t\rho_*\left(\frac{\partial}{\partial t}p_{\rho_*} - \partial_{\rho_*}\frac{1}{2}(\dot{u}\dot{u} + u'u' - uVu - \nabla_{S^2}u\cdot V_L\nabla_{S^2}u) - \nabla_{S^2}\cdot(u'V_L\nabla_{S^2}u)\right)d^3\mu \tag{3.37}$$
$$- \int 2t\rho_*\left(\frac{1}{2}uV'u + \frac{1}{2}\nabla_{S^2}u\cdot V_L'\nabla_{S^2}u\right)d^3\mu \tag{3.38}$$
$$= \int (t^2+\rho_*^2)\left(\frac{\partial}{\partial t}e[u(t),\dot{u}(t)] - \partial_{\rho_*}p_{\rho_*}\right)d^3\mu \tag{3.39}$$
$$+ \int 2t\rho_*\left(\frac{\partial}{\partial t}p_{\rho_*} - \partial_{\rho_*}\frac{1}{2}(\dot{u}\dot{u} + u'u' - uVu - \nabla_{S^2}u\cdot V_L\nabla_{S^2}u)\right)d^3\mu \tag{3.40}$$
$$- \int 2t\rho_*\left(\frac{1}{2}uV'u + \frac{1}{2}\nabla_{S^2}u\cdot V_L'\nabla_{S^2}u\right)d^3\mu \tag{3.41}$$

This is further simplified by integrating by parts in the radial variable and isolating pure time derivatives.

$$0 = \int \frac{\partial}{\partial t}\left((t^2+\rho_*^2)e[u(t),\dot{u}(t)] + 2t\rho_* p_{\rho_*}\right)d^3\mu \tag{3.42}$$
$$+ \int -2te[u(t),\dot{u}(t)] + 2\rho_* p_{\rho_*}d^3\mu \tag{3.43}$$
$$+ \int -2\rho_* p_{\rho_*} + 2t\frac{1}{2}(\dot{u}\dot{u} + u'u' - uVu - \nabla_{S^2}u\cdot V_L\nabla_{S^2}u)d^3\mu \tag{3.44}$$
$$- \int 2t\rho_*\left(\frac{1}{2}uV'u + \frac{1}{2}\nabla_{S^2}u\cdot V_L'\nabla_{S^2}u\right)d^3\mu \tag{3.45}$$
$$=\frac{d}{dt}\int (t^2+\rho_*^2)e[u(t),\dot{u}(t)] + 2t\rho_* p_{\rho_*}d^3\mu \tag{3.46}$$
$$- \int 2t(Vu^2 + \nabla_{S^2}u\cdot V_L\nabla_{S^2}u)d^3\mu \tag{3.47}$$
$$- \int 2t\rho_*\left(\frac{1}{2}uV'u + \frac{1}{2}\nabla_{S^2}u\cdot V_L'\nabla_{S^2}u\right)d^3\mu \tag{3.48}$$

□

3.2 Sobolev estimates

Our goal in this section is to bound the L^6 norm by the energy, the conformal charge, and a negative power of t. The main step in this is to prove a Sobolev estimate, which, roughly speaking, controls the

L^6 norm by the H^1 norm. The analogous result in Minkowski space,$\mathbb{R}^{3+1}$, is that the spatial L^6 norm can be controlled by a third of a factor of the radial component of the H^1 norm and two thirds of the angular component of the inhomogeneous H^1 norm. The energy controls the radial component of the H^1 norm, and the conformal charge controls the product of a positive power of t and the angular component of the H^1 norm. It is not obvious from the definition that the conformal charge controls the weighted L^2 norm needed in the inhomogeneous part of the angular H^1 norm, but it does [18]. A similar result holds for the Reissner-Nordstrøm solution, but, because of the weight appearing in the L^6 norm, we require two estimates on weighted L^2 norms instead of one.

The two estimates in the following lemma are the estimates needed for the Sobolev estimate. Note that the terms appearing on the right are independent of w, the second argument of $E_{\mathcal{C}}$.

Lemma 3.6. *There is a constant such that, for all $(v,w) \in \mathbb{S}\times\mathbb{S}$*

$$E_{\mathcal{C}}[v,w] \geq C\langle v, \frac{t^2+\rho_*^2}{\rho_*^2+1}v\rangle, \tag{3.49}$$

$$E_{\mathcal{C}}[v,w]t^{-2} \geq C\langle v, \frac{1}{\rho_*^2+1}v\rangle. \tag{3.50}$$

Proof. This is a smoothed version of the argument used in $\mathbb{R}^n$ [18].

The ingoing and outgoing wave terms can be isolated and rearranged. We introduce the notation $e_{\mathcal{C},(t,\rho_*)}$ to denote these terms.

$$\begin{aligned} e_{\mathcal{C},(t,\rho_*)}[v,w] =& \|\rho_* w\|^2 + \|tw\|^2 + \|\rho_* v\|^2 + \|tv\|^2 + \langle w, 4\rho_* tv\rangle & (3.51)\\ =& \|tw+\rho_* v'\|^2 + \|\rho_* w + tv'\|^2 & (3.52)\end{aligned}$$

The weighted term $hu = h(\rho_*)u$ is introduced into these terms.

$$\begin{aligned} e_{\mathcal{C},(t,\rho_*)}[v,w] =& \|tw+\rho_* v' - \rho_* hv\|^2 + \|\rho_* w + tv' + thv\|^2 & (3.53)\\ &+ 2\langle \rho_* v', \rho_* hv\rangle - 2\langle tv', thv\rangle - \langle \rho_* hv, \rho_* hv\rangle - \langle thv, thv\rangle & (3.54)\end{aligned}$$

Dropping the first two terms, which are strictly positive, and integrating by parts in the following pair yields:

$$e_{\mathcal{C},(t,\rho_*)}[v,w] \geq -\langle v, (2\rho_* h + (\rho_*^2-t^2)h')v\rangle - \langle v, (\rho_*^2+t^2)h^2 v\rangle \tag{3.55}$$

A particular choice of h can now be made in terms of a parameter a.

$$\begin{aligned} h(\rho_*) =& \epsilon\frac{\rho_*}{\rho_*^2+a} & (3.56)\\ h'(\rho_*) =& \epsilon\frac{a-\rho_*^2}{(\rho_*^2+a)^2} & (3.57)\end{aligned}$$

We now substitute this choice of h into the previous calculations.

$$\begin{aligned} -(2\rho_* h + (\rho_*^2-t^2)h') - (\rho_*^2+t^2)h^2 =& -\epsilon(\frac{2\rho_*^2}{\rho_*^2+a} + (\rho_*^2-t^2)\frac{a-\rho_*^2}{(\rho_*^2+a)^2}) - \epsilon^2\frac{(\rho_*^2+t^2)\rho_*^2}{(\rho_*^2+a)^2} & (3.58)\\ =& -\epsilon\frac{\rho_*^4+3a\rho_*^2-t^2a+t^2\rho_*^2}{(\rho_*^2+a)^2} - \epsilon^2\frac{\rho_*^4+t^2\rho_*^2}{(\rho_*^2+a)^2} & (3.59)\\ =& -(\epsilon+\epsilon^2)\frac{\rho_*^4}{(\rho_*^2+a)^2} - \epsilon\frac{3a\rho_*^2}{(\rho_*^2+a)^2} - \epsilon\frac{-t^2a}{(\rho_*^2+a)^2} - (\epsilon+\epsilon^2)\frac{t^2\rho_*^2}{(\rho_*^2+a)^2} & (3.60)\end{aligned}$$

For $\epsilon \in (-1,0)$ the first, second, and forth terms are positive. For $\epsilon \in (-\frac{1}{2},0)$ and $\rho_*^2 > \frac{a}{4}$, the forth term dominates the third by a factor of 2. ϵ can be chosen sufficiently close to zero so that for $\rho_*^2 < \frac{a}{4}$,

$$0 \leq -\epsilon\frac{-t^2a}{(\rho_*^2+a)^2} + V(\rho_*^2+t^2) \leq -\epsilon\frac{-t^2a}{(\rho_*^2+a)^2} + H_2(\rho_*^2+t^2) \tag{3.61}$$

Thus, with $a = 1$,

$$E_{\mathcal{C}}[v,w] \geq e_{\mathcal{C},(t,\rho_*)}[v,w] + \langle v, H_2 v\rangle \geq (-\epsilon - \epsilon^2)\langle v, \frac{t^2+\rho_*^2}{\rho_*^2+1} v\rangle \tag{3.62}$$

This proves the first statement in the theorem. The second part follows trivially by dropping the $\rho_*^2/(\rho_*^2+1)$ term from the first estimate and dividing by t^2. □

We now turn to the Sobolev estimate. Because our main interest is making estimates in terms of the energy and conformal estimate, the estimate is expressed both in terms of L^2 norms of the derivatives of the function under consideration and in terms of the energy and conformal charge. Once again, the energy and conformal charge take a second argument w which does not appear in the quantity estimated.

Lemma 3.7. *There is a constant C, such that if $(v,w) \in \mathbb{S}\times\mathbb{S}$, then*

$$\int |v|^6 F^3 r^{-4} d^3\mu \leq C(E[v,w] + E_{\mathcal{C}}[v,w]t^{-2})E_{\mathcal{C}}[v,w]^2 t^{-4}, \tag{3.63}$$

$$\|F^{\frac{1}{2}} r^{\frac{-2}{3}} v\|_{L^6(\mathfrak{M})} \leq C(E[v,w] + E_{\mathcal{C}}[v,w]t^{-2})^{\frac{1}{6}} E_{\mathcal{C}}[v,w]^{\frac{1}{3}} t^{-\frac{2}{3}}. \tag{3.64}$$

Proof. We begin by proving a Sobolev estimate which controls the L^6 norm by weighted H^1 norms.

Following the standard argument [30, 16], the proof starts with a $W^{1,1} \hookrightarrow L^{\frac{3}{2}}$ estimate. Take $\psi(\rho_*,\omega)$ infinitely differentiable and of compact support for simplicity. To this function ψ associate a function on the sphere

$$I_1(\omega) \equiv \int_{\mathbb{R}} |\frac{\partial}{\partial r_*}\psi(\rho_*,\omega)| dr_* \tag{3.65}$$

Integration and the Sobolev estimates for $\mathbb{R}$ and S^2 can be applied. The Sobolev estimate in 1 dimension follows directly from the fundamental theorem of Calculus.

$$|\psi(\rho_*,\omega)| \leq I_1(\omega) \tag{3.66}$$

$$|\psi(\rho_*,\omega)|^{\frac{3}{2}} \leq I_1(\omega)^{\frac{1}{2}} |\psi(\rho_*,\omega)| \tag{3.67}$$

$$\int_{S^2} |\psi(\rho_*,\omega)|^{\frac{3}{2}} d\omega \leq \int_{S^2} I_1(\omega)^{\frac{1}{2}} |\psi(\rho_*,\omega)| d\omega \tag{3.68}$$

The second term in the final product on the right side is estimated by the spherical Sobolev estimate. The Sobolev estimate on S^2 follows from using a partition of unity on S^2 into co-ordinate charts and then applying the Sobolev estimate on $\mathbb{R}^2$ [16]. The notation $\|\psi\|_{\binom{1}{\omega}}(\rho_*)$ is introduced to denote the $L^1(S^2)$ norm of $\psi(\rho_*,\omega)$ with ρ_* fixed.

$$\int_{S^2} |\psi(\rho_*,\omega)|^{\frac{3}{2}} d\omega \leq (\int_{S^2} I_1(\omega) d\omega)^{\frac{1}{2}} (\int_{S^2} |\psi(\rho_*,\omega)|^2 d\omega)^{\frac{1}{2}} \tag{3.69}$$

$$\leq \|\frac{\partial}{\partial r_*}\psi\|_1^{\frac{1}{2}} (\|(-\Delta_{S^2})^{\frac{1}{2}}\psi\|_{\binom{1}{\omega}}(\rho_*) + \|\psi\|_{\binom{1}{\omega}}(\rho_*)) \tag{3.70}$$

A radial weight $f^\alpha(\rho_*)$ can be introduced before integrating with respect to dr_*. For this proof, the exponents α and β are used. They have no relation to the value of $r-\alpha$ governing the location of the photon sphere.

$$\int_{\mathbb{R}}\int_{S^2} f^\alpha(\rho_*)|\psi(\rho_*,\omega)|^{\frac{3}{2}} d\omega dr_* \leq \|\frac{\partial}{\partial r_*}\psi\|_1^{\frac{1}{2}} \int_{\mathbb{R}} f^\alpha(\|(-\Delta_{S^2})^{\frac{1}{2}}\psi\|_{\binom{1}{\omega}}(\rho_*) + \|\psi\|_{\binom{1}{\omega}}(\rho_*)) dr_* \tag{3.71}$$

$$\|f^{\frac{2\alpha}{3}}\psi\|_{\frac{3}{2}} \leq \|\frac{\partial}{\partial r_*}\psi\|_1^{\frac{1}{3}} (\|f^\alpha(-\Delta_{S^2})^{\frac{1}{2}}\psi\|_1 + \|f^\alpha\psi\|_1)^{\frac{2}{3}} \tag{3.72}$$

The substitution $\psi = f^\beta|v_1|^4$ transforms this to a $H_1 \hookrightarrow L^6$ Sobolev estimate.

$$\|f^{\frac{2\alpha}{3}}\psi\|_{\frac{3}{2}} \le (4\int f^\beta|v_1|^3|\frac{\partial}{\partial r_*}v_1| + \beta f^{\beta-1}f'|v_1|^4 d^3\mu)^{\frac{1}{3}} \tag{3.73}$$

$$\times (4\int f^{\alpha+\beta}|v_1|^3|(-\Delta_{S^2})^{\frac{1}{2}}v_1|d^3\mu + \int f^{\alpha+\beta}|v_1|^4 d^3\mu)^{\frac{2}{3}} \tag{3.74}$$

$$(\int f^{\alpha+\frac{3}{2}\beta}|v_1|^6 d^3\mu)^{\frac{2}{3}} \le 4(\int f^{2\beta}|v_1|^6 d^3\mu)^{\frac{1}{6}}(\int |\frac{\partial}{\partial r_*}v_1|^2 d^3\mu + \int (\beta\frac{f'}{f})^2|v_1|^2 d^3\mu)^{\frac{1}{6}} \tag{3.75}$$

$$\times (\int f^{2\alpha+2\beta-1}|v_1|^6 d^3\mu)^{\frac{1}{3}}(\int f|(-\Delta_{S^2})^{\frac{1}{2}}v_1|^2 d^3\mu + \int f|v_1|^2 d^3\mu)^{\frac{2}{6}} \tag{3.76}$$

We now choose

$$\alpha + \frac{3}{2}\beta = 2\beta = 2\alpha + 2\beta - 1, \tag{3.77}$$

$$\alpha = \frac{1}{2}, \tag{3.78}$$

$$\beta = 1, \tag{3.79}$$

so that $\|f^{\frac{1}{3}}v_1\|_6$ can be cancelled.

$$\|f^{\frac{1}{3}}v_1\|_6 \le 4(\|\frac{\partial}{\partial r_*}v_1\|_2 + \|\frac{f'}{f}v_1\|^2)^{\frac{1}{3}}(\|f^{\frac{1}{2}}(-\Delta_{S^2})^{\frac{1}{2}}v_1\|_2 + \|f^{\frac{1}{2}}v_1\|_2)^{\frac{2}{3}} \tag{3.80}$$

The weight $f = r^{-2}$ is now taken to give a result analogous to the Sobolev estimate in $\mathbb{R}^3$.

$$\|r^{\frac{-2}{3}}v_1\|_6 \le C(\|\frac{\partial}{\partial r_*}v_1\|_2 + \|-2Fr^{-1}v_1\|^2)^{\frac{1}{3}}(\|r^{-1}(-\Delta_{S^2})^{\frac{1}{2}}v_1\|_2 + \|r^{-1}v_1\|_2)^{\frac{2}{3}} \tag{3.81}$$

To complete the proof of the Sobolev estimate for the Reissner-Nordstrøm solution, the substitution $v_1 = F^{\frac{1}{2}}v$ and the inequality $Fr^{-1} \le (1+\rho_*^2)^{-\frac{1}{2}}$ are used.

$$\|F^{\frac{1}{2}}r^{\frac{-2}{3}}v\|_6 \le C(\|F^{\frac{1}{2}}\frac{\partial}{\partial r_*}u\|_2 + \|\frac{1}{2}F^{\frac{1}{2}}(2Mr^{-2} - 2Q^2)v\| + \|-2F^{\frac{3}{2}}r^{-1}u\|^2)^{\frac{1}{3}} \tag{3.82}$$

$$\times (\|F^{\frac{1}{2}}r^{-1}(-\Delta_{S^2})^{\frac{1}{2}}v\|_2 + \|F^{\frac{1}{2}}r^{-1}v\|_2)^{\frac{2}{3}} \tag{3.83}$$

$$\le C(\|F^{\frac{1}{2}}\frac{\partial}{\partial r_*}v\|_2 + \|(1+\rho_*^2)^{-\frac{1}{2}}v\|)^{\frac{1}{3}} \tag{3.84}$$

$$\times (\|F^{\frac{1}{2}}r^{-1}(-\Delta_{S^2})^{\frac{1}{2}}v\|_2 + \|F^{\frac{1}{2}}r^{-1}v\|_2)^{\frac{2}{3}} \tag{3.85}$$

We introduce the dummy function w to act as the second argument of the energy and conformal charge. From the definition of the energy, the conformal charge, and lemma 3.6,

$$\|F^{\frac{1}{2}}\frac{\partial}{\partial r_*}v\|_2^2 \le \|\frac{\partial}{\partial r_*}v\|_2^2 \le E[v,w], \tag{3.86}$$

$$\|F^{\frac{1}{2}}r^{-1}(-\Delta_{S^2})^{\frac{1}{2}}v\|_2 \le E_{\mathcal{C}}[v,w]t^{-2}, \tag{3.87}$$

$$\|F^{\frac{1}{2}}r^{-1}v\|_2 \le \|(1+\rho_*^2)^{-\frac{1}{2}}v\| \le E_{\mathcal{C}}[v,w]t^{-2}. \tag{3.88}$$

Substituting these into equation (3.85) proves the first two results. The second is simply the sixth root of the first. □

We remark that if we repeat the same argument with $f = Fr^{-2}$ and $v_1 = v$, the Sobolev type result, analogous to equation (3.85), would be

$$\|F^{\frac{1}{3}}r^{\frac{-2}{3}}v\|_6 \le C(\|\frac{\partial}{\partial r_*}v\|_2 + \|(-2r^{-1} + 6Mr^{-2} - 4Q^2r^{-3})v\|)^{\frac{1}{3}} \tag{3.89}$$

$$\times (\|F^{\frac{1}{2}}r^{-1}(-\Delta_{S^2})^{\frac{1}{2}}v\|_2 + \|F^{\frac{1}{2}}r^{-1}v\|_2)^{\frac{2}{3}}. \tag{3.90}$$

The new weighted L^2 norm is also controlled by lemma 3.6, for $t > 1$, although, there is no longer the additional factor of t^{-2}.

$$\|(-2r^{-1} + 6Mr^{-2} - 4Q^2r^{-3})v\|^2 \leq \|(\frac{t^2+\rho_*^2}{\rho_*^2+1}u)^{\frac{1}{2}}v\| \leq E_{\mathcal{C}}[v,w]. \tag{3.91}$$

Combining these results, we have control of a norm which decays less rapidly towards the event horizon but at the cost of less time decay on the right hand side.

$$\int |v|^6F^3r^{-4}d^3\mu \leq C(E[v,w] + E_{\mathcal{C}}[v,w])E_{\mathcal{C}}[v,w]^2t^{-2}, \tag{3.92}$$

3.3 Local support of the trapping terms

We refer to terms of the form $2V + \rho_*V_l'$, for both the potential and the angular potential, as trapping terms. In this section, we show that the trapping terms are positive only in a finite interval of ρ_* values in the subcritical case. The functions W and W_l will refer to the positive part of the trapping terms. Through the conformal identity and the Sobolev estimates, this reduces the problem of finding point wise, weighted L^6 estimates to proving local decay estimates of the form $\int\int \chi|u|^2d^3\mu dt + \int\int \chi|\nabla_{S^2}u|^2d^3\mu dt \leq C$.

One of the factors in the derivative of the potential will require careful attention, both here and in subsection 6.2. We introduce it with the notation $P_Q(r)$.

Definition 3.8. *For $Q \in [0, M]$, $P_Q(r)$ is defined by*

$$P_Q(r) = 3Mr^3 - 4(Q^2 + 2M^2)r^2 + 15MQ^2r - 6Q^4 \tag{3.93}$$

We now show that the trapping term for the potential is positive only in a bounded set of ρ_* values by computing the limit at $\pm\infty$.

Lemma 3.9. *The derivative of the potential is given by*

$$V' = -2Fr^{-7}P_Q(r) \tag{3.94}$$

For $|\rho_*|$ sufficiently large, $2V + \rho_*V'$ is negative.

There is a compactly supported, positive, bounded function, W, such that $W > 2V + \rho_*V'$

Proof. The derivative is computed from the definition of V in equation 2.27.

$$\partial_{\rho_*}V = \frac{dr}{d\rho_*}\frac{\partial}{\partial r}(r^{-1}F\frac{\partial F}{\partial r}) \tag{3.95}$$

$$= F\frac{\partial}{\partial r}((r^{-1} - 2Mr^{-2} + Q^2r^{-3})(2Mr^{-2} - 2Q^2r^{-3})) \tag{3.96}$$

$$= F\frac{\partial}{\partial r}(2Mr^{-3} - 2(Q^2 + 2M^2)r^{-4} + 6MQ^2r^{-5} - 2Q^4r^{-6}) \tag{3.97}$$

$$= -2Fr^{-7}(3Mr^3 - 4(Q^2 + 2M^2)r^2 + 15MQ^2r - 6Q^4) \tag{3.98}$$

From this,

$$2V + \rho_*V' = 2r^{-1}F\left(\frac{dF}{dr} - \frac{\rho_*}{r^6}P_Q(r)\right) \tag{3.99}$$

$$= 2r^{-1}F\left(\left(\frac{2M}{r^2} - \frac{2Q^2}{r^3}\right) - \frac{\rho_*}{r^6}P_Q(r)\right) \tag{3.100}$$

$$= 2r^{-1}F\left(\left(\frac{2M}{r^2} - \frac{2Q^2}{r^3}\right) - \frac{\rho_*}{r}\left(\frac{3M}{r^2} - \frac{4(Q^2-2M^2)}{r^3} + \frac{15MQ^2}{r^4} - \frac{6Q^4}{r^5}\right)\right) \tag{3.101}$$

To show that this is negative for sufficiently large values of $|\rho_*|$, we will multiply by a positive factor and then show that the resulting quantity has a negative limit as $\rho_* \to \pm\infty$. As $\rho_* \to -\infty$, the subcritical and critical cases must be dealt with separately.

In both the subcritical and critical cases, for $\rho_* \to \infty$, $\frac{\rho_*}{r} \to 1$ and $F \to 1$, so

$$r^3(2V + \rho_* V') \to 2M - 3M = -M \tag{3.102}$$

For $\rho_* \to \infty$, the original radial variable has limit $r \to r_+ = M + \sqrt{M^2 - Q^2}$, and the limiting value of $P_Q(r_+)$ is

$$P_Q(M + \sqrt{M^2 - Q^2}) = -2(2M^2 - Q^2)(M^2 - Q^2) - 4M\sqrt{M^2 - Q^2}(M^2 - Q^2) \tag{3.103}$$

In the subcritical case, the other factors in the trapping term are positive, so it is sufficient to look at $P_Q(r_+)$.

$$\begin{aligned} \frac{r^7}{-2F\rho_*}(2V + \rho_* V') \to& P_Q(r) & (3.104)\\ =& -2(2M^2 - Q^2)(M^2 - Q^2) - 4M\sqrt{M^2 - Q^2}(M^2 - Q^2) & (3.105)\\ <& 0 & (3.106) \end{aligned}$$

In the critical case, since $P_M(M) = 0$, it is necessary to multiply by a different positive factor before taking the limit. For $\rho_* \to -\infty$, from the asymptotic behaviour of ρ_* given in equation (2.20)

$$\begin{aligned} \frac{r^4}{2F}\frac{1}{r-M}(2V + \rho_* V') =& 2M\frac{r-M}{r-M} + \frac{\rho_*}{r-M}\frac{1}{r^3}P_M(r) & (3.107)\\ =& 2M + \rho_*(r-M)\frac{3M}{r^3}(r-2M) & (3.108)\\ \to& 2M - 3M & (3.109)\\ =& -M & (3.110) \end{aligned}$$

From these limits, it follows that $2V + \rho_* V'$ is negative for sufficiently large values of ρ_*. Since V and $\rho_* V'$ are continuous, it follows that $2V + \rho_* V'$ can be bounded above by a compactly supported, positive, bounded function. □

In the subcritical case, a similar calculation of limits as $\rho_* \to \pm\infty$ shows that $2V_l' + \rho_* V_l'$ is positive only on a bounded set of ρ_* values. In the critical case, the angular trapping term is positive as $\rho_* \to -\infty$, and we compute the rate at which it vanishes as $r \to r_+ = M$.

Lemma 3.10. *In the subcritical case, for $|\rho_*|$ sufficiently large, $2V_L + \rho_* V_L'$ is negative, and there is a compactly supported, positive, bounded function, W_L, such that $W_L > 2V_L + \rho_* V_L'$.*

In the critical case, there is a positive, bounded function, W_L, such that $W_L > 2V_L + \rho_* V_L'$, W_L is identically zero for sufficiently large ρ_*, and W_L decays like $(r-M)F$ as $\rho_* \to -\infty$.

Proof. As in the previous lemma, $2V_L + \rho_* V_L'$ will be shown to be negative from the fact that when a positive factor is applied, the limit as $\rho_* \to \pm\infty$ is strictly negative or the quantity diverges to negative infinity. For this lemma, it is simplest to separate the subcritical and critical cases, when evaluating the limits.

From the definition of V_L,

$$\partial_{\rho_*} V_L = \frac{-2F}{r^3}\left(1 - \frac{3M}{r} + \frac{2Q^2}{r^2}\right) \tag{3.111}$$

so that

$$2V - \rho_* V' = \frac{2F}{r^3}\left(r - \rho_*(1 - \frac{3M}{r} + \frac{2Q^2}{r^2})\right) \tag{3.112}$$

In the subcritical, as $\rho_* \to \infty$, by the explicit expansion of ρ_* given in equation 2.15,

$$\begin{aligned} \frac{r^3}{2F}(2V - \rho_* V') =& r - (r + \frac{r_+^2}{r_+ - r_-}\log(\frac{r - r_+}{M}) - \frac{r_-}{r_+ - r_-}\log(\frac{r - r_-}{M}) + C)(1 - \frac{3M}{r} + \frac{2Q^2}{r^2}) & (3.113)\\ =& -\frac{r_+^2}{r_+ - r_-}\log(\frac{r - r_+}{M}) + \frac{r_-^2}{r_+ - r_-}\log(\frac{r - r_-}{M}) + \mathrm{O}(1) & (3.114)\\ \to& -\infty & (3.115) \end{aligned}$$

In the critical case, as $\rho_* \to \infty$, by the expansion in 2.18,

$$\frac{r^3}{2F}(2V - \rho_* V') = r - (r + 2M\log(\frac{r-M}{M}) - \frac{M^2}{r-M} + C)(1 - \frac{3M}{r} + \frac{2Q^2}{r^2}) \tag{3.116}$$

$$= -2M\log(\frac{r-M}{M}) + \mathrm{O}(1) \tag{3.117}$$

$$\to -\infty \tag{3.118}$$

For $\rho_* \to -\infty$, we use both the asymptotics of ρ_* from subsection 2.1 and the geometry of the angular potential from subsection 2.2. The critical points of V_l are the roots of $r^2 - 3Mr + 2Q^2$, which we denote by

$$\alpha_\pm = \frac{3M \pm \sqrt{9M^2 - 8Q^2}}{2}. \tag{3.119}$$

Only one of these, $\alpha_+ = \alpha$ is in the exterior region $r > r_+$. We now multiply the angular trapping term by a positive term.

$$\frac{r^5}{2F}\frac{1}{-\rho_*}(2V_L + \rho_* V_L') = \frac{r^3}{-\rho_*} + r^2 - 3Mr + 2Q^2 \tag{3.120}$$

$$= \frac{r^3}{-\rho_*} + (r - \alpha)(r - \alpha_-) \tag{3.121}$$

In the subcritical case, the relevant terms are ordered $\alpha_- < r_+ < \alpha_+$, so that in the limit $\rho_* \to -\infty$, $r \to r_+$, and

$$\frac{r^5}{2F}\frac{1}{-\rho_*}(2V_L + \rho_* V_L') \to (r_+ - \alpha_-)(r_+ - \alpha_+) < 0 \tag{3.122}$$

In the critical case, the relevant terms are not distinct, $\alpha_- = r_+ = M < 2M = \alpha_+$, so the previous argument does not hold. In fact, the angular trapping term is positive as $r- \to r_+ = M$. Instead, equation (3.121) can be rearranged as

$$(r^5)(2V_L + \rho_* V_L') = (r^3 - \rho_*(r - \alpha)(r - \alpha_-)) \tag{3.123}$$

From the asymptotics of ρ_* in equation (2.18),

$$r^3 - \rho_*(r - \alpha_-)(r - \alpha_+) = r^3 - (r + 2M\ln\left(\frac{r-M}{M}\right) - \frac{M^2}{r-M} + C_*)(r - M)(r - 2M) \tag{3.124}$$

$$= (r^3 + M^2(r - 2M)) + \ln\left(\frac{r-M}{M}\right)(r - M)2M(2M - r) + (r - M)(r + C_*)(r - 2m) \tag{3.125}$$

On the right, the last term clearly vanishes like $r - M$, and the second term is negative as $\rho_* \to -\infty$. The first term is a polynomial in r which vanishes at $r = M$, so it must vanish at least linearly. Thus the right hand side is bounded above by a term which vanishes linearly in $r - M$. The potential term vanishes at a rate which is F times faster. $\square$

Together, the results from this section show that a weighted L^6 norm is controlled, point wise in time, by weighted space- time integrals of a solution and its derivative.

Proposition 3.11. *In the subcritical case, $|Q| < M$, there are bounded, compactly supported functions W and W_L such that if $u \in \mathbb{S}$ is a solution to the wave equation, $\ddot{u} + Hu = 0$, then*

$$\int |u|^6(t, \rho_*, \omega)F^3 r^{-4} d^3\mu \le C(E[u_0, u_1] + E_{\mathcal{C}}[u(t), \dot{u}(t)]t^{-2})E_{\mathcal{C}}[u(t), \dot{u}(t)]^{\frac{2}{3}} t^{\frac{-4}{3}} \tag{3.126}$$

$$E_{\mathcal{C}}[u(t), \dot{u}(t)] \le E_{\mathcal{C}}[u_0, u_1] + \int\int 2t(W|u|^2 + W_L|\nabla_{S^2} u|^2) d^3\mu dt \tag{3.127}$$

In the critical case, the same result holds with W_L zero for r sufficiently large and vanishing linearly in $r - M$ as $r \to r_+ = M$.

4 Relativistic considerations on the event horizon

To study the behaviour of waves on the event horizon, it is common to introduce the Eddington-Finkelstein co-ordinates [3][9, 25, 33]

$$s_- = t - \rho_* \in \mathbb{R}, \tag{4.1}$$
$$s_+ = t + \rho_* \in \mathbb{R}, \tag{4.2}$$
$$S_- = -e^{-\frac{s_-}{4M}} \in (-\infty, 0), \tag{4.3}$$
$$S_+ = e^{\frac{s_+}{4M}} \in (0, \infty). \tag{4.4}$$

In fact, this range of co-ordinates only describes the exterior region of the black hole, , and the subcritical Reissner-Nordstrøm solutions extend smoothly to positive S_- and negative S_+ [4]. The outer event horizon corresponds to both the lines $S_- = 0$ and $S_+ = 0$. The surface $S_- = 0$ is the future event horizon, $\mathcal{H}_+$, and $S_+ = 0$ is the past event horizon, $\mathcal{H}_-$. To approach $\mathcal{H}_+$, t must diverge to infinite. The bifurcation sphere is the sphere given by $(S_-, S_+) = (0, 0)$, with the spherical co-ordinates free.

In this section, we discuss how solutions must decay near the bifurcation sphere if they have finite conformal charge.

4.1 Decay on the bifurcation sphere

To get $u \in L^2(\mathfrak{M})$, we made several transformations and change of variables in section 2. To determine the physical constraints imposed by finite energy and finite conformal charge, we must return to the original function $\tilde{u} = r^{-1}u$, and use normalised vectors and their duals,

$$\partial_T = F^{\frac{-1}{2}} \partial_t, \tag{4.5}$$
$$\partial_R = F^{\frac{-1}{2}} \partial_{\rho_*}, \tag{4.6}$$
$$dT = F^{\frac{1}{2}} dt, \tag{4.7}$$
$$dR = F^{\frac{1}{2}} d\rho_*. \tag{4.8}$$

From this, the natural measure on $(\rho_* \omega) \in \mathbb{R} \times S^2$ is

$$d\mu_{\text{normalised}} = F^{\frac{1}{2}} r^2 d\rho_* d\mu_{S^2}. \tag{4.9}$$

Since there is no bifurcation sphere in the critical case, we will only consider the subcritical cases in this subsection.

We begin by writing the energy in terms of $\tilde{u}$.

$$E[u, \dot{u}] = \int |\dot{u}|^2 + |u'|^2 + Fr^{-2}|\nabla_{S^2} u|^2 + 2MFr^{-3}|u|^2 d^3\mu \tag{4.10}$$
$$= \int (|\dot{\tilde{u}}|^2 + |Fr^{-1}\tilde{u} + \tilde{u}'| + Fr^{-2}|\nabla_{S^2}\tilde{u}|^2 + V|\tilde{u}|^2) r^2 d^3\mu \tag{4.11}$$
$$= \int (|\dot{\tilde{u}}|^2 + |\tilde{u}'|^2 + Fr^{-2}|\nabla_{S^2}\tilde{u}|^2) r^2 d^3\mu \tag{4.12}$$
$$+ \int 2Fr\tilde{u}\tilde{u}' + F^2|\tilde{u}|^2 + Vr^2|\tilde{u}|^2 d^3\mu. \tag{4.13}$$

The co-ordinates t and ρ_* are singular at the bifurcation sphere, so we must rewrite the derivatives in terms

[3]The co-ordinates (s_-, s_+) are typically denoted (u, v) and the term Eddington-Finkelstein co-ordinates properly refers to the mixed co-ordinate systems (u, r, θ, ϕ) or (v, r, θ, ϕ).

[4]In the critical case, $(S_-, S_+) = (0, 0)$ is a singular point. Otherwise, the critical solution can be extended to an open set containing $S_- > 0, S_+ > 0$ and $S_- < 0, S_- < 0$.

of $\tilde{u}_T = \partial_T \tilde{u}$, $\tilde{u}_R = \partial_R \tilde{u}$, and the four gradient $\tilde{\nabla}_4 \tilde{u}$.

$$E[u,\dot{u}] = \int (F^{-1}|\dot{\tilde{u}}|^2 + F^{-1}|\tilde{u}'|^2 + r^{-2}|\nabla_{S^2}\tilde{u}|^2) Fr^2 d\rho_* d^2\omega \tag{4.14}$$

$$+ \int -(F\frac{dF}{dr}r + F^2)|\tilde{u}'|^2 + (F^2 + rF\frac{dF}{dr})|\tilde{u}|^2 d^3\mu \tag{4.15}$$

$$= \int (|\tilde{u}_T|^2 + |\tilde{u}_X|^2 + r^{-2}|\nabla_{S^2}\tilde{u}|^2) Fr^2 d\rho_* d^2\omega \tag{4.16}$$

$$= \int |\tilde{\nabla}_4 \tilde{u}|^2 F^{\frac{1}{2}} d\mu_{\text{normalised}}. \tag{4.17}$$

For the energy to be bounded, it is sufficient that, far from the bifurcation sphere the four gradient of $\tilde{u}$ is integrable and that near the bifurcation sphere,

$$|\tilde{\nabla}_4 \tilde{u}|^2 F^{\frac{1}{2}} < C, \tag{4.18}$$

$$|\tilde{\nabla}_4 \tilde{u}| < C(r - r_+)^{\frac{-1}{2}}. \tag{4.19}$$

In particular, any function which has a continuous gradient at the bifurcation sphere (and is integrable far from the bifurcation sphere), has finite energy.

The calculations and conditions for the conformal charge are similar. Once again, we begin be rewriting the conformal charge in terms of $\tilde{u}$.

$$E_{\mathcal{C}}[u,\dot{u}] = \int (\rho_*^2 + 1)(|\dot{u}|^2 + |u'|^2 + Fr^{-2}|\nabla_{S^2}u|^2 + 2MFr^{-3}|u|^2) d^3\mu \tag{4.20}$$

$$= \int (\rho_*^2 + 1)(|\tilde{u}_T|^2 + |\tilde{u}_R|^2 + r^{-2}|\nabla_{S^2}\tilde{u}|^2) Fr^2 d\rho_* d^2\omega \tag{4.21}$$

$$+ \int (\rho_*^2 + 1)(2Fr\tilde{u}\tilde{u}' + F^2|\tilde{u}|^2 + Vr^2|\tilde{u}|^2) d^3\mu \tag{4.22}$$

$$= \int (\rho_*^2 + 1)(|\tilde{u}_T|^2 + |\tilde{u}_R|^2 + r^{-2}|\nabla_{S^2}\tilde{u}|^2) Fr^2 d\rho_* d^2\omega \tag{4.23}$$

$$+ \int -2Fr\rho_*|\tilde{u}|^2 d^3\mu \tag{4.24}$$

$$= \int (\rho_*^2 + 1)(|\tilde{u}_T|^2 + |\tilde{u}_R|^2 + r^{-2}|\nabla_{S^2}\tilde{u}|^2) Fr^2 d\rho_* d^2\omega \tag{4.25}$$

$$+ \int -\frac{2\rho_*}{r}|\tilde{u}|^2 Fr^2 d\rho_* d^2\omega \tag{4.26}$$

$$= \int |\tilde{\nabla}_4 \tilde{u}|^2 (\rho_*^2 + 1) F^{\frac{1}{2}} d\mu_{\text{normalised}} + \int |\tilde{u}|^2 (-\frac{2\rho_*}{r}) F^{\frac{1}{2}} d\mu_{\text{normalised}} \tag{4.27}$$

Since, towards the event horizon, $-\frac{2\rho_*}{r}$ is positive, both of the integrands are positive. In the subcritical case, as $r \to r_+$, $\rho_* \sim \ln(r - r_+)$, so that sufficient conditions near the bifurcation sphere for the conformal charge to be bounded are

$$|\tilde{\nabla}_4 \tilde{u}|^2 (\rho_*^2 + 1) F^{\frac{1}{2}} \leq C, \qquad |\tilde{u}|^2 (-\frac{2\rho_*}{r}) F^{\frac{1}{2}} \leq C, \tag{4.28}$$

$$|\tilde{\nabla}_4 \tilde{u}| \leq C|\ln(r - r_+)|^{-1}(r - r_+)^{\frac{-1}{4}}, \qquad |\tilde{u}| \leq C|\ln(r - r_+)|^{-\frac{1}{2}}(r - r_+)^{\frac{-1}{4}}. \tag{4.29}$$

In particular, if $\tilde{u}$ and $\tilde{\nabla}_4 \tilde{u}$ are continuous at the bifurcation sphere, and decay sufficiently rapidly at ∞, then the conformal charge is bounded.

5 The Heisenberg-type relation

For the Schrödinger equation from quantum mechanics,

$$-i\psi_t + H\psi = 0, \tag{5.1}$$

there is the well known Heisenberg type relation for the time derivative of a the expectation value of a self-adjoint operator, A,

$$\frac{d}{dt}\langle\psi, A\psi\rangle =\langle\psi, i[H,A]\psi\rangle. \tag{5.2}$$

This formulation is central to the standard interpretation of quantum mechanics which associates operators to physically observable quantities, and the expectation value to the mean observed value. This formulation was also used in the original proof of scattering for the quantum n-body problem [12, 20, 28, 29].

We begin by defining the commutator in the form sense.

Definition 5.1. *If H and A are two self-adjoint operators, and $D(A)\cap D(H)$ is dense in $L^2(\mathfrak{M})$, then the commutator $[H,A]$ is defined to be the form Ω given by*

$$\Omega(v,w) = \langle Hv, Aw\rangle - \langle Av, Hw\rangle. \tag{5.3}$$

The Heisenberg-type relation follows from the wave equation and this definition.

Lemma 5.2. *If A is a time-independent, self-adjoint operator, and $u\in\mathbb{S}$ is a solution to the wave equation, $\ddot{u}+Hu=0$, then*

$$\frac{d}{dt}(\langle u, A\dot{u}\rangle - \langle\dot{u}, Au\rangle) =\langle u, [H,A]u\rangle, \tag{5.4}$$

where $\langle u,[H,A]u\rangle$ is understood to mean the quadratic form $[H,A]$ evaluated on the pair (u,u).

Proof. We begin by computing the left hand side.

$$\frac{d}{dt}(\langle u, A\dot{u}\rangle - \langle\dot{u}, Au\rangle) =\langle\dot{u}, A\dot{u}\rangle + \langle u, A\ddot{u}\rangle - \langle\ddot{u}, Au\rangle - \langle\dot{u}, A\dot{u}\rangle \tag{5.5}$$

$$=\langle u, -AHu\rangle + \langle Hu, Au\rangle. \tag{5.6}$$

Since A is self-adjoint,

$$\frac{d}{dt}(\langle u, A\dot{u}\rangle - \langle\dot{u}, Au\rangle) = -\langle Au, Hu\rangle + \langle Hu, Au\rangle. \tag{5.7}$$

The right hand side is exactly the commutator. □

Our method will be to find bounded propagation observables. A simple example of a propagation observable, A, which majorates an operator G, is one for which

$$[H,A] = G^*G \tag{5.8}$$

(where G^* represents the adjoint of G). If A is a bounded operator on the energy space, then, from the Heisenberg-type relation,

$$\int \|Gu\|^2 dt \leq \int \frac{d}{dt}(\langle u, A\dot{u}\rangle - \langle\dot{u}, Au\rangle)dt \leq 4\|\dot{u}\|\|Au\| \leq 4E[u,\dot{u}]. \tag{5.9}$$

The standard definition of a propagation observable is broader than this, and our definition will be even broader.

Definition 5.3. *Given a air of operators, A and G, with $D(A)\cap D(H)\cap D(G)$ dense in $L^2(\mathfrak{M})$, the operator A is a propagation observable which majorates G if there is a pair of bounded, non-zero operators, X_1 and X_2, for which*

$$X_1 + X_2 =Id, \tag{5.10}$$

$$[H,A] =G^*X_1G + \textit{lower order terms}, \tag{5.11}$$

where "lower order terms" refers to the sum of operators R for which either

1. R is a bounded operator and for all $u \in \mathbb{S}$ which solve the wave equation,

$$\int \langle u, Ru\rangle dt \leq C, \tag{5.12}$$

2. or R is an operator with domain $D(R) \subset D((G^*G)^{\frac{1}{2}})$, and for all $u \in D((G^*G)^{\frac{1}{2}})$,

$$\langle u, Ru\rangle \leq C\langle u, (G^*G)^{1-\epsilon}u\rangle. \tag{5.13}$$

If A maps $\mathbb{S}$ to $\mathbb{S}$, even if it is not self-adjoint, then, since H maps $\mathbb{S}$ to $\mathbb{S}$, for $v \in \mathbb{S}$, $H(Av) - A(Hv)$ is well-defined on $\mathbb{S}$. Since H is self-adjoint, by a similar calculation,

$$\frac{d}{dt}(\langle u, A\dot{u}\rangle - \langle \dot{u}, Au\rangle) = \langle u, (HA - AH)u\rangle. \tag{5.14}$$

Since products of smooth functions, integer powers of the radial derivative, and integer powers of angular derivatives map $\mathbb{S}$ to $\mathbb{S}$, if A is of this form, we can write

$$\frac{d}{dt}(\langle u, A\dot{u}\rangle - \langle \dot{u}, Au\rangle) = \langle u, [H, A]u\rangle, \tag{5.15}$$

$$[H, A] = HA - AH. \tag{5.16}$$

For anti-self-adjoint operators, since multiplication by i commutes with H, we can define

$$\langle u, [H, A]u\rangle = \langle u, [H, -iA]iu\rangle. \tag{5.17}$$

From this, for an anti-self-adjoint operator, we have the same Heisenberg-type relation.

6 Morawetz Estimates

The goal of this section is to prove bounds on weighted space-time norms of solutions. We begin by introducing a propagation observable, γ, determine which weighted quantities it majorates, and conclude with a Gronwall's type argument to integrate the Heisenberg-type relation.

In $\mathbb{R}^{3+1}$, the radial derivative can be used as a propagation observable to control the time integral of $|u(t, \vec{0})|^2$. This can be thought of as a weighted space- time integral with the δ- function as a weight. Essentially, our propagation observable is a smooth version of the radial derivative, which leads to a smooth weight. This follows [22, 4].

The Morawetz-type operator is defined in terms of the radial co-ordinate ρ_*. The definition involves a weight, g_σ, which is defined for $\sigma > \frac{1}{2}$ so that g_σ remains bounded.

Definition 6.1. *Given $\sigma > \frac{1}{2}$, the Morawetz-type multiplier γ_σ is defined by*

$$g_\sigma(\rho_*) \equiv \int_0^{\frac{\rho_*}{2M}} \frac{1}{(1+\tau^2)^\sigma} d\tau \tag{6.1}$$

$$\gamma_\sigma \equiv \frac{1}{2}(g_\sigma(\rho_*)\frac{\partial}{\partial r_*} + \frac{\partial}{\partial r_*} g_\sigma(\rho_*)) \tag{6.2}$$

$$= g_\sigma(\rho_*) + \frac{1}{2} g'_\sigma(\rho_*) \tag{6.3}$$

In all cases σ will not vary so the notation $\gamma = \gamma_\sigma$ and $g = g_\sigma$ will be used.

6.1 Preliminary bounds

Lemma 6.2. *If $\sigma > \frac{1}{2}$, then there is a constant C_σ such that for all $u \in \mathbb{S}$,*

$$\langle u, \gamma_\sigma u\rangle = 0 \tag{6.4}$$

$$\|\gamma u\|_{L^2} \leq C_\sigma \|u'\|_{L^2} + \frac{1}{4M}\|(1 + (\frac{\rho_*}{2M})^2)^{-\sigma} u\|_{L^2} \tag{6.5}$$

$$\leq C_\sigma \sqrt{E[u]} + \frac{1}{4M}\|(1 + (\frac{\rho_*}{2M})^2)^{-\sigma} u\|_{L^2} \tag{6.6}$$

Proof. Since $u \in \mathbb{S}$, derivatives can be moved about freely.

$$\langle u, \gamma u\rangle = \langle u, \frac{1}{2}(g\frac{\partial}{\partial r_*} + \frac{\partial}{\partial r_*}g)u\rangle \tag{6.7}$$
$$= \frac{1}{2}(\langle u, g\frac{\partial}{\partial r_*}u\rangle - \langle g\frac{\partial}{\partial r_*}u, u\rangle) \tag{6.8}$$
$$= 0 \tag{6.9}$$

The last equality holds since the L^2 inner product is symmetric on real valued functions.

Since $g = \int_0^{\frac{\rho_*}{2M}}(1+\tau^2)^{-\sigma d\tau}$, g' is bounded by $\frac{1}{2M}(1+(\frac{\rho_*}{2M}))^{-\sigma}$. Since the integrand is continuous and decays as $\tau^{-2\sigma}$ for $\sigma > \frac{1}{2}$, g is bounded by some constant C. Using this and the fact that $u \in \mathbb{S}$, the following holds.

$$\|\gamma u\| = \|gu' + \frac{1}{2}g'u\| \tag{6.10}$$
$$\leq \|gu'\| + \|\frac{1}{2}g'u\| \tag{6.11}$$
$$\leq C_\sigma\|u'\| + \frac{1}{4M}\|(1+(\frac{\rho_*}{2M})^2)^{-\sigma}u\| \tag{6.12}$$

□

6.2 Computation of Morawetz commutators

We show that the propagation observable, γ, majorates $\left(1+\left(\frac{\rho_*}{2M}\right)^2\right)^{-\sigma-1}$ by computing the commutator with H. Some additional terms involving the derivatives are also majorated. Since the commutator is linear in each argument, we compute the commutator of γ with each component of H, H_i. We begin with H_3, because the computation is simple. The commutators involving H_1 and H_2 are more complicated to and must be combined to show they dominate positive operators.

Lemma 6.3. *If $Q \leq M$, and $\sigma > \frac{1}{2}$, then for all $u \in \mathbb{S}$,*

$$\langle u, [H_3, \gamma]u\rangle \geq \langle u, 2Fr^{-5}(r-M)(r-\alpha)g(-\Delta_{S^2})u\rangle \geq 0 \tag{6.13}$$

Proof. Since $H_3 = V_l(-\Delta_S^2)$, and $-\Delta_{S^2}$ commutes with radial functions and derivatives, it is sufficient to consider $[V_l, \gamma]$.

$$[V_l, \gamma] = -g\frac{\partial}{\partial r_*}(Fr^{-2}) = -gF\frac{\partial}{\partial r}(r^{-2}F) = 2gFr^{-5}(r^2 - 3Mr + 2Q^2) \tag{6.14}$$

The zeroes of $(r^2 - 3Mr + 2Q^2)$ are

$$2\alpha_\pm = 3M \pm \sqrt{9M^2 - 8Q^2} \tag{6.15}$$

The lower value, α_-, is an increasing function of Q and hence has its maximum value at $Q = M$, where $\alpha_- = M$. The outer even horizon, $r_+ = M + \sqrt{M^2 - Q^2}$ is a decreasing function of Q and has minimum value at M. Hence $r_+ \geq \alpha_-$, so that $\alpha = \alpha_+$ is the unique zero of $[V_l, \gamma]$ in (r_+, ∞). The product $2Fr^{-5}(r-\alpha_-)$ is positive on (r_+, ∞). The function g and $(r - \alpha_+)$ each go from negative to positive at $r = \alpha$. Therefore, the product is non negative, ie

$$[V_l, \gamma] = 2Fr^{-5}(r-\alpha_-)(r-\alpha_+)g \geq 0 \tag{6.16}$$

Since $2Fr^{-5}(r-\alpha_+)g$ is a positive function on $r \in (r_+, \infty)$ and $r - \alpha_-$ is decreasing in Q with minimum value $r - M$, it follows that

$$[V_l, \gamma] \geq 2Fr^{-5}(r-M)(r-\alpha)g \geq 0 \tag{6.17}$$

Multiplying through by $-\Delta_{S^2}$ and taking the expectation value gives the desired result. □

We now show that the commutator with H_1 dominates the weight $\left(1+(\frac{\rho_*}{2M})^2\right)^{-\sigma-1}$.

Lemma 6.4. *If $\sigma > \frac{1}{2}$, then for all $u \in \mathbb{S}$,*

$$\langle u, [H_1,\gamma]u\rangle \geq \langle u, \frac{\sigma}{(1+(\frac{\rho_*}{2M})^2)^{\sigma+2}}\frac{1}{(2M)^3}[5+(3-2\sigma)(\frac{\rho_*}{2M})^2]u\rangle \tag{6.18}$$

Proof. Since $u \in \mathbb{S}$ and $H_1 = -\frac{\partial^2}{\partial r_*^2}$ is a linear operator, the commutator can be computed formally, and then the expectation value can be taken.

With out using the choice of $g(\rho_*)$,

$$[-\frac{\partial^2}{\partial r_*^2},\gamma] = \frac{-1}{2}(\frac{\partial^2}{\partial r_*^2}g\frac{\partial}{\partial r_*}+\frac{\partial^3}{\partial\rho_*^3}g - g\frac{\partial^3}{\partial\rho_*^3}-\frac{\partial}{\partial r_*}g\frac{\partial^2}{\partial r_*^2}) = -2\frac{\partial}{\partial r_*}g'\frac{\partial}{\partial r_*}+\frac{1}{2}g''' \tag{6.19}$$

To transform this expression to the sum of a positive operator and a local decay term, we consider the action of this operator on a function, u, and to transform this to the action on the function $\rho_*^{-1}u$.

$$\begin{aligned}[-\frac{\partial^2}{\partial r_*^2},\gamma]u =&(-2\frac{\partial}{\partial r_*}g'\frac{\partial}{\partial r_*}-\frac{1}{2}g''')\rho_*(\rho_*^{-1}u) &(6.20)\\ =&\rho_*(-2\frac{\partial}{\partial r_*}g'\frac{\partial}{\partial r_*}-4\rho_*^{-1}g'\frac{\partial}{\partial r_*}-2\rho_*^{-1}g''-\frac{1}{2}g''')(\rho_*^{-1}u) &(6.21)\end{aligned}$$

The $L^2(\mathfrak{M})$ inner product of this with u is now taken to compute the expectation value. This is rewritten as an expectation value with respect to $\rho_*^{-1}u$ and the measure $\rho_*^2 d\rho_* d^2\mu_{S^2}$.

$$\begin{aligned}\langle u,[-\frac{\partial^2}{\partial r_*^2},\gamma]u\rangle =& \int_{\mathfrak{M}}(\rho_*^{-1}u)(-2\frac{\partial}{\partial r_*}g'\frac{\partial}{\partial r_*}-4\rho_*^{-1}g'\frac{\partial}{\partial r_*}-2\rho^{-1}g''-\frac{1}{2}g''')(\rho_*^{-1}u)\rho_*^2 d\rho_* d^2\mu_{S^2} &(6.22)\\ =&\int_{S^2}\int_{(0,\infty)}(\rho_*^{-1}u)(-2\frac{\partial}{\partial r_*}g'\frac{\partial}{\partial r_*}-4\rho_*^{-1}g'\frac{\partial}{\partial r_*})(\rho_*^{-1}u)\rho_*^2 d\rho_* d^2\mu_{S^2} &(6.23)\\ &+\int_{S^2}\int_{(-\infty,0)}(\rho^{-1}u)(-2\frac{\partial}{\partial r_*}g'\frac{\partial}{\partial r_*}-4\rho_*^{-1}g'\frac{\partial}{\partial r_*})(\rho_*^{-1}u)\rho_*^2 d\rho_* d^2\mu_{S^2} &(6.24)\\ &+\int_{\mathfrak{M}}u(-2\rho_*^{-1}g''-\frac{1}{2}g''')ud^3\mu_{S^2} &(6.25)\end{aligned}$$

Each of these integrals is dealt with separately; although, the first and second are very similar. The first is dealt with by noting $\rho_*^2 d\rho_* d^2\mu_{S^2}$ is the volume form on $(0,\infty)\times S^2$ corresponding to spherical co-ordinates in $\mathbb{R}^3$ with ρ_* as the radial parameter. Since the adjoint in $\mathbb{R}^3$ of $\frac{\partial}{\partial r_*}$ is $-\frac{\partial}{\partial r_*}-2\rho_*^{-1}$, the first integral can be written in terms of $v(\rho_*,\omega)=\rho_*^{-1}u(\rho_*,\omega)$ and the inner product in $L^2(\mathbb{R}^3)$, $\langle\bullet,\bullet\rangle_{L^2(\mathbb{R}^3)}$.

$$\int_{S^2}\int_{(0,\infty)}(\rho_*^{-1}u)(-2\frac{\partial}{\partial r_*}g'\frac{\partial}{\partial r_*}-4\rho_*^{-1}g'\frac{\partial}{\partial r_*})(\rho_*^{-1}u)\rho_*^2 d\rho_* d\mu_{S^2} = 2\langle\frac{\partial}{\partial\rho_*}v, g'\frac{\partial}{\partial\rho_*}v\rangle_{L^2(\mathbb{R}^3)} \geq 0 \tag{6.26}$$

A similar argument holds for the second integral, taking $-\rho_*$ as the radial co-ordinate in $\mathbb{R}^3$.

Turning to the third integral in equation 6.22, direct computation shows

$$-2\rho_*^{-1}g''-\frac{1}{2}g''' = \frac{\sigma}{(1+(\frac{\rho_*}{2M})^2)^{\sigma+2}}\frac{1}{(2M)^3}[5+(3-2\sigma)(\frac{\rho_*}{2M})^2] \tag{6.27}$$

This calculation is shown in more detail in the proof of lemma 7.5. □

The commutator with $H_2 = V$ still must be dealt with. Unfortunately, this is not positive. While it can be bounded by the contribution from $H_1 = -\frac{\partial^2}{\partial r_*^2}$ so that $[-\frac{\partial^2}{\partial r_*^2}+V,\gamma]$ is positive, some tedious calculations are necessary to show this. The work is to show that the polynomial, $P_Q(r)$, defined below, is positive in two regions and bounded in absolute value in the third. Nothing more than Calculus is needed to show this. The three regions are naturally divided by α and the root of $P_Q(r)$. Since the root is hard to locate, we replace it with an approximate root. The problematic region lies between α and the approximate root.

Lemma 6.5. *If $Q \in [0, M]$, and $\sigma \in (\frac{1}{2}, \frac{3}{2})$, then there is a constant C such that for all $u \in \mathbb{S}$,*

$$\langle u, [H_1 + H_2, \gamma]u\rangle \geq C\langle u, (1 + (\frac{\rho_*}{2M})^2)^{-\sigma-1}u\rangle \tag{6.28}$$

Proof. Having already proven a lower bound for $[H_1, \gamma]$ of the form $(1 + (\frac{\rho_*}{2M})^2)^{-\sigma-1}$, this result would immediately follow if the potential satisfied the repulsivity condition

$$[H_2, \gamma] = [V, \gamma] = -g\frac{\partial}{\partial r_*}V \geq 0. \tag{6.29}$$

However, there is an attractive region where $[H_2, \gamma] < 0$. In this region, the contributions from H_1 and H_2 must be combined.

The commutator involving H_2 is given by

$$[H_2, \gamma] = [V, \gamma] = -g\frac{\partial}{\partial r_*}V = -gF\frac{\partial}{\partial r}V = 2gFr^{-7}P_Q(r) \tag{6.30}$$

where

$$P_Q(r) = 3Mr^3 - 4(Q^2 + 2M^2)r^2 + 15MQ^2r - 6Q^4 \tag{6.31}$$

Since $2Fr^{-7}$ is clearly positive, the repulsivity condition is equivalent to $gP_q \geq 0$. From its definition, g has a single root at $r = \alpha$. In the case $Q = 0$, $P_Q = 3Mr^3 - 8M^2r^2$ has a single root at $r = \frac{8}{3}M < 3M = \alpha$, so gP_Q is negative only in the interval $\frac{8}{3}M < r < 2M$. Although a similar result is true for all $Q \in [0, M]$, because the root of P_Q is hard to work with, it is easier to introduce an approximate root of P_Q given by

$$\beta_{\text{approx}} = \frac{8}{3}M - \frac{2}{3}Q. \tag{6.32}$$

This is equal to the root at the end points $Q = 0$ and $Q = M$ and should be thought of as a lower bound for the root. Note that $\beta_{\text{approx}} \leq \alpha$.

With this background, we proceed to prove the result by working in the three regions.

Step 1- $gP_q(r)$ is positive above $r = \alpha$

For $r \geq \alpha$, from its definition, $g \geq 0$. At $r = \alpha$, we have $r^2 = 3Mr - 2Q^2$, and

$$\begin{aligned} P_Q(r) =&(3M)^2r^2 - 6MQ^2r - 12(Q^2 + 2M^2)Mr + 8(Q^2 + 2M^2)Q^2 + 15MQ^2r - 6Q^4 &(6.33)\\ =&27M^3r - 18M^2Q^2 - 12MQ^2r - 24M^3r + 2Q^4 + 16M^2Q^2 + 9MQ^2r &(6.34)\\ =&(3M^3 - 3MQ^2)r + (-2M^2Q^2 + 2Q^4) &(6.35)\\ =&(3Mr - 2Q^2)(M^2 - Q^2), &(6.36)\end{aligned}$$

which is nonnegative at $r = \alpha$, since $\alpha \geq 2M$. Since the r derivative of P_Q is

$$\partial_r P_Q(r) =9Mr^2 - 8(2M^2 + Q^2)r + 15MQ^2 \geq (9r - 16M)(M^2 - Q^2) \geq 0, \tag{6.37}$$

which is positive for $r \geq \alpha \geq 2M$, P_Q is nonnegative for $r \geq \alpha$, and the condition $gP_Q \geq 0$ holds.

Step 2- gP_Q is positive beneath β_{approx}, the approximate root of P_Q

The region under consideration is $r_+ < r \leq \beta_{\text{approx}}$. Since $\beta_{\text{approx}} \leq \alpha$, g is non positive and it is sufficient to show that P_Q is non positive. This will be done showing there are no critical points in this region and that P_Q is non positive on the boundary.

Let Q_Q and Q_r be the values of Q, as a function of r, which satisfy $\frac{\partial}{\partial Q}P_Q = 0$ and $\frac{\partial}{\partial r}P_Q = 0$ respectively.

$$\begin{aligned} \frac{\partial}{\partial Q}P_Q =& - 8Qr^2 + 30MQr - 24Q^3 = -2Q_Q(4r^2 - 15Mr + 12Q_Q^2) = 0 &(6.38)\\ Q_Q^2 \in& \{0, \frac{-4r + 15M}{12}r\} &(6.39)\\ \frac{\partial}{\partial r}P_Q =&9Mr^2 - 8(Q^2 + 2M^2)r + 15MQ^2 = 0 &(6.40)\\ Q_r^2 =&\frac{9r - 16M}{8r - 15M}Mr. &(6.41)\end{aligned}$$

The condition for a critical point to exist is $Q_Q = Q_r$.

Since $r \geq r_+ \geq M \geq Q$ equation 6.41 can only hold if $r = M = Q_r$, but this contradicts equation 6.39, so there can be no critical points in the strip $Q \in [0, M]$.

The values of P_Q are now computed on the boundary.

$$P_0(r) = 3Mr^2(r - \frac{8}{3}M) \leq 0 \qquad \text{for } r \in [2M, \frac{8}{3}M] \tag{6.42}$$
$$P_M(r) = 3M(r-M)^2(r-2M) \leq 0 \qquad \text{for } r \in [M, 2M] \tag{6.43}$$
$$P_Q(\frac{8}{3}M - \frac{2}{3}Q) = \frac{2}{9}Q(M-Q)(-64M^2 + 20MQ + 35Q^2) \leq 0 \qquad \text{for } Q \in [0,1] \tag{6.44}$$
$$P_Q(M + \sqrt{M^2 - Q^2}) = -2(2M^2 - Q^2 + 2M\sqrt{M^2-Q^2})(M^2 - Q^2) \leq 0 \qquad \text{for } Q \in [0,1] \tag{6.45}$$

Step 3- Estimate $[H_2, \gamma]$ from below between α and $\beta_{\mathbf{approx}}$

Since g is non positive in this region, it is sufficient to get an upper bound on P_Q. Using the decomposition $2Fr^{-7}P_Qg = 2Fr^{-5}(r^{-2}P_Q)g$, it is sufficient to get upper bounds on the factors $2F$, r^{-5}, $r^{-2}P_Q$ and $-g$.

Since F is increasing in r and Q, and r^{-5} is decreasing in r,

$$2F \leq 2(1 - \frac{2M}{3M} + \frac{M^2}{9M^2}) \tag{6.46}$$
$$\leq 2(1 - \frac{2}{3} + \frac{1}{9}) \tag{6.47}$$
$$\leq \frac{8}{9} \tag{6.48}$$
$$r^{-5} \leq (2M)^{-5} \tag{6.49}$$

Since, in the previous step, it was shown that P_Q is non positive at β_{approx}, $r^{-2}P_Q$ can be bounded by $(r - \beta_{\text{approx}})\sup(\frac{\partial}{\partial r}r^{-2}P_Q)$.

$$r^{-2}P_Q = 3Mr - 4(Q^2 + 2M^2) + 15MQ^2r^{-1} - 6Q^4 \tag{6.50}$$
$$\frac{\partial}{\partial r}(r^{-2}P_Q) = 3M - 15MQ^2r^{-2} + 12Q^4r^{-3} \tag{6.51}$$
$$\leq 3M \tag{6.52}$$
$$r^{-2}P_Q \leq 3M(r - \beta_{\text{approx}}) \tag{6.53}$$

The term $-g$ can be estimated in terms of ρ_*, since the integrand in the definition of g is less than 1,

$$-g \leq \frac{|\rho_*|}{2M}. \tag{6.54}$$

An estimate for ρ_* can be found from estimating F^{-1}, which is maximised where F is minimised. In the region under consideration and for fixed Q, F is minimised along the lower edge of the region, at $r = \beta_{\text{approx}}$. Along this edge of the region,

$$F^{-1}(\frac{8}{3}M - \frac{2}{3}Q) = \frac{(\frac{8}{3}M - \frac{2}{3}Q)^2}{(\frac{8}{3}M - \frac{2}{3}Q)^2 - 2M(\frac{8}{3}M - \frac{2}{3}Q) + Q^2} \tag{6.55}$$
$$= \frac{\frac{4}{9}(16M^2 - 8MQ + Q^2)}{\frac{4}{9}(4M^2 - 5MQ + \frac{13}{4}Q^2)} \tag{6.56}$$
$$\frac{\partial}{\partial Q}(F^{-1}(\frac{8}{3}M - \frac{2}{3}Q)) = \frac{3M(16M^2 - 32MQ + 7Q^2)}{(4M^2 - 5MQ + \frac{13}{4}Q^2)^2} \tag{6.57}$$

The derivative has one zero in $[0, M]$ at $Q = \frac{4}{7}M$. At $Q = 0$ or $Q = M$, $F^{-1} = 4$, so by evaluating at the critical point, it follows that in the region $0 \leq Q \leq M$ and $\beta_{\text{approx}} \leq r \leq \alpha$,

$$\max F^{-1} = \frac{16}{3} \tag{6.58}$$

This bound on F gives a bound on ρ_* and thus g,

$$-\rho_* \leq \sup(F^{-1})|r-\alpha| \tag{6.59}$$
$$\leq \frac{16}{3}|r-\alpha| \tag{6.60}$$
$$-g \leq \frac{1}{2M}\frac{16}{3}|r-\alpha| \tag{6.61}$$

Combining all these results,

$$(2F)(r^{-5})(r^{-2}P_Q)(-g) \leq \frac{8}{9}\frac{1}{(2M)^5}3M(r-\beta_Q)\frac{1}{2M}\frac{16}{3}(\alpha-r) \tag{6.62}$$
$$\leq \frac{1}{(2M)^3}\frac{8}{9}\frac{1}{4M^2}8(r-\beta_Q)(-(r-\alpha)) \tag{6.63}$$

The maximum value of the quadratic $-(x-\beta_{\text{approx}})(x-\alpha)$ occurs at $\frac{\alpha+\beta_{\text{approx}}}{2}$ and is $\frac{1}{4}(\alpha-\beta_{\text{approx}})^2$. The distance $\alpha-\beta_{\text{approx}}$ is now estimated to find an upper bound.

$$\alpha-\beta_{\text{approx}} = \frac{3M}{2}+\frac{\sqrt{9M^2-8Q^2}}{2}-\frac{8M}{3}+\frac{2Q}{3} \tag{6.64}$$
$$= \frac{-7M}{6}+\frac{\sqrt{9M^2-8Q^2}}{2}+\frac{2Q}{3} \tag{6.65}$$
$$\frac{\partial}{\partial Q}|\alpha-\beta_{\text{approx}}| = \frac{-4Q}{\sqrt{9M^2-8Q^2}}+\frac{2}{3} \tag{6.66}$$

The critical point is at $Q=\frac{3}{2}\frac{1}{\sqrt{11}}M$. At the endpoints, $Q=0$ and $Q=M$, the distance $|\alpha-\beta_{\text{approx}}|$ is $\frac{1}{3}M$ and 0 respectively. At the critical point, the distance is $(\frac{-7}{6}+\frac{\sqrt{11}}{2})M$. The estimate on $|\alpha-\beta_{\text{approx}}|$ is therefore

$$\alpha-\beta_{\text{approx}} \leq \frac{-7}{6}M+\frac{\sqrt{11}}{2}M \tag{6.67}$$
$$< .5M \tag{6.68}$$

Using this information on the quadratic and $\alpha-\beta_{\text{approx}}$,

$$(2F)(r^{-5})(r^{-2}P_Q)(-g) \leq \frac{1}{(2M)^3}\frac{8}{9}\frac{1}{4M^2}8\frac{1}{4}(\frac{M}{2})^2 \tag{6.69}$$
$$\leq \frac{1}{(2M)^3}\frac{1}{9} \tag{6.70}$$

Step 4- Estimate the contribution of $[H_1,\gamma]$ between α and β_{approx}

Using the lower bound for $[H_1,\gamma]$ from lemma 6.4 and the bounds on ρ_* and r in equation (6.60) and (6.68),

$$\langle u,[H_1,\gamma]u\rangle \geq \langle u,\frac{1}{(2M)^3}\frac{\sigma}{(1+(\frac{\rho_*}{2M})^2))^{\sigma+2}}(5+(3-2\sigma)(\frac{\rho_*}{2M})^2)u\rangle \tag{6.71}$$
$$\geq \langle u,\frac{1}{(2M)^3}(\frac{5}{2}(1+(\frac{8}{3}(\frac{\alpha-\beta_{\text{approx}}}{M})^2))^{-3})u\rangle \tag{6.72}$$
$$\geq \langle u,\frac{1}{(2M)^3}(\frac{5}{2}(1+(\frac{4}{3})^2)^{-3}))u\rangle \tag{6.73}$$
$$\geq \langle u,\frac{1}{(2M)^3}(\frac{3^6}{2(5^5)})u\rangle. \tag{6.74}$$

Step 5- Sum the contributions

For $r \geq \alpha$ or $r \leq \beta_{\text{approx}}$, since the contribution from $[H_2, \gamma]$ is nonnegative, the lower bound on $[H_1, \gamma]$ in lemma 6.4 is sufficient.

In the region $\beta_{\text{approx}} < r < \alpha$, combining the results from steps 4 and 5,

$$\langle u, [H_1 + H_2, \gamma]u\rangle = \langle u, [H_1, \gamma]u\rangle + \langle u, [H_2, \gamma]u\rangle \tag{6.75}$$

$$\geq \langle u, \frac{1}{(2M)^3}(\frac{3^6}{2(5^5)} - \frac{1}{9})u\rangle \tag{6.76}$$

$$\geq \langle u, \frac{1}{(2M)^3}(5.5 \times 10^{-3})u\rangle. \tag{6.77}$$

The quantity in the expectation value is strictly positive so it dominates some multiple of any bounded function, in particular $(1 + \frac{\rho_*}{2M})^{-\sigma-1}$. The constant in equation 6.77 is very small because crude estimates involving σ were made. In fact, estimates made with the help of computer algebra packages show that $[H_1, \gamma]$ quite easily dominates $[H_2, \gamma]$.

Combining the results in all the regions considered gives the existence of a constant C so that

$$\langle u, [H_1 + H_2, \gamma]u\rangle \geq C\langle u, (1 + (\frac{\rho_*}{2M})^2)^{-\sigma-1}u\rangle. \tag{6.78}$$

□

6.3 L^2 local decay estimate

We begin with a type of Gronwall's estimate.

Lemma 6.6. *If $\theta : [0, \infty) \to [0, \infty)$ with uniformly bounded derivative, $\epsilon \in (0, \frac{1}{3})$, and there are constants C_1, C_2, and T, such that for $t > T$*

$$\int_0^t \theta(\tau)^2 d\tau \leq C_1 + C_2 t^\epsilon \theta(t)^{1-\epsilon} \tag{6.79}$$

then there is a sequence $\{t_i\}$ such that $\lim_{i\to\infty} t_i = \infty$ and

$$\lim_{i\to\infty} t_i^\epsilon \theta(t)^{1-\epsilon} = 0 \tag{6.80}$$

Proof. Successively stronger bounds on $\theta(t)$ will be proven.

The bound on the derivative implies $\theta(t)$ and $t^\epsilon\theta(t)^{1-\epsilon}$ are linearly bounded above.

Suppose there is no sequence $\{t_i\}$ on which $\theta(t_i) \to 0$, then $\theta(t)$ is bounded below by a constant. By the integral condition $t^\epsilon\theta(t)^{1-\epsilon}$ is bounded below by a linear function. From this $\theta(t)^2$ is bounded below by a quadratic, and its integral is bounded below by a cubic. This contradicts the linear upper bound of the integral. Therefore, there is a sequence $\{t_i\}$ such that $t_i \to \infty$ and $\theta(t_i) \to 0$.

Since $\epsilon < \frac{1}{3}$, there exists $r < 1$ such that

$$-\frac{1-\epsilon}{2 - r + r\epsilon} < -\frac{\epsilon}{1-\epsilon}$$

Now choose δ negative such that $\delta \geq \frac{-\epsilon}{1-\epsilon}$, from which it follows that

$$r\delta \quad < \quad 1 + \frac{2\delta}{1-\epsilon} \tag{6.81}$$

Suppose $\exists K > 0, S > 0 : \forall t > S : \theta(t) > Kt^\delta$, then by the integral condition (6.79) there is a positive C_3 such that

$$C_3 t^{1+2\delta} \leq C_1 + C_2 t^\epsilon \theta(t)^{1-\epsilon} \tag{6.82}$$

Since $\delta \geq \frac{-\epsilon}{1-\epsilon} > \frac{-1}{2}$, $1 + 2\delta$ is strictly positive. Thus for sufficiently large t, there are constants C_4, C_5 such that

$$\begin{aligned} C_4 t^{1+2\delta} &\leq t^\epsilon \theta(t)^{1-\epsilon} \\ C_5 t^{1+\frac{2\delta}{1-\epsilon}} &\leq \theta(t) \end{aligned}$$

If δ is sufficiently close to zero, then this contradicts the previous result that $\theta(t)$ goes to zero on a subsequence. If δ is larger, then we use the fact that by equation (6.81), $\theta(t) \geq C_5 t^{r\delta}$. This implies the original assumed lower bound is replaced by the larger lower bound $t^{r\delta}$. Repeated iterations of this process shows that $\theta(t)$ is bounded below by $t^{r^n\delta}$ for any n. Since $r < 1$, this reduces the situation to the δ close to zero case, which led to a contradiction. Thus $\theta(t)$ can not be bounded below by a function of the form Kt^δ for $\delta \geq \frac{-\epsilon}{1-\epsilon}$. In particular, $\theta(t)$ can not be bounded below by a function of the form $Kt^{\frac{-\epsilon}{1-\epsilon}}$ and so the desired subsequence must exist. □

The propagation observable γ majorates the weight $\left(1+\left(\frac{\rho_*}{2M}\right)^2\right)^{\frac{-\beta}{2}}$ for $\beta > \frac{3}{2}$. This is a regularised version of the Morawetz estimate. As a consequence, the solution is space-time integrable in any bounded region, and must decay in that region.

Theorem 6.7 (Local Decay Estimate). *If $\beta > \frac{3}{2}$, then there are positive constants K_{LD2} and K_{LD3} such that for $u \in \mathbb{S}$ a solution to the wave equation, $\ddot{u} + Hu = 0$,*

$$\int_1^\infty K_{LD2}\|(1+(\frac{\rho_*}{2M})^2)^{\frac{-\beta}{2}}u\|^2 \tag{6.83}$$

$$+ K_{LD3}\langle u, 2Fr^{-5}(r-M)(r-\alpha)g(-\Delta_{S^2})u\rangle \leq \sqrt{E}(\sqrt{E}+\|u(1)\|) \tag{6.84}$$

where $E = E[u_0, u_1]$.

Proof. Throughout this proof the letters C_i will refer to constants independent of u. They will be used as constants within a particular equation, but will vary in value from one equation to the next.

First a crude estimate on the norm growth of $\|\left(1+\left(\frac{\rho_*}{2M}\right)^2\right)^{\frac{-\beta}{2}}u\|$ is made. In this paragraph, we use the notation $f \equiv \left(1+\left(\frac{\rho_*}{2M}\right)^2\right)^{\frac{-\sigma-1}{2}} \leq 1$.

$$\frac{\partial}{\partial t}\|fu\|^2 = \frac{\partial}{\partial t}\langle fu, fu\rangle \tag{6.85}$$

$$2\|fu\|\frac{\partial}{\partial t}\|fu\| = 2\langle fu, fu\rangle \tag{6.86}$$

$$\frac{\partial}{\partial t}\|fu\| \leq \|f\dot{u}\| \leq \|\dot{u}\| \leq \sqrt{E[u(1)]} \tag{6.87}$$

Initially β is restricted to $(\frac{3}{2}, 2]$, and the variable $\sigma = \beta - 1 \in (\frac{1}{2}, 1]$ is used. From the commutator estimates lemmas 6.3 and 6.5, there are constants C_2 and C_3 such that

$$\int_1^T C_2\|\left(1+\left(\frac{\rho_*}{2M}\right)^2\right)^{\frac{-\beta}{2}}u\|^2 dt \leq \int_1^T C_2\|\left(1+\left(\frac{\rho_*}{2M}\right)^2\right)^{\frac{-\beta}{2}}u\|^2 \tag{6.88}$$

$$+ C_3\langle u, 2Fr^{-5}(r-M)(r-\alpha)g(-\Delta_{S^2})u\rangle dt \tag{6.89}$$

$$\leq -\int_1^T \langle u, [\sum_{i=1}^3 H_i, \gamma]u\rangle dt \tag{6.90}$$

From integrating the Heisenberg relation, the final integral on the right is bounded by

$$\int_1^T \frac{\partial}{\partial t}(\langle u, \gamma\dot{u}\rangle - \langle\dot{u}, \gamma u\rangle)dt \leq [\langle u, \gamma\dot{u}\rangle - \langle\dot{u}, \gamma u\rangle]_{t=1}^T \tag{6.91}$$

$$\leq [\frac{\partial}{\partial t}\langle u, \gamma u\rangle - 2\langle\dot{u}, \gamma u\rangle]_{t=1}^T \tag{6.92}$$

$$\leq [-2\langle\dot{u}, \gamma u\rangle]_{t=1}^T \tag{6.93}$$

$$\leq [\|\dot{u}\|\|\gamma u\|]_{t=1} + [\|\dot{u}\|\|\gamma u\|]_{t=T} \tag{6.94}$$

From energy conservation and the bound on $\|\gamma u\|$ in lemma 6.2, these norms are bounded by

$$[\|\dot{u}\|\|\gamma u\|]_{t=1} + [\|\dot{u}\|\|\gamma u\|]_{t=T} \leq C_1\sqrt{E}(\sqrt{E} + \| \left(1 + \left(\frac{\rho_*}{2M}\right)^2\right)^{\frac{-\sigma}{2}} u(1)\|) \tag{6.95}$$

$$+ C_2\sqrt{E}(\sqrt{E} + \| \left(1 + \left(\frac{\rho_*}{2M}\right)^2\right)^{\frac{-\sigma}{2}} u(T)\|) \tag{6.96}$$

Since $\sigma \in (\frac{1}{2}, 1]$, q can be chosen in $(\frac{1}{2\sigma} + \frac{1}{2}, \frac{3}{2}) \subset (1, \frac{3}{2})$. If p is the conjugate exponent to q and $\kappa \equiv \frac{2}{p}$, then

$$\frac{1}{p} > 1 - \frac{2}{3} = \frac{1}{3} \tag{6.97}$$

$$(\frac{2-\kappa}{2})q = 1 \tag{6.98}$$

$$\sigma q > \frac{\sigma + 1}{2} \tag{6.99}$$

Hölder's inequality can be applied to the last term in equation 6.96 using conjugate exponent p and q. Continuing to use $\| \bullet \|$ as the L^2 norm gives,

$$\| \left(1 + \left(\frac{\rho_*}{2M}\right)^2\right)^{-\sigma} u\|^2 = \int_{\mathfrak{M}} \frac{|u|^{\kappa}|u|^{2-\kappa}}{\left(1 + \left(\frac{\rho_*}{2M}\right)^2\right)^{2\sigma}} d^3\mu \tag{6.100}$$

$$\leq (\int_{\mathfrak{M}} |u|^{p\kappa} d^3\mu)^{\frac{1}{p}} (\int_{\mathfrak{M}} \frac{|u|^{(2-\kappa)q}}{\left(1 + \left(\frac{\rho_*}{2M}\right)^2\right)^{2\sigma q}} d^3\mu)^{\frac{1}{q}} \tag{6.101}$$

$$\leq \||u|^{\frac{p\kappa}{2}}\|^{\frac{2}{p}} \| \frac{|u|^{\frac{2-\kappa}{2}q}}{\left(1 + \left(\frac{\rho_*}{2M}\right)^2\right)^{\sigma q}} \|^{\frac{2}{q}} \tag{6.102}$$

The fact that $\sigma q > \frac{\sigma+1}{2}$ can now be used.

$$\| \left(1 + \left(\frac{\rho_*}{2M}\right)^2\right)^{-\sigma} u\| \leq \|u\|^{\frac{1}{p}} \| \left(1 + \left(\frac{\rho_*}{2M}\right)^2\right)^{\frac{-\sigma-1}{2}} u\|^{1-\frac{1}{p}} \tag{6.103}$$

Since $\|\dot{u}\|^2 \leq E$, the norm of u is controlled, $\|u(t)\| \leq \|u(1)\| + tE^{\frac{1}{2}}$. The computations from the beginning of 6.88 to 6.103 give an integral inequality for the weighted norm of u.

$$\int_1^T \| \left(1 + \left(\frac{\rho_*}{2M}\right)^2\right)^{\frac{-\sigma-1}{2}} u\|^2 dt \leq C_1(E[u] + \|u\|^2) + C_2 E[u]^{\frac{1}{2p}} t^{\frac{1}{p}} \| \left(1 + \left(\frac{\rho_*}{2M}\right)^2\right)^{\frac{-\sigma-1}{2}} u\|^{1-\frac{1}{p}} \tag{6.104}$$

Since $\frac{\partial}{\partial t}\| \left(1 + \left(\frac{\rho_*}{2M}\right)^2\right)^{\frac{-\sigma-1}{2}} u\|$ is uniformly bounded by equation (6.87), and since $\frac{1}{p} \in (0, \frac{1}{3})$, the Gronwall's estimate lemma 6.6 applies, and there is a subsequence on which

$$t^{\frac{1}{p}} \| \left(1 + \left(\frac{\rho_*}{2M}\right)^2\right)^{\frac{-\sigma-1}{2}} u\|^{1-\frac{1}{p}} \to 0 \tag{6.105}$$

Using the calculations starting from the left hand side of equation 6.88, which is monotonically increasing in time, to equation 6.103, which is sequentially decreasing, the desired result holds for $\beta \in (\frac{3}{2}, 2)$. Since $\left(1 + \left(\frac{\rho_*}{2M}\right)^2\right)^{\frac{-\beta}{2}} |u|$ is monotonically decreasing in β, the Lebesgue dominated convergence theorem extends the result to all $\beta > \frac{3}{2}$. □

Since the weights in the local decay theorem will dominate any compactly supported function, if u is a radial function, the trapping terms can be integrated in time to control the growth of the conformal charge and the weighted L^6 norm.

Corollary 6.8. *In the non super critical cases, $|Q| \leq M$, if $u \in \mathbb{S}$ is a solution to the wave equation, $\ddot{u} + Hu = 0$, and u_0 and u_1 are radial functions, then there is a constant, C, depending only on $\|u_0|^2$, $E[u_0, u_1]$, and $E_{\mathcal{C}}[u_0, u_1]$ such that, then*

$$\left(\int |u|^6(t, \rho_*, \omega)F^3 r^{-4} d^3\mu\right)^{\frac{1}{6}} \leq C t^{\frac{-1}{3}} \tag{6.106}$$

Proof. Since the initial conditions are radial, the solution u will remain radial, and

$$\int W_L |\nabla_{S^2} u|^2 d\mu = 0. \tag{6.107}$$

Since the potential trapping term, W, is compactly supported, it can be dominated by the weights in the local decay result. This provides a bound on the conformal charge.

$$E_{\mathcal{C}}[u(T), \dot{u}(T)] \leq E_{\mathcal{C}}[u_0, u_1] + \int^t \int 2\tau W |u|^2 d^3\mu d\tau \tag{6.108}$$

$$\leq E_{\mathcal{C}}[u_0, u_1] + t \int \int 2W |u|^2 d^3\mu d\tau \tag{6.109}$$

$$\leq E_{\mathcal{C}}[u_0, u_1] + t E^{\frac{1}{2}}(E^{\frac{1}{2}} + \|u_0\|) \tag{6.110}$$

From proposition 3.11,

$$\int |u|^6(t, \rho_*, \omega)F^3 r^{-4} d^3\mu \leq C(E[u_0, u_1] + E_{\mathcal{C}}[u(t), \dot{u}(t)]t^{-2}) E_{\mathcal{C}}[u(t), \dot{u}(t)] t^{-4} \tag{6.111}$$

$$\leq C t^{-2} \tag{6.112}$$

□

For non-radial solutions, the local decay result controls also controls the angular energy away from the photon sphere, $r = 0$.

Corollary 6.9. *In the non super critical case, $|Q| \leq M$, for any positive, smooth, compactly supported function χ_α which doesn't vanish at $r = \alpha$, there is a constant C, such that if $u \in \mathbb{S}$ is a solution to the wave equation, $\ddot{u} + Hu = 0$, then*

$$\int |u|^6(t, \rho_*, \omega)F^3 r^{-4} d^3\mu \leq C(E[u_0, u_1] + E_{\mathcal{C}}[u(t), \dot{u}(t)]t^{-2}) E_{\mathcal{C}}[u(t), \dot{u}(t)]^{\frac{2}{3}} t^{\frac{-4}{3}} \tag{6.113}$$

$$E_{\mathcal{C}}[u(t), \dot{u}(t)] \leq E_{\mathcal{C}}[u_0, u_1] + C E[u-0, u_1]^{\frac{1}{2}}(E[u_0, u_1]^{\frac{1}{2}} + \|u_0\|) \tag{6.114}$$

$$+ Ct \int^t \int \chi_\alpha |\nabla_{S^2} u|^2 d^3\mu d\tau \tag{6.115}$$

Proof. Given χ_α, there are constants so that

$$C_1 2F r^{-5}(r - M)(r - \alpha)g + C_2 \chi_\alpha \geq W_L. \tag{6.116}$$

This controls angular term in proposition 3.11.

$$\int^t \int 2\tau (W|u|^2 + W_L |\nabla_{S^2} u|^2) d^3\mu d\tau \leq \int^t \int C_1 \tau \left(1 + \left(\frac{\rho_*}{2M}\right)^2\right)^{\frac{-\beta}{2}} |u|^2 d^3\mu d\tau \tag{6.117}$$

$$+ \int^t \int C_1 \tau 2F r^{-5}(r - M)(\frac{r - \alpha}{r} g) |\nabla_{S^2} u|^2 d^3\mu d\tau \tag{6.118}$$

$$+ \int^t \int C\tau \chi_\alpha |\nabla_{S^2} u|^2 d^3\mu d\tau \tag{6.119}$$

$$\leq C E[u-0, u_1]^{\frac{1}{2}}(E[u-0, u_1]^{\frac{1}{2}} + \|u_0\|) \tag{6.120}$$

$$+ Ct \int^t \int \chi_\alpha |\nabla_{S^2} u|^2 d^3\mu d\tau. \tag{6.121}$$

□

7 Angular Modulation

To close the conformal estimate it is necessary to estimate one angular derivative in L^2 localised near the photon sphere $r = \alpha$. Although this will not be possible, we will be able to bound fractional powers of the angular derivative. To do this, we introduce a propagation observable, which is an analogue of γ but is rescaled, on each spherical harmonic, by a fractional power of $1 - \Delta_{S^2}$.

Inspired by, but deviating from, the standard notation from Quantum Mechanics, L is notationally used to refer to the operator square root of $1 - \Delta_{S^2}$; however, since an explicit decomposition of the function space into eigenfunctions of $1 - \Delta_{S^2}$ is known, at this point it can still be thought of as a purely notational convenience. A strictly positive version of $-\Delta_{S^2}$ is used so that inverse powers can be taken in section 8.

Definition 7.1. *L is defined to be the operator square root of $1 - \Delta_{S^2}$. On each spherical harmonic this acts as multiplication by $\sqrt{1 + l(l + 1)}$.*

The local decay result, theorem 6.7, provides control on the angular energy away from $\rho_* = 0$, and, by corollary 6.9, the angular component of the energy only needs to be controlled in a compact set around $\rho_* = 0$. This is the region near $r = \alpha$. We introduce the weight χ_α to work in this region; although, the particular choice of χ_α will not matter.

Definition 7.2. *The function χ_α is defined to be a smooth, compactly supported, radial function, such that $\chi_\alpha \le 1$, and $\chi_\alpha = 1$ in a neighbourhood of $\rho_* = 0$.*

7.1 Angular Modulation and Initial Estimates

A new operator γ_{L^m} is introduced to give better estimates near $\rho_* = 0$. Because there is no optimisation over σ, as in the previous local decay estimate, $\sigma = 1$ is used.

Definition 7.3. *The projection operator onto the l^{th} spherical harmonic is denoted $\mathbf{P}_l$. The angularly modulated multiplier γ_{L^m} is defined for $m \in [0, 1]$ by*

$$g_{L^m} \equiv \int_0^{\frac{L^m \rho_*}{2M}} (1 + \tau^2)^{-1} d\tau = \arctan(\frac{L^m \rho_*}{2M}) \tag{7.1}$$

$$= \sum_0^\infty \int_0^{\frac{(1+l(l+1))^{\frac{m}{2}} \rho_*}{2M}} (1 + \tau^2) d\tau \mathbf{P}_l \tag{7.2}$$

$$\gamma_{L^m} \equiv \frac{1}{2}(g_{L^m} \frac{\partial}{\partial r_*} + \frac{\partial}{\partial r_*} g_{L^m}) \tag{7.3}$$

(Note that if v is a radial function, then $g_{L^m} v = gv$ since $L = 1$ in the definition of g_{L^m}.)

Lemma 7.4. *If $m \in [0, 1]$, then for all $u \in \mathbb{S}$, $\|\gamma_{L^m} u\|^2 \le C_m(E[u] + \| \left(1 + (\frac{\rho_*}{2M})^2\right)^{-1} u\|^2)$*

Proof. Since $g_{L^m} = \arctan(L^m \rho_*)$, it is bounded in absolute value by $\pi/2$. The derivative is bounded on each spherical harmonic shell. $g'_{L^m} = L^m(1 + (\frac{L^m \rho_*}{2M})^2)^{-1} \le C(L^m F^{-\frac{1}{2}} r^{-1} + \left(1 + (\frac{\rho_*}{2M})^2\right)^{-1})$ (This constant is independent of the spherical harmonic shell). The norm $\|g'_{L^m} u\|^2$ is bounded by $\|LF^{\frac{1}{2}} r^{-1} u\|^2 + \| \left(1 + (\frac{\rho_*}{2M})^2\right)^{-1} u\|^2$ which are bounded by the energy and local decay norm.

$$\|\gamma_{L^m} u\|^2 \le 2(\|g_{L^m} u'\|^2 + \frac{1}{2}\|g'_{L^m} u\|^2) \tag{7.4}$$

$$\le C_m(E[u] + \| \left(1 + \left(\frac{\rho_*}{2M}\right)^2\right)^{-1} u\|^2). \tag{7.5}$$

□

7.2 Direct Angular Momentum Bounds

The family of multipliers γ_{L^m} is not sufficient to prove a local decay estimate for the angular energy $\langle u, \chi_\alpha L^2 u\rangle$. Instead only an estimate for $\langle u, \chi_\alpha L^{\frac{3}{2}} u\rangle$ can be proven directly without phase space analysis. The same method as in the proof of local decay is used, and the best result is found by optimising over m.

Whenever a commutator involving χ_α, or its derivatives, appears in a calculation, that term can be controlled by the previous local decay result. The commutators are dealt with in index order, not order of simplicity as in section 6.

Lemma 7.5. *If $m \in [0,1]$, then there are constants C_1 and C_2 such that for all $u \in \mathbb{S}$,*

$$\langle u, [H_1, \gamma_{L^m}]u\rangle \geq C_1 \langle u, \frac{L^{3m}}{(2M)^3(1+(\frac{L^m\rho_*}{2M})^2)^2} u\rangle \tag{7.6}$$

$$+ C_2\langle u', \frac{2L^m}{1+(\frac{L^m\rho_*}{2M})^2} u'\rangle \tag{7.7}$$

Proof. We begin by proving the intermediate results:

$$\langle u, [H_1, \gamma_{L^m}]u\rangle = \langle u', \frac{L^m}{1+(\frac{L^m\rho_*}{2M})^2} u'\rangle \tag{7.8}$$

$$+ \langle u, \frac{2L^{3m}}{(2M)^3} \frac{1}{(1+(\frac{L^m\rho_*}{2M})^2)^3}(1-4(\frac{L^m\rho_*}{2M})^2)u\rangle \tag{7.9}$$

$$\langle u, [H_1, \gamma_{L^m}]u\rangle \geq \langle u, \frac{L^{3m}}{(2M)^3(1+(\frac{L^m\rho_*}{2M})^2)^2} u\rangle \tag{7.10}$$

The argument from lemma 6.4 is repeated using the scaling $\rho_* \mapsto L^m\rho_*$ on each spherical harmonic. Substituting g_{L^m} for g and γ_{L^m} for γ in equation (6.19) gives

$$[-\frac{\partial^2}{\partial r_*^2}, \gamma_{L^m}] = -\frac{1}{2}(4\frac{\partial}{\partial r_*} g'_{L^m} \frac{\partial}{\partial r_*} + g'''_{L^m}) \tag{7.11}$$

The same decomposition as in equation (6.25) can be made to write the expectation value of this as a sum of three terms

$$\langle u, [H_1, \gamma_{L^m}]u\rangle = \int_{S^2}\int_{[0,\infty)} (\rho^{-1}u)(-2\frac{\partial}{\partial r_*} g'_{L^m}\frac{\partial}{\partial r_*} - 4\rho^{-1} g_{L^m}\frac{\partial}{\partial r_*})(\rho_* u)\rho_*^2 d\rho_* d\mu_{S^2} \tag{7.12}$$

$$+ \int_{S^2}\int_{(\infty,0]} (\rho^{-1}u)(-2\frac{\partial}{\partial r_*} g'_{L^m}\frac{\partial}{\partial r_*} - 4\rho^{-1} g_{L^m}\frac{\partial}{\partial r_*})(\rho_* u)\rho_*^2 d\rho_* d\mu_{S^2} \tag{7.13}$$

$$+ \langle u, (-2\rho^{-1} g''_{L^m} - \frac{1}{2} g'''_{L^m})u\rangle \tag{7.14}$$

This is the same decomposition as in equation (6.26), and the first two terms can be shown to be positive by the same transformation to an $\mathbb{R}^3$ integral by the same argument as in lemma 6.26.

For the third term, direct computation shows

$$g''_{L^m} = \frac{-2\rho_*}{(2M)^2} \frac{L^{3m}}{(1+(\frac{L^m\rho_*}{2M})^2)^2} \tag{7.15}$$

$$g'''_{L^m} = \frac{-2}{(2M)^3} \frac{L^{3m}}{(1+(\frac{L^m\rho_*}{2M})^2)^3} + \frac{8\rho_*^2}{(2M)^4}\frac{L^{5n}}{(1+(\frac{L^m\rho_*}{2M})^2)^3} \tag{7.16}$$

$$-2\frac{g''_{L^m}}{\rho_*} - \frac{g'''_{L^m}}{2} = \frac{L^{3m}}{(2M)^3}\frac{1}{(1+(\frac{L^m\rho_*}{2M})^2)^2}(4+1-(\frac{L^m\rho_*}{2M})^2 4\frac{1}{(1+(\frac{L^m\rho_*}{2M})^2)}) \tag{7.17}$$

$$= \frac{L^{3m}}{(2M)^3}\frac{1}{(1+(\frac{L^m\rho_*}{2M})^2)^3}(5+(\frac{L^m\rho_*}{2M})^2) \tag{7.18}$$

$$\geq \frac{L^{3m}}{(2M)^3}\frac{1}{(1+(\frac{L^m\rho_*}{2M})^2)^2} \tag{7.19}$$

This establishes equation (7.10).

Equation (7.9) follows from substituting g'_{L^m} and g'''_{L^m} into equation (7.11).

Since $\frac{1-4(\frac{L^m\rho_*}{2M})^2}{1+(\frac{\rho_* L^m}{2M})^2}$ is bounded, taking a weighted sum of equations (7.10) and (7.9) gives the desired result. □

Lemma 7.6. *If $m \in [0,1]$, then there is a positive constant C such that for all $u \in \mathbb{S}$,*

$$|\langle u, [H_2, \gamma_{L^m}]u\rangle| \leq C\langle u, \left(1 + \left(\frac{\rho_*}{2M}\right)^2\right)^{-2} u\rangle \tag{7.20}$$

Proof. From lemma 3.9, the derivative of the potential is $-V' = 2Fr^{-7}P_Q(r)$ where P_Q is a cubic polynomial in r.

Since for $\rho_* \to \infty$, $r \sim \rho_*$, and for $\rho_* \to -\infty$, F decays exponentially, there is a constant C so that $|V'| < C\left(1 + (\frac{\rho_*}{2M})^2\right)^{-2}$. Since $|g_{L^m}|$ is bounded by $\pi/2$ and $[H_2, \gamma_{L^m}] = -g_{L^m}V'$, the result holds. □

The next theorem has a factor of $g_{L^m}\rho_*\chi_\alpha$ in its statement. The additional factor of $g_{L^m}\rho_*$ is not redundant. It describes the rate of decay at $\rho_* = 0$. These factors are not relevant for large values of ρ_* since χ_α has compact support.

Lemma 7.7. *If $m \in [0,1]$, then there is a constant C such that for all $u \in \mathbb{S}$,*

$$\langle u, [H_3, \gamma_{L^m}]u\rangle \geq \langle u, g_{L^m}(L^2-1)2Fr^{-5}(r-M)(r-\alpha)u\rangle \tag{7.21}$$

$$\geq C(\langle u, (L^2-1)(g_{L^m}\rho_*)\chi_\alpha u\rangle) \tag{7.22}$$

Proof. From the proof of lemma 6.3,

$$[V_l, \gamma_{L^m}] \geq 2Fr^{-5}(r-M)(r-\alpha)g_{L^m} \tag{7.23}$$

which is non negative since $r-\alpha$ and g_{L^m} both go from negative to positive at $\rho_* = 0$, and the other terms are positive. Since $H_2 = (L^2-1)V_l$ and L^2 commutes with γ, equation (7.21) holds. Since $2Fr^{-5}(r-M)$ is continuous and doesn't vanish in $\mathfrak{M}$, it is bounded below on supp(χ_α). Furthermore, since $r-\alpha$ vanishes linearly in ρ_*, and since $(r-\alpha)g_{L^m}$ and $\rho_* g_{L^m}$ are both non negative, there is a constant so that $2Fr^{-5}(r-M)(r-\alpha)g_{L^m} \geq C\rho_* g_{L^m}\chi_\alpha$, proving equation (7.22). □

The last sequence of lemmas shows that γ_{L^m} is a propagation observable which majorates powers of L. That is, ignoring terms which are space- time integrable, the commutator is bounded below by the product of powers of L and localisation functions. These localisation functions are functions of $L^m\rho_*$. In the following theorem, the Heisenberg-type relation with γ_{L^m} is integrated in time. The theorem shows that γ_{L^m} is a bounded operator on the energy space.

The localising function χ_α is included to restrict attention to the region ρ_* small. In the theorem, taking the optimal value $m = \frac{1}{2}$ gives an estimate for $\langle u, \chi_\alpha L^{\frac{3}{2}} u\rangle$. The estimate for other values of m will be used later, in the phase space analysis.

Theorem 7.8. *If $m \in [0,1]$, then there is a constant C such that for all $u \in \mathbb{S}$ which satisfy the linear wave equation, $\ddot{u} + Hu = 0$,*

$$\int_1^\infty \langle u, L^2 g_{L^m}\rho_*\chi_\alpha u\rangle d\tau + \int_1^\infty \langle u, \frac{L^{3m}}{(1+(\frac{L^m\rho_*}{2M})^2)^2} u\rangle d\tau \leq C(E[u] + \|u(1)\|^2) \tag{7.24}$$

$$\int_1^\infty \langle u, L^{\frac{3}{2}}\chi_\alpha u\rangle \leq C(E[u] + \|u(1)\|^2) \tag{7.25}$$

Proof. Applying the Heisenberg identity to the operator γ_{L^m} gives

$$\frac{d}{dt}(\langle \dot{u}, \gamma_{L^m} u\rangle - \langle u, \gamma_{L^m}\dot{u}\rangle) = \langle u, \sum_{i=1}^{3} H_i u\rangle \tag{7.26}$$

Using lemma 7.4 and the fact that γ_{L^m} is antisymmetric with respect to the $L^2(\mathfrak{M})$ norm when acting on C^∞ functions, this equation can be integrated to get an estimate on the commutators.

$$\int_1^t \langle u, [\sum_{i=1}^3 H_i, \gamma_{L^m}]u\rangle d\tau \leq 2\|\dot{u}\|\|\gamma_{L^m} u\||_1^t \tag{7.27}$$

$$\leq C(E[u] + \|\left(1+\left(\frac{\rho_*}{2M}\right)^2\right)^{-1} u(1)\|^2 + \|\left(1+\left(\frac{\rho_*}{2M}\right)^2\right)^{-1} u(t)\|^2) \tag{7.28}$$

By the local decay estimate, theorem 6.7, $\|\left(1+(\frac{\rho_*}{2M})^2\right)^{-1} u\| \to 0$ sequentially. On a sequence of times,

$$\int_1^t \langle u, [\sum_{i=1}^3 H_i, \gamma_{L^m}]u\rangle d\tau \leq 2\|\dot{u}\|\|\gamma_{L^m} u\||_1^t \tag{7.29}$$

$$\leq C(E[u] + \|\left(1+\left(\frac{\rho_*}{2M}\right)^2\right)^{-1} u(1)\|^2) \tag{7.30}$$

By the local decay result, theorem 6.7, and lemma 7.6,

$$\int_1^\infty |\langle u, [H_2, \gamma_{L^m}]u\rangle| d\tau \leq \int_1^\infty \|\left(1+\left(\frac{\rho_*}{2M}\right)^2\right)^{-1} u\|^2 d\tau \leq C(E[u] + \|u(1)\|^2) \tag{7.31}$$

From the estimates for $[H_1, \gamma_{L^m}]$ and $[H_3, \gamma_{L^m}]$ in equations (7.10) and (7.22) respectively, the following estimate holds on a sequence of times:

$$\int_1^t \langle u, \frac{L^{3m}}{(1+(\frac{L^m\rho_*}{2M})^2)^2} u\rangle + \langle u, (L^2-1) g_{L^m}\rho_*\chi_\alpha u\rangle d\tau \tag{7.32}$$

$$\leq C(E[u] + \|u(1)\|^2) \tag{7.33}$$

$$+ C(E[u] + \|\left(1+\left(\frac{\rho_*}{2M}\right)^2\right)^{-1} u(1)\|^2)| \tag{7.34}$$

By the local decay estimate, theorem 6.7, $\langle u, g_{L^m}\rho_*\chi_\alpha u\rangle$ can be integrated in time. This means that the L^2-1 term in the second term in the integral can be replaced with L^2.

$$\int_1^t \langle u, \frac{L^{3m}}{(1+(\frac{L^m\rho_*}{2M})^2)^2} u\rangle + \langle u, L^2 g_{L^m}\rho_*\chi_\alpha u\rangle d\tau \leq C(E[u] + \|u(1)\|^2) \tag{7.35}$$

Since the left hand side is monotonically increasing and the right hand side is decreasing on a sequence, equation (7.24) holds for all t, not just a sequence.

Equation (7.25) is proven by taking $m = \frac{1}{2}$, and estimating $L^{\frac{3}{2}}\chi_\alpha$ in the regions $|\rho_*| \leq L^{-\frac{1}{2}}$ and $|\rho_*| \geq L^{-\frac{1}{2}}$. In the first region $|\rho_*| \leq L^{-\frac{1}{2}}$

$$\frac{L^{3m}}{(1+(\frac{L^m\rho_*}{2M})^2)^2} = \frac{L^{\frac{3}{2}}}{(1+(\frac{L^{\frac{1}{2}}\rho_*}{2M})^2)^2} \geq CL^{\frac{3}{2}} \geq CL^{\frac{3}{2}}\chi_\alpha \tag{7.36}$$

In the second region, $|\rho_*| \geq L^{-\frac{1}{2}}$,

$$L^2 g_{L^m}\rho_*\chi_\alpha \geq L^2 \arctan(L^{\frac{1}{2}}\rho_*)\rho_*\chi_\alpha \geq \frac{\pi}{4}L^{\frac{3}{2}}\chi_\alpha \geq CL^{\frac{3}{2}}\chi_\alpha \tag{7.37}$$

Combining the estimates in the two regions with equation 7.24 gives the desired result

$$\int_1^\infty \langle u, L^{\frac{3}{2}}\chi_\alpha u\rangle \leq C(E[u] + \|u(1)\|^2) \tag{7.38}$$

□

This has proven an estimate for $\langle u, L^{\frac{3}{2}}\chi_\alpha u\rangle$. In the later phase space analysis, it will be necessary to estimate mixed derivative norms involving $\frac{\partial}{\partial r_*}$ and functions of L and ρ_*. The following corollary is a result of theorem 7.8 and the first part of lemma 7.5. It estimates a mixed derivative norm.

Corollary 7.9. *If $m \in [0,1]$ then there is a positive constant C such that for all $u \in \mathbb{S}$ which satisfy the linear wave equation, $\ddot{u} + Hu = 0$,*

$$\int_1^\infty \langle u', \frac{L^m}{(1+(\frac{L^m\rho_*}{2M})^2)}u'\rangle d\tau \leq C(E[u] + \|u(1)\|^2) \tag{7.39}$$

In particular, if u is a solution of the linear wave equation, $\ddot{u} + Hu = 0$, and χ_1 is a smooth, compactly supported function, then $\exists C$:

$$\int_1^\infty \langle u', \chi_1^2 u'\rangle d\tau \leq C(E[u] + \|u(1)\|^2) \tag{7.40}$$

$$\int_1^\infty \|\frac{\partial}{\partial r_*}(\chi_1 u)\|^2 d\tau \leq C(E[u] + \|u(1)\|^2) \tag{7.41}$$

$$\tag{7.42}$$

Proof. If in the proof of theorem 7.8, $[H_1, \gamma]$ is estimated by equation (7.7) instead of (7.6), it follows that

$$\int_1^\infty \langle u', \frac{2L^m}{(1+(\frac{L^m\rho_*}{2M})^2)}u'\rangle d\tau \leq C(E[u] + \|u(1)\|^2) \tag{7.43}$$

To get the second result, the first result is applied with $m = 0$.

$$\int_1^\infty \langle u', \frac{1}{(1+(\frac{\rho_*}{2M})^2)}u'\rangle d\tau \leq C(E[u] + \|u(1)\|^2) \tag{7.44}$$

If χ_1 is smooth and compactly supported, then there is a constant C such that $\chi^2 \leq C\frac{1}{(1+(\frac{\rho_*}{2M})^2)}$. Taking the expectation value of this and integrating in time gives equation 7.40.

$$\int_1^\infty \langle u', \chi_1^2 u'\rangle d\tau \leq \int_1^\infty \langle u', \frac{1}{(1+(\frac{\rho_*}{2M})^2)}u'\rangle d\tau \tag{7.45}$$

$$\leq C(E[u] + \|u(1)\|^2) \tag{7.46}$$

For the estimate involving $\|\frac{\partial}{\partial r_*}(\chi_1 u)\|$, the Leibniz rule can be applied on $\frac{\partial}{\partial r_*}(\chi_1 u)$. The resulting term involving $\chi_1' u$ can be controlled by the local decay theorem, since χ' is compactly supported, and the $\chi u'$ term can be controlled by the result just proven.

$$\int_1^\infty \|\frac{\partial}{\partial r_*}(\chi_1 u)\|^2 d\tau = \int_1^\infty \|\chi_1' u + \chi_1 u'\|^2 d\tau \tag{7.47}$$

$$\leq \int_1^\infty \|\chi_1' u\|^2 d\tau + \int_1^\infty \|\chi_1 u'\|^2 d\tau \tag{7.48}$$

$$\leq C(E[u] + \|u(1)\|^2) \tag{7.49}$$

□

8 Phase Space Analysis

The main conclusion of this section is the phase space induction theorem. It does not control the time integral of the angular energy, but instead loses an arbitrarily small power of L. The bounds are still in terms of the initial energy and the L^2 norm.

To prove this result, we introduce the operators $\Gamma_{n,m}$ to majorate $L^{\frac{\frac{3}{2}+m}{2}}$. Each $\Gamma_{n,m}$ is a phase space localised version of γ_{L^m}. Our localisation will use the radial variable and the radial derivative rescaled using the powers of L. We call these rescaled quantities the phase space variables. Because of the localisation, the commutator of $\Gamma_{n,m}$ and H only dominates $L^{\frac{\frac{3}{2}+m}{2}}$ in regions of phase space. This is where our definition of propagation observable differs from the standard one. In the standard definition, the commutator of a propagation observable must dominate the operator it is said to majorate; whereas, we allow this domination to occur in a region of phase space. Under our definition, but not the standard one, $\Gamma_{n,m}$ majorates $L^{\frac{\frac{3}{2}+m}{2}}$.

The definition of majoration always permits "lower order terms". For $\Gamma_{n,m}$, these will be $L^{\frac{1+2n}{2}}$. So that the lower order terms truly are lower order, we require $0 \leq n \leq m \leq \frac{1}{2}$. By combining various m and n choices, we eventually show that $\|L^{1-\varepsilon}\chi_\alpha u\|^2$ is integrable in time.

In subsection 1.1, *Structure of the Paper*, there is already an outline of this section, which we repeat in brief. Subsection 8.1 introduces the phase space variables and $\Gamma_{n,m}$. Subsection 8.2 introduces the commutator expansion theorem, which allows is to expand commutators of functions of the phase space variables, and several lemmas for common cases. Subsection 8.3 computes the commutators to show $\Gamma_{n,m}$ dominates powers of L with localisation in both phase space variables. Subsection 8.4 combines these estimates to prove estimates which are localised in the rescaled radial derivative alone. Finally, subsection 8.5 combines these in a finite induction to eliminate all phase space localisation. This is sufficient to prove point wise in time L^6 decay.

8.1 Phase Space Variables, Localisation, and Multipliers

Previously, we introduced a radial variable which was rescaled by the angular momentum. Now, we also introduce a rescaled radial derivative. We refer to these as the phase space variables $\mathbf{x}_m$ and $\boldsymbol{\xi}_n$ respectively. These will be rescaled separately with different parameters m and n. By varying these parameters, we will be able to control powers of L in various regions of phase space.

Definition 8.1. *The phase space variables are*

$$\mathbf{x}_m = L^m \rho_* \tag{8.1}$$

$$\boldsymbol{\xi}_n = -iL^{n-1}\frac{\partial}{\partial r_*} \tag{8.2}$$

For these phase space variables, we have a notion of lower order.

Notation 8.2. *The notation $O(L^1)$ denotes functions of time which are integrable, and for which the L^2 norm in time is bounded by a combination of constants, the energy of u, and its initial $L^2(\mathfrak{M})$ norm.*

The notation B denotes an arbitrary bounded operator, just as the notation C denotes an arbitrary constant. As with arbitrary constants, the value of B may vary from line to line in an argument. The notation B_i is used to refer to a bounded operator which is referred to later in the same argument.

For $n \leq \frac{1}{2}$, the notation $O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1)$ denotes inner products and norms of u which are either of the following form

$$\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, O(L^1), \tag{8.3}$$

or bounded by norms of this form. Note that for $q \leq 1+2n$, this includes any term of the form

$$\|L^{\frac{q}{2}}B\chi_\alpha u\|^2. \tag{8.4}$$

Note that in particular if $m \leq \frac{1}{2}$, then $m+n \leq \frac{1}{2}+n$ and

$$\|L^{m+n}B\chi_\alpha u\|^2 = O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1). \tag{8.5}$$

To begin with, it is not possible to use sharp cut off functions, so more regular functions with infinite support must be used. The rate of decay for these functions is important, and various secondary functions derived from the first appear.

In our estimate on the angular energy, we lose powers of L from our choice of localisation. We lose a factor of L^ϵ from our choice of decay for $\Phi_{a,\epsilon}(x)$ and a factor of L^δ from our choice of width for $\Psi_{L^{-\delta}\leq|x|\leq 1}$.

Definition 8.3. *The near and far localisation functions for $\mathbf{x}_m$ are*

$$X_\downarrow(x) = \frac{1}{1+\left(\frac{x}{2M}\right)^2} \tag{8.6}$$

$$X_\uparrow(x) = x\sqrt{\frac{\arctan(x)}{x}}. \tag{8.7}$$

From this, $X_\downarrow(\mathbf{x}_m)$ and $X_\uparrow(\mathbf{x}_m)$ are defined by the spectral theorem.

Definition 8.4. *The smooth, near localisation for $\boldsymbol{\xi}_n$ and two functions derived from it are defined by*

$$\Phi_{a,\epsilon}(x) = (1+x^2)^{-\frac{1-\epsilon}{4}} \tag{8.8}$$

$$\Phi_{b,\epsilon}(x) = x\Phi'_{a,\epsilon}(x) \tag{8.9}$$

$$\Phi_{c,\epsilon}(x) = \sqrt{\Phi_{a,\epsilon}(x)(\Phi_{a,\epsilon}(x)+2x\Phi'_{a,\epsilon}(x))}. \tag{8.10}$$

(The definition of $\Phi_{c,\epsilon}(x)$ requires the argument of the square root to be positive, which is proven in lemma 8.6.)

We use the notation $\chi(A,x)$ to denote the characteristic function of A which is 1 if $x\in A$ and 0 otherwise.

The sharp, near localisation and sharp, interval localisation for $\boldsymbol{\xi}_n$ are

$$\Phi_{|x|\le 1} = \chi([0,1],x) \tag{8.11}$$

$$\Psi_{L^{-\delta}\le|x|\le 1} = \chi([l^{-\delta},1],x). \tag{8.12}$$

Smooth, interval localisations will be defined in section 8.4. The functions $\Phi_{|x|\le 1}$ and $\Psi_{L^{-\delta}\le|x|\le 1}$ are extended as even functions.

From these, $\Phi_{a,\epsilon}(\boldsymbol{\xi}_n), \Phi_{b,\epsilon}(\boldsymbol{\xi}_n), \dots$ can be defined by the spectral theorem.

To prove a phase space localised version of the Morawetz estimate, we introduce the operator $\Gamma_{n,m}$. The parameters n and m are restricted to $0\le n\le m\le \frac{1}{2}$, because this is the range on which an estimate can be proven.

Definition 8.5. *For $0\le n\le m\le\frac{1}{2}$, the phase space induction multiplier is defined to be*

$$\Gamma_{n,m} = \chi_\alpha\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)L^{n-\epsilon}\gamma_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha \tag{8.13}$$

$$= \frac{1}{2}\chi_\alpha\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)i\left(\arctan(\mathbf{x}_m)\boldsymbol{\xi}_n + \boldsymbol{\xi}_n\arctan(\mathbf{x}_m)\right)\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha \tag{8.14}$$

We now prove some preliminary results, starting with some estimates on the localisations.

Lemma 8.6. *For all $\epsilon\ge 0$, $n\in[0,\frac{1}{2}]$, and $v\in\mathbb{S}$*

$$\|\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)v\| \ge \epsilon^{\frac{1}{2}}\|\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)v\|. \tag{8.15}$$

For $0\le n\le m\le\frac{1}{2}$ and $v\in\mathbb{S}$,

$$\|L^{1-\epsilon}\boldsymbol{\xi}_n\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)^2 v\| \le \|L^{\frac{1+n\epsilon-2\epsilon}{1-\epsilon}}v\| + \|\frac{\partial}{\partial r_*}v\| \tag{8.16}$$

Proof. The function $\Phi_{c,\epsilon}(x)$ is computed explicitly.

$$\Phi_{c,\epsilon}(x)^2 = \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)(\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)+2x\Phi'_{a,\epsilon}(x)) \tag{8.17}$$

$$= (1+x^2)^{-\frac{1-\epsilon}{4}}(1+x^2)^{-\frac{5-\epsilon}{4}}((1+x^2)-2\frac{1-\epsilon}{4}2x^2) \tag{8.18}$$

$$= (1+x^2)^{-\frac{6-2\epsilon}{4}}(1+\epsilon x^2) \tag{8.19}$$

$$\ge \epsilon(1+x^2)^{-\frac{1-\epsilon}{2}} \tag{8.20}$$

$$= \epsilon\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)^2 \tag{8.21}$$

Since $\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)(\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)+2x\Phi'_{a,\epsilon}(x))$ is strictly positive, the square root function, $\Phi_{c,\epsilon}(x)$ is well defined. Since $\Phi_{c,\epsilon}(x)^2\geq\epsilon\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)^2$, the spectral theorem also determines that $\|\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)v\|\geq\epsilon^{\frac{1}{2}}\|\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)v\|$.

Equation (8.16) is proven first for Schwartz class data. For this data, in a representation where L and $-i\frac{\partial}{\partial r_*}$ both act as multiplication operators, $\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)^2\leq|\boldsymbol{\xi}_n|^{-1+\epsilon}=|-i\frac{\partial}{\partial r_*}L^{n-1}|^{-1+\epsilon}$. This can be used to start estimating the norm of $L^{n-\epsilon}\frac{\partial}{\partial r_*}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)^2v$. This gives equation (8.16).

$$\begin{aligned}\|L^{1-\epsilon}\boldsymbol{\xi}_n\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)^2v\|\leq&\|L^{1-\epsilon}\boldsymbol{\xi}_n|\boldsymbol{\xi}_n|^{-1+\epsilon}v\| &(8.22)\\ \leq&\|L^{1-\epsilon}|\boldsymbol{\xi}_n|^{\epsilon}v\| &(8.23)\\ \leq&\|L^{1+n\epsilon-2\epsilon}|-i\frac{\partial}{\partial r_*}|^{\epsilon}v\| &(8.24)\end{aligned}$$

Continuing in a representation where both L and $-i\frac{\partial}{\partial r_*}$ act as multiplication operators, the identity, for positive f and g, $fg\leq f^{\frac{1}{1-\epsilon}}+g^{\frac{1}{\epsilon}}$ can be applied with $f=L^{1+n\epsilon-2\epsilon}$ and $g=|-i\frac{\partial}{\partial r_*}|^{\epsilon}$.

$$\|L^{1-\epsilon}\boldsymbol{\xi}_n\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)^2v\|\leq\|L^{\frac{1+n\epsilon-2\epsilon}{1-\epsilon}}v\|+\|\frac{\partial}{\partial r_*}v\| \tag{8.25}$$

□

Since $\Gamma_{n,m}$ is a product of operators, there is a Leibniz (or product) rule for computing the commutator with another operator. It is possible to further simplify this by expressing a sum of complex conjugates as a real part. The main application will be when the other operator is H or one of the subterms H_i.

Lemma 8.7. *For $u\in\mathbb{S}$, $0\leq n\leq m\leq\frac{1}{2}$, and any self-adjoint operator G which commutes with L*

$$\begin{aligned}\langle u,[G,\Gamma_{n,m}]u\rangle=&\langle u,\chi_\alpha\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)[G,\gamma_{L^m}]L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\rangle &(8.26)\\ &+\langle u,\chi_\alpha[G,\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)]\gamma_{L^m}L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\rangle &(8.27)\\ &+\langle u,\chi_\alpha\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\gamma_{L^m}L^{n-\epsilon}[G,\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)]\chi_\alpha u\rangle &(8.28)\\ &+\langle u,\chi_\alpha\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\gamma_{L^m}L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)[G,\chi_\alpha]u\rangle &(8.29)\\ &+\langle u,[G,\chi_\alpha]\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\gamma_{L^m}L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\rangle &(8.30)\end{aligned}$$

Proof. This is simply an application of the Leibniz rule for commutators. □

8.2 Commutator Expansions

To apply the Heisenberg like relation, it is necessary to expand commutators involving localisation in $\boldsymbol{\xi}_n$. This is done through a version of the commutator expansion lemma previously used in scattering theory[29]. We consider the special case of this expansion for commutators involving localisation in the phase space variables $\boldsymbol{\xi}_n$ and ρ_* or $\mathbf{x}_m$. These expansions are as finite order power series with an error term which involves the Fourier transform of a kth order derivative of one of the localising functions.

First, the adjoint is defined. These are iterated commutators and will appear in the commutator expansion, both as terms in the finite power series expansion and as a term in the remainder.

Definition 8.8. *For two operators H and A, the kth commutator of A with respect to H, $Ad_A^k(H)$ is defined recursively by*

$$\begin{aligned}Ad_A^1(H)=&[H,A] &(8.31)\\ Ad_A^k(H)=&[Ad_A^{k-1}(H),A] &(8.32)\end{aligned}$$

Initially the commutator $[H,A]$ is defined only as a form on the domain of A intersect the domain of H by the formula

$$\langle u,[H,A]u\rangle=\langle Hu,Au\rangle-\langle Au,Hu\rangle \tag{8.33}$$

If $Ad_A^{k-1}(H)$ extends to a bounded operator, then $Ad_A^k(H)$ is defined on the domain of A.

The commutator expansion theorem expands the commutator of an operator, F_1 , with a function of a self-adjoint operator, $F_2(A)$. The expansion is as a power series in the adjoint and has remainder involving the L^1 norm of Fourier transform of $F_2^{[k]}$.

Theorem 8.9 (Commutator Expansion Theorem). *If $n > 0$ is an integer, A is a self-adjoint operator, F_1 is a self-adjoint operator satisfying*

1. *for $1 \leq k \leq n$, $Ad_A^k(F_1)$ extends to a bounded operator,*

and $F_2(x)$ is a smooth function satisfying

1. $\|\mathcal{F}[F_2^{[n]}]\|_1 \leq \infty$,

then if $[F_1, F_2(A)]$ is defined as a form on the domain of A^n,

$$[F_1, F_2(A)] = \sum_{k=1}^{n-1} \frac{1}{k!} F_2^{[k]}(A) Ad_A^k(F_1) + R_n \tag{8.34}$$

in the form sense with the remainder R_n satisfying

$$\|R_n\| \leq C \|\mathcal{F}[F_2^{[n]}]\|_1 \|Ad_A^n(F_1)\| \tag{8.35}$$

Consequently, $[F_1, F_2(A)]$ defines an operator on the domain of A^{n-1}.

Proof. This is proven in [29]. □

The commutator expansion theorem is now specialised to the localising functions in the phase space variables. In terms of the previous theorem, $A = \boldsymbol{\xi}_n = -i\frac{\partial}{\partial r_*}L^{n-1}$ and F_1 is a function of ρ_* or of $\mathbf{x}_m$.

There are two applications of this lemma. The first application is the expansion of the commutator of the Hamiltonian, H, and the phase space induction multiplier, $\Gamma_{n,m}$, as a power series in the adjoint. The remainder is a higher order term, in the sense that the commutator minus the power series expansion can be multiplied by a positive power of L and still remains a bounded operator. The second application is using the expansion with no terms and using the remainder to directly estimate the entire commutator as a higher order term. This second application is expanded upon in the lemmas which follow this one.

Lemma 8.10 (Commutator order reduction lemma). *Let k be a positive integer, $F_1(x)$ be a smooth function satisfying*

1. *for $1 \leq j \leq k$, $\|F_1^{[j]}\|_\infty \leq \infty$,*

and $F_2(x)$ be a smooth function satisfying

1. $\|\mathcal{F}[F_2^{[k]}]\|_1 \leq \infty$

then there is a constant, C_k, depending only on k, such that

$$L^{k(1-n)}[F_1(\rho_*), F_2(\boldsymbol{\xi}_n)] = \sum_{j=1}^{k-1} L^{k(1-n)} F_2^{[j]}(\boldsymbol{\xi}_n)(iL^{n-1})^j F_1^j(\rho_*) + R_k \tag{8.36}$$

$$\|R_k\| \leq C_k \|\mathcal{F}[F_2^{[k]}]\|_1 \|F_1^{[k]}\|_\infty \tag{8.37}$$

and

$$L^{k(1-m-n)}[F_1(\mathbf{x}_m), F_2(\boldsymbol{\xi}_n)] = \sum_{j=1}^{k-1} L^{k(1-m-n)} F_2^{[j]}(\boldsymbol{\xi}_n)(iL^{m+n-1})^j F_1^j(\rho_*) + R_k \tag{8.38}$$

$$\|R_k\| \leq C_k \|\mathcal{F}[F_2^{[k]}]\|_1 \|F_1^{[k]}\|_\infty \tag{8.39}$$

Proof. The main idea in this proof is to apply the commutator expansion theorem, theorem 8.9.

The $m = 0$ case is a special case of the formula for general $m \geq 0$.

All quantities in equation (8.39) are composed of powers of L and functions of operators which commute with L. Therefore, they preserve the spherical harmonic decomposition, and it is sufficient to prove the power series expansion of the commutator on each spherical harmonic. We will use l to denote the action of L on each spherical harmonic, rather than the standard spherical harmonic index.

On a fixed spherical harmonic, take $x = \mathbf{x}_m = l^m\rho_*$ and $A = \boldsymbol{\xi}_n = -il^{n-1}\frac{\partial}{\partial r_*}$ and apply the commutator expansion theorem.

$$[F_1(\mathbf{x}_m), F_2(\boldsymbol{\xi}_n)] = \sum_{j=1}^{k-1} \frac{1}{j!} F_2^{[k]}(\boldsymbol{\xi}_n)\mathrm{Ad}^j_{\boldsymbol{\xi}_n}(F_1(\mathbf{x}_m)) + R_k \tag{8.40}$$

$$\|R_k\| \leq C_k \|\mathcal{F}[F_2^{[k]}]\|_1 \|\mathrm{Ad}^k_{\boldsymbol{\xi}_n}(F_1)\|_\infty \tag{8.41}$$

The adjoint can be rewritten in terms of l, ρ_*, and $\frac{\partial}{\partial r_*}$.

$$\mathrm{Ad}^j_{\boldsymbol{\xi}_n}(F_1(\mathbf{x}_m)) = [\mathrm{Ad}^{j-1}_{\boldsymbol{\xi}_n}(F_1(\mathbf{x}_m)), \boldsymbol{\xi}_n] \tag{8.42}$$

$$= [\mathrm{Ad}^{j-1}_{\boldsymbol{\xi}_n}(F_1(l^m\rho_*), -il^{n-1}\frac{\partial}{\partial r_*}] \tag{8.43}$$

$$= il^{n-1}\frac{\partial}{\partial r_*}\mathrm{Ad}^{j-1}_{\boldsymbol{\xi}_n}(F_1(l^m\rho_*) \tag{8.44}$$

Taking a derivative with respect to ρ_* of a function of $l^m\rho_*$ will introduce an additional factor of l^m. For $j = 1$, we have

$$\mathrm{Ad}^1_{\boldsymbol{\xi}_n}(F_1(\mathbf{x}_m)) = il^{n-1}\frac{\partial}{\partial r_*}F_1(l^m\rho_*) \tag{8.45}$$

$$= il^{m+n-1}F_1'(l^m\rho_*) \tag{8.46}$$

Each adjoint acts as a derivative, so, by induction,

$$\mathrm{Ad}^j_{\boldsymbol{\xi}_n}(F_1(\mathbf{x}_m)) = (il^{m+n-1})^j F_1^{[j]}(l^m\rho_*) \tag{8.47}$$

$$= (il^{m+n-1})^j F_1^{[j]}(\mathbf{x}_m) \tag{8.48}$$

Applying this, and multiplying by $l^{k(1-n-m)}$, gives that on each spherical harmonic,

$$l^{k(m+n-1)}[F_1(\mathbf{x}_m), F_2(\boldsymbol{\xi}_n)] = \sum_{j=1}^{k-1} l^{k(m+n-1)}\frac{1}{j!}F_2^{[k]}(\boldsymbol{\xi}_n)(il^{1-n-m})^j F_1^{[j]}(\mathbf{x}_m) + R_k \tag{8.49}$$

$$\|R_k\| \leq C_k \|\mathcal{F}[F_2^{[k]}]\|_1 \|F_1^{[j]}\|_\infty \tag{8.50}$$

Since the constant in the commutator expansion theorem, theorem 8.9, is independent of the choice of x, A, F_1, and F_2, the constant C_k is independent of l. The result on each spherical harmonic can be extended across all harmonics with out any change. In particular, the operator R_k is uniformly bounded across all spherical harmonics. □

The commutator order reduction lemma, lemma 8.10, is used to estimate commutators of localisation in the phase space variables. Usually it is sufficient to know that the commutator is a bounded operator times negative powers of L, and that is what the following lemmas assert for certain classes of localising operators.

The following lemma shows that the commutator of phase space localising functions is lower order in the common situation when the localisation in $\boldsymbol{\xi}_n$ is a Schwartz class function.

Lemma 8.11. *There is a constant C, such that if $0 \leq n \leq m < \frac{1}{2}$, $F_1 \in C^1(\mathbb{R})$, $F_1' \in L^\infty$, and F_2 is of Schwartz class, then there is a bounded operator, B, satisfying the following*

$$[F_1(\mathbf{x}_m), F_2(\boldsymbol{\xi}_n)] = L^{n+m-1}B_1 \tag{8.51}$$

$$\|B_1\| \leq C\|\mathcal{F}[F_2']\|_1 \|F_1'\|_\infty. \tag{8.52}$$

Proof. Since the Schwartz class is preserved by differentiation and the Fourier transform, $\mathcal{F}[F_2']$ is Schwartz class and, hence, in L^1. The expansion to order 0 from the commutator order reduction lemma, lemma 8.10, is

$$L^{1-m-n}[F_1(\mathbf{x}_m), F_2(\boldsymbol{\xi}_n)] = R_1 \tag{8.53}$$

$$\|R_1\| \leq C\|\mathcal{F}[F_2']\|_{L^1}\|F_2'\|_{L^\infty} \tag{8.54}$$

Since L is bounded below by 1, and $0 \leq n \leq m < \frac{1}{2}$, the operator L^{1-m-n} is bounded. The operators L^{1-m-n} and R_1 are bounded, so they can be composed. Applying L^{1-m-n} to (8.53) proves the desired result. □

The previous lemma can not be applied with the smooth, near localisations for $\boldsymbol{\xi}_n$, which are not in Schwartz class. To prove a similar result for the smooth, near localisations, it is sufficient to show that each satisfies the condition $\mathcal{F}[F_2'] \in L^1$. The following lemma also proves a commutator order reduction result for two unbounded functions of $\boldsymbol{\xi}_n$, $F_2 = \xi$ and $F_2 = \xi\Phi_{a,\epsilon}(\xi)$, using a first order expansion in lemma 8.10.

Lemma 8.12. *There is a constant C, such that if $0 \leq n \leq m \leq \frac{1}{2}$, $F_1 \in C^2(\mathbb{R})$, $\epsilon > 0$, and $F_2(x)$ is one of the following functions: $x, \Phi_{a,\epsilon}(x), \Phi_{b,\epsilon}(x), \Phi_{c,\epsilon}(x)$, then there is a bounded operator, B_{F_2}, satisfying the following*

$$[F_1(\mathbf{x}_m), F_2(\boldsymbol{\xi}_n)] = L^{n+m-1}B_{F_2} \tag{8.55}$$

$$\|B_{F_2}\| \leq C\|F_1'\|_\infty. \tag{8.56}$$

If $F_2(x) = x\Phi_{a,\epsilon}(x)$ and the other conditions apply, then there is a bounded operator, B_{F_2}, satisfying

$$[F_1(\mathbf{x}_m), F_2(\boldsymbol{\xi}_n)] = L^{n+m-1}B_{F_2} \tag{8.57}$$

$$\|B_{F_2}\| \leq C(\|F_1'\|_\infty + \|F_1''\|_\infty). \tag{8.58}$$

Proof. The fundamental problem here is that the permitted F_2 functions are not in Schwartz class. There are finitely many functions F_2 considered in this theorem, so if a C can be found for each of them, then the largest one can be applied to make the result hold uniformly.

For the function $F_2(x) = x$, $F_2(\boldsymbol{\xi}_n) = \boldsymbol{\xi}_n = -i\frac{\partial}{\partial r_*}L^{n-1}$. All the operators involved commute with the spherical harmonic decomposition, and on each spherical harmonic, the commutator can be explicitly computed

$$[F_1(\mathbf{x}_m), -i\frac{\partial}{\partial r_*}L^{n-1}] = [F_1(L^m\rho_*), -i\frac{\partial}{\partial r_*}L^{n-1}] \tag{8.59}$$

$$= iL^{n+m-1}F_1'(L^m\rho_*) = iL^{n+m-1}F_1'(\mathbf{x}_m) \tag{8.60}$$

The commutator is a product of a power of L and a multiplication operator. The L^∞ norm of the function is the operator norm of the associated multiplication operator.

To apply the commutator order reduction lemma, lemma 8.10 to the remaining functions, we prove: If $g \in C^2(\mathbb{R}) \cap L^2(\mathbb{R})$ and $g'' \in L^2(\mathbb{R})$, then $\|\mathcal{F}[g]\|_1 \leq \infty$.

Since $g \in C^2$, g'' is well defined, and since $g, g'' \in L^2(\mathbb{R})$, both $\mathcal{F}[g]$ and $\mathcal{F}[g'']$ are well defined and in $L^2(\mathbb{R})$. Let

$$f(\xi) \equiv |(1+\xi^2)\mathcal{F}[g](\xi)|, \tag{8.61}$$

$$f_{\text{big}}(\xi) \equiv f(\xi)\chi([1,\infty), f(\xi)), \tag{8.62}$$

$$f_{\text{small}}(\xi) \equiv f(\xi)\chi([0,1), f(\xi)), \tag{8.63}$$

and observe that

$$f(\xi) = f_{\text{big}}(\xi) + f_{\text{small}}(\xi) \tag{8.64}$$

$$|\mathcal{F}[g](\xi)| \leq |\frac{f_{\text{big}}(\xi)}{1+\xi^2}| + |\frac{f_{\text{small}}(\xi)}{1+\xi^2}| \leq |f_{\text{big}}(\xi)| + |\frac{f_{\text{small}}(\xi)}{1+\xi^2}|. \tag{8.65}$$

From properties of the Fourier transform it is known that

$$|\mathcal{F}[g](\xi)| + |\mathcal{F}[g''](\xi)| = |\mathcal{F}[g](\xi)| + |\xi^2\mathcal{F}[g](\xi)| = |f(\xi)|, \tag{8.66}$$

and hence that f is in $L^2(\mathbb{R})$. Since $f_{\text{big}} \leq f_{\text{big}}^2 \leq f^2$, f_{big} is in $L^1(\mathbb{R})$, and since $f_{\text{small}} \leq 1$, $\frac{f_{\text{small}}(\xi)}{1+\xi^2}$ is also in L^1. The sum of these bounds $|\mathcal{F}[g]|$, so that $\mathcal{F}[g]$ is in $L^1(\mathbb{R})$.

For the three functions, $\Phi_{a,\epsilon}(x)$, $\Phi_{b,\epsilon}(x)$, and $\Phi_{c,\epsilon}(x)$, each of these is a power of a polynomial and decays like $x^{\frac{-1-\epsilon}{2}}$. Therefore, F_2' and $(F_2')''$ are each in L^2, since they decay like $x^{\frac{-3+\epsilon}{2}}$ and $x^{\frac{-7+\epsilon}{2}}$ respectively, and $\mathcal{F}[F_2']$ is in L^1. By the same argument as in lemma 8.11,

$$[F_1(\mathbf{x}_m), F_2(\boldsymbol{\xi}_n)] = L^{n+m-1}B \tag{8.67}$$

$$\|B\| \leq C(\|F_1'\|_\infty). \tag{8.68}$$

The last, and most difficult piece, is for $F_2(x) = x\Phi_{a,\epsilon}(x)$. In the previous arguments, the commutator was directly estimated. In this case it is expanded to first order, and both the first order term and the remainder are estimated. The derivative of $x\Phi_{a,\epsilon}(x)$ decays too slowly to be in L^2 and may not have an L^1 Fourier transform.

As usual, all the operators involved commute with the spherical harmonic decomposition. On each spherical harmonic, by lemma 8.10

$$[F_1(\mathbf{x}_m), F_2(\boldsymbol{\xi}_n)] = F_2'(\boldsymbol{\xi}_n)(iL^{n+m-1})(F_1'(\mathbf{x}_m)) + L^{2n+2m-2}R_2. \tag{8.69}$$

The first term is a product of operators which commute with powers of L and each can be estimated in norm.

$$\|F_2'(\boldsymbol{\xi}_n)(-iL^{n+m-1})(-F_1'(\mathbf{x}_m))\| = L^{n+m-1}\|F_2'(\boldsymbol{\xi}_n)\|\|F_1'(\mathbf{x}_m)\| \tag{8.70}$$

$$\leq L^{n+m-1}\|F_2'\|_\infty\|F_1'\|_\infty \tag{8.71}$$

Since $F_2(x) = x\Phi_{a,\epsilon}(x)$, F_2' is bounded.

The second term is estimated by lemma 8.10.

$$\|R_2\| = C\|\mathcal{F}[F_2'']\|_1\|F_1''\|_\infty \tag{8.72}$$

Since $F_2 = x\Phi_{a,\epsilon}(x)$, F_2'' is a smooth function, which is in L^2 and has its second derivative, $(F_2'')''$, also in L^2, we conclude that $\mathcal{F}[F_2'']$ is in L^1.

Since $L^{2n+2m-2} \leq L^{n+m-1}$, the estimate on each of the terms in the expression for the commutator can be combined to give

$$\|[F_1(\mathbf{x}_m), F_2(\boldsymbol{\xi}_n)]\| \leq L^{n+m-1}\|F_2'\|_\infty\|F_1'\|_\infty + L^{2n+2m-2}C\|\mathcal{F}[F_2'']\|_1\|F_1''\|_\infty \tag{8.73}$$

$$\leq CL^{n+m-1}(\|F_1'\|_\infty + \|F_1''\|_\infty) \tag{8.74}$$

On each spherical harmonic there is a bounded operator, B, such that

$$[F_1(\mathbf{x}_m), F_2(\boldsymbol{\xi}_n)] = L^{n+m-1}B \tag{8.75}$$

$$\|B\| \leq C(\|F_1'\|_\infty + \|F_1''\|_\infty). \tag{8.76}$$

Therefore, B can be extended as an operator on L^2 satisfying the same condition. □

8.3 Phase Space Estimates

In this subsection, we compute the commutators between $\Gamma_{n,m}$ and the components of the Hamiltonian, H_i. This is analogous to subsection 6.2 with additional factors of L. The H_1 commutator dominates the radial derivative and a constant both localised around small $\mathbf{x}_m$, with localisation like $X_\downarrow(\mathbf{x}_m)$. The H_2 commutator is simply lower order. The H_3 commutator dominates powers of L localised away from small $\mathbf{x}_m$, with localisation $X_\uparrow(\mathbf{x}_m)$.

We begin with the H_1 commutator which, as in lemma 7.5 for γ_{L^m}, dominates two terms.

Proposition 8.13. *There are constants C_1 and C_2, such that for $u \in \mathbb{S}$ and $0 \leq n \leq m \leq \frac{1}{2}$,*

$$\langle u, [H_1, \Gamma_{n,m}]u\rangle = C_1 \|L^{\frac{m+n-\epsilon}{2}} X_\downarrow(\mathbf{x}_m)^{\frac{1}{2}} \frac{\partial}{\partial r_*} \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\|^2 \tag{8.77}$$
$$+ C_2 \|L^{\frac{3m+n-\epsilon}{2}} X_\downarrow(\mathbf{x}_m)\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\|^2 + O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) \tag{8.78}$$

Proof. The commutator is expanded using the Leibniz rule, lemma 8.7. Since $\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)$ and $\frac{\partial}{\partial r_*}$ commute and $H_1 = -\frac{\partial^2}{\partial r_*^2}$, the commutator $[H_1, \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)]$ is zero. Therefore, the expectation value of the commutator is the the sum of three non zero terms.

$$\langle u, [H_1, \Gamma_{n,m}], u\rangle = \langle u, \chi_\alpha \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)[H_1, \gamma_{L^m}]L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\rangle \tag{8.79}$$
$$+ \langle u, [H_1, \chi_\alpha]\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\gamma_{L^m}L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\rangle \tag{8.80}$$
$$+ \langle u, \chi_\alpha\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\gamma_{L^m}L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)[H_1, \chi_\alpha]u\rangle \tag{8.81}$$

The commutator in the first term, $[H_1, \gamma_{L^m}]$ only involves the angularly modulated multiplier, γ_{L^m}. Commutators involving the angularly modulated multiplier were calculated in section 7, and the commutator with H_1 was computed in lemma 7.5. For these calculations, the expectation value is taken with respect to $L^{\frac{n-\epsilon}{2}}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha$.

$$\langle u, \chi_\alpha\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)[-\frac{\partial^2}{\partial r_*^2}, \gamma_{L^m}]L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\rangle \geq C_1\|L^{\frac{m+n-\epsilon}{2}} X_\downarrow(\mathbf{x}_m)^{\frac{1}{2}}\frac{\partial}{\partial r_*}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\|^2 \tag{8.82}$$
$$+ C_2\|L^{3m+n-\epsilon}X_\downarrow(\mathbf{x}_m)\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\|^2 \tag{8.83}$$

It now remains to show that the commutators involving χ_α gives only lower order terms. There are two of these. The one involving $\chi_\alpha\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\gamma_{L^m}L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)[H_1, \chi_\alpha]$ will be dealt with first.

Since χ_α is a function and $H_1 = -\frac{\partial^2}{\partial r_*^2}$ is a derivative operator, their commutator, $[H_1, \chi_\alpha]$, can be explicitly calculated and substituted into the relevant expectation value.

$$\langle u, \chi_\alpha\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\gamma_{L^m}L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)[-\frac{\partial^2}{\partial r_*^2}, \chi_\alpha]u\rangle \tag{8.84}$$
$$= \langle u, \chi_\alpha\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\gamma_{L^m}L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)(-2\frac{\partial}{\partial r_*}\chi'_\alpha - \chi''_\alpha)u\rangle \tag{8.85}$$

To analyse this term, γ_{L^m} is expanded into two terms so that the most significant term can be dealt with first. The slightly less common expansion $\gamma_{L^m} = \frac{\partial}{\partial r_*}g_{L^m} - \frac{1}{2}g'_{L^m}$ is used. There are a total of four terms to consider, two from expanding $[H_1, \chi_\alpha]$ times two from expanding γ_{L^m}.

The term that appear to be highest order is the one involving $\frac{\partial}{\partial r_*}g_{L^m}$ from γ_{L^m} and $-2\frac{\partial}{\partial r_*}\chi'_\alpha$ from $[-\frac{\partial^2}{\partial r_*^2}, \chi_\alpha]$. It will be referred to as $I_{1,1}$. It can be simplified by moving one of the derivative operators through $\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)$ (since they commute), and then moving it and a factor of χ_α to the other side of the inner product.

$$I_{1,1} \equiv \langle u, \chi_\alpha\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\frac{\partial}{\partial r_*}g_{L^m}L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)(-2\frac{\partial}{\partial r_*}\chi'_\alpha)u\rangle \tag{8.86}$$
$$= -\langle\frac{\partial}{\partial r_*}\chi_\alpha u, \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)g_{L^m}L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)(-2\frac{\partial}{\partial r_*}\chi'_\alpha)u\rangle \tag{8.87}$$

Applying the Cauchy-Schwartz estimate gives the sum of two terms. One of these is $\|\frac{\partial}{\partial r_*}\chi u\|^2$ which is $O(L^1)$ (integrable in time) by corollary 7.9. The other can be rewritten in terms of $\boldsymbol{\xi}_n$.

$$I_{1,1} \geq -\|\frac{\partial}{\partial r_*}\chi_\alpha u\|^2 - \|\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)g_{L^m}L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\frac{\partial}{\partial r_*}\chi'_\alpha u\|^2 \tag{8.88}$$
$$\geq -\|L^{1-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)g_{L^m}\boldsymbol{\xi}_n\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi'_\alpha u\|^2 + O(L^1) \tag{8.89}$$

The remaining norm is going to be manipulated with the goal of applying lemma 8.6 to control $\boldsymbol{\xi}_n \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)^2$ by the energy. This manipulation will introduce additional error terms from commuting the phase space localisations. To commute $\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)$ and g_{L^m} while avoiding any error terms involving factors of $\boldsymbol{\xi}_n$ outside a commutator, it is necessary to commute $\boldsymbol{\xi}_n$ through g_{L^m} and then commute $\boldsymbol{\xi}_n\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)$ back through g_{L^m}.

$$\begin{aligned}
I_{1,1} \geq & -\|L^{1-\epsilon}\boldsymbol{\xi}_n\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)g_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha' u\|^2 && (8.90)\\
& -\|L^{1-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)[g_{L^m},\boldsymbol{\xi}_n]\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha' u\|^2 + O(L^1) && (8.91)\\
\geq & -\|L^{1-\epsilon}g_{L^m}\boldsymbol{\xi}_n\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)^2\chi_\alpha' u\|^2 - \|L^{1-\epsilon}[\boldsymbol{\xi}_n\Phi_{a,\epsilon}(\boldsymbol{\xi}_n), g_{L^m}]\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha' u\|^2 && (8.92)\\
& -\|L^{1-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)[g_{L^m},\boldsymbol{\xi}_n]\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha' u\|^2 + O(L^1) && (8.93)
\end{aligned}$$

By lemma 8.12, the commutators are of the form $L^{m+n-1}B$. Since $\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)$ is also a bounded operator, all the error terms from commuting are lower order terms. In the remaining term, the factor of g_{L^m} can be dropped for a constant and then lemma 8.16 can be applied to control $\boldsymbol{\xi}_n\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)^2$.

$$\begin{aligned}
I_{1,1} \geq & -C\|L^{1-\epsilon}g_{L^m}\boldsymbol{\xi}_n\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)^2\chi_\alpha' u\|^2 - C\|L^{1-\epsilon}L^{n+m-1}B\chi_\alpha' u\|^2 + O(L^1) && (8.94)\\
\geq & -C(\|L^{\frac{1+n\epsilon-2\epsilon}{1-\epsilon}}\chi_\alpha' u\|^2 + \|\frac{\partial}{\partial r_*}\chi_\alpha' u\|^2) - O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) && (8.95)
\end{aligned}$$

The angular energy away from $r=\alpha$ is time integrable by theorem 6.7. Since χ_α is compactly supported and identically 1 in a neighbourhood of $r=\alpha$, $|\chi_\alpha'|^2$ is dominated by a multiple of $(\rho_*\arctan(\rho_*))$ and $\|L\chi_\alpha' u\|$ is integrable in time. Interpolating between the local decay result and the angular energy decay away from the photon sphere result, both from theorem 6.7, $\|L^{\frac{1+n\epsilon-2\epsilon}{1-\epsilon}}\chi_\alpha' u\|^2$ is time integrable, and hence in $O(L^1)$. By corollary 7.9, $\|\frac{\partial}{\partial r_*}\chi_\alpha' u\|^2$ is time integrable and hence in $O(L^1)$. All the terms in $I_{1,1}$ are now controlled.

$$I_{1,1} \equiv \langle u, \chi_\alpha\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\frac{\partial}{\partial r_*}g_{L^m}L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)(-2\frac{\partial}{\partial r_*}\chi_\alpha')u\rangle \geq O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1). \qquad (8.96)$$

The term that appears to be the next highest order is also $O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1)$ not merely $O(L^1)$. This is the term involving g_{L^m}' from γ_{L^m} and $-2\frac{\partial}{\partial r_*}\chi_\alpha'$ from $[-\frac{\partial^2}{\partial r_*^2}, \chi_\alpha]$. It is estimated by moving most of the localisation to the other side of the inner product, applying the Cauchy-Schwartz inequality, replacing bounded operators by a constant, and then finding that the remaining terms are each $O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1)$.

$$\begin{aligned}
&\langle u, \chi_\alpha\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)(-\frac{1}{2}\frac{L^{m+n-\epsilon}}{1+(\frac{\rho_* L^m}{2M})^2})\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\frac{\partial}{\partial r_*}\chi_\alpha' u\rangle && (8.97)\\
&= -\frac{1}{2}\langle \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)L^{m+n-\epsilon}X_\downarrow(\mathbf{x}_m)\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u, \frac{\partial}{\partial r_*}\chi_\alpha' u\rangle && (8.98)\\
&\geq -C\|L^{m+n-\epsilon}\chi_\alpha u\|\|\frac{\partial}{\partial r_*}\chi_\alpha' u\| && (8.99)\\
&\geq -C(\|L^{m+n-\epsilon}\chi_\alpha u\|^2 + \|\frac{\partial}{\partial r_*}\chi_\alpha' u\|^2) && (8.100)\\
&\geq -O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) && (8.101)
\end{aligned}$$

The terms involving χ_α'' from $[-\frac{\partial^2}{\partial r_*^2}, \chi_\alpha]$ are the lowest order terms and are quickly shown to be lower order. The same method as was used for the term involving g_{L^m}' and $-2\frac{\partial}{\partial r_*}\chi_\alpha'$ can be applied.

$$\begin{aligned}
\langle u, \chi_\alpha\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\frac{\partial}{\partial r_*}L^{n-\epsilon}g_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha'' u\rangle &= \langle g_{L^m}\frac{\partial}{\partial r_*}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u, L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha'' u\rangle && (8.102)\\
&\geq -C\|\frac{\partial}{\partial r_*}\chi_\alpha u\|\|L^{n-\epsilon}\chi_\alpha'' u\| && (8.103)\\
&\geq -O(L^1) && (8.104)\\
&= -O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) && (8.105)
\end{aligned}$$

$$\langle u, \chi_\alpha \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \frac{L^{m+n-\epsilon}}{1+(\frac{\rho_* L^m}{2M})^2} \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \chi''_\alpha u\rangle \tag{8.106}$$
$$= \langle \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u, L^{m+n-\epsilon} X_\downarrow(\mathbf{x}_m) \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \chi''_\alpha u\rangle \tag{8.107}$$
$$\geq - C \| L^{\frac{m+n-\epsilon}{2}} \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \chi_\alpha u\| \| L^{\frac{m+n-\epsilon}{2}} \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \chi_\alpha u\| \tag{8.108}$$
$$\geq - C \| L^{\frac{m+n-\epsilon}{2}} \chi_\alpha u\|^2 \tag{8.109}$$
$$\geq - O(\| L^{\frac{1+2n}{2}} \chi_\alpha u\|^2, L^1) \tag{8.110}$$

There is another term in the Leibniz rule expansion of $\langle u, [H_1, \Gamma_{n,m}]u\rangle$. From the symmetry properties of $\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)$, $L^{n-\epsilon}$, and χ_α and the anti-symmetry properties of γ_{L^m} and $[H_1,\chi_\alpha]$, this term can be rewritten as the complex conjugate of the one just shown to be $O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1)$.

$$\langle u, [H_1, \chi_\alpha] \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \gamma_{L^m} L^{n-\epsilon} \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \chi_\alpha u\rangle \tag{8.111}$$
$$= \langle \chi_\alpha \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \gamma_{L^m} L^{n-\epsilon} \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) [H_1, \chi_\alpha] u\rangle \tag{8.112}$$
$$= (\langle u, \chi_\alpha \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \gamma_{L^m} L^{n-\epsilon} \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) [H_1, \chi_\alpha] u\rangle)^* \tag{8.113}$$
$$= O(\| L^{\frac{1+2n}{2}} \chi_\alpha u\|^2, L^1) \tag{8.114}$$

□

The commutator with H_2 is now computed. Because H_2 is a function of ρ_* and contains no derivatives, all the commutators can be shown to lower order by lemma 8.12.

Lemma 8.14. *For $u \in \mathbb{S}$ and $0 \leq n \leq m \leq \frac{1}{2}$,*

$$\langle u, [H_2, \Gamma_{n,m}]u\rangle = O(L^1) \tag{8.115}$$

Proof. Lemma 8.7, the Leibniz formula for commutators, is used to calculate $[H_2, \Gamma_{n,m}]$ and then the commutators are shown to be lower order by lemma 8.12. Since χ_α and V commute, $[H_2, \chi_\alpha] = 0$ and only three terms are left in the expansion of the $[H_2, \Gamma_{n,m}]$.

$$\langle u, [H_2, \Gamma_{n,m}]u\rangle = \langle u, \chi_\alpha \; L^{n-\epsilon}([V, \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)] \gamma_{L^m} \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.116}$$
$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) [V, \gamma_{L^m}] \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.117}$$
$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \gamma_{L^m} [V, \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)]) \chi_\alpha u\rangle \tag{8.118}$$

The commutator $[V, \gamma_{L^m}]$ only involves the angularly modulated multiplier, and was shown to be a bounded function in lemma 7.6. Applying additional bounded operators and powers of L yields a bounded operator multiplied by the same power of L.

$$L^{n-\epsilon} \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) [V, \gamma_{L^m}] \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) = L^{n-\epsilon} B \tag{8.119}$$

Since $n \leq \frac{3}{2}$, by the angular modulation theorem, theorem 7.8, the expectation value of this operator is $O(L^1)$.

$$\langle u, \chi_\alpha \; L^{n-\epsilon}[V, \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)] \gamma_{L^m} \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \chi_\alpha u\rangle = \langle u, L^{n-\epsilon} B u\rangle \tag{8.120}$$
$$= O(L^1). \tag{8.121}$$

The remaining terms are complex conjugates of each other. Only one is considered here, and the other can be dealt with in the same way. The commutator is shown to be a bounded operator times a negative power of L using lemma 8.12. γ_{L^m} can be expanded and the bounded functions involved in this expansion can be absorbed into the bounded operator from the commutator expansion.

$$L^{n-\epsilon} \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \gamma_{L^m} [V, \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)] = L^{n-\epsilon} \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) (\frac{\partial}{\partial r_*} g_{L^m} - \frac{1}{2} g'_{L^m}) L^{n-1} B \tag{8.122}$$
$$= L^{2n-1-\epsilon} \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \frac{\partial}{\partial r_*} B_1 + L^{2n-1-\epsilon} L^m \frac{1}{2} X_\downarrow(\mathbf{x}_m) B_2 \tag{8.123}$$
$$= L^{2n-1-\epsilon} \Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \frac{\partial}{\partial r_*} B_1 + L^{m+2n-1-\epsilon} B_3. \tag{8.124}$$

This operator can be substituted into the relevant expectation value. This gives a sum of two terms, both of which can be estimated by the Cauchy-Schwartz inequality.

$$\langle u, \chi_\alpha L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\gamma_{L^m}[V,\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)]\chi_\alpha u\rangle \tag{8.125}$$

$$=\langle -\frac{\partial}{\partial r_*}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u, L^{2n-1-\epsilon}B_1\chi_\alpha u\rangle + \langle \chi_\alpha u, L^{m+2n-1-\epsilon}B_3\chi_\alpha u\rangle \tag{8.126}$$

$$\geq -C(\|\frac{\partial}{\partial r_*}\chi_\alpha u\|\|L^{\frac{2n-1-\epsilon}{2}}B_1\chi_\alpha u\| + \|L^{\frac{3-\epsilon}{2}n}\chi_\alpha u\|^2) \tag{8.127}$$

$$\geq -C(\|\frac{\partial}{\partial r_*}\chi_\alpha u\|^2 + \|L^{\frac{2n-1-\epsilon}{2}}\chi_\alpha u\|^2 + \|L^{\frac{3-\epsilon}{2}n}\chi_\alpha u\|^2) \tag{8.128}$$

Since $2n-1-\epsilon \leq \frac{3}{2}$ and $(3-\epsilon)n \leq \frac{3}{2}$, by the angular modulation theorem, theorem 7.8, the two terms involving powers of L are time integrable, ie $O(L^1)$. By corollary 7.9, $\|\frac{\partial}{\partial r_*}\chi_\alpha u\|^2 = O(L^1)$. Therefore the expectation value being considered is time integrable

$$\langle u, \chi_\alpha L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\gamma_{L^m}[V,\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)]\chi_\alpha u\rangle = O(L^1) \tag{8.129}$$

□

The commutator with H_3 can now be computed. In the angular modulation argument, lemma 7.7 was used to show that $L^2 g_{L^m}\rho_*$ was bounded in expectation value by the commutator. Here, up to localisation in $\boldsymbol{\xi}_n$, $L^{2+n-\epsilon}g_{L^m}\rho_*$ is again controlled. For $m=\frac{1}{2}$ and $\mathbf{x}_{\frac{1}{2}}$ large, this controls $L^{\frac{3}{2}+n-\epsilon}$. The same power was controlled in the complementary region by lemma 8.13.

Because $H_3 = (-\Delta_{S^2})V_l = (L^2-1)V_l$ contains two angular derivatives, which are exactly what is under consideration, this commutator requires the most care. The difference between L^2-1 and L^2 is a local decay term. The commutator is first expanded using the product rule in lemma 8.7. The commutator with γ_{L^m} is computed for the angular modulation argument in lemma 7.7. The commutator with the localisation $\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)$ can be expanded to second order using the commutator order reduction lemma, lemma 8.10. The first order term is of the same order as the commutator with γ_{L^m}. It involves a derivative of $\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)$ and is multiplied by the $\frac{\partial}{\partial r_*}$ term from γ_{L^m}. There are two such terms, and when rearranged and combined with the term from the commutator with γ_{L^m}, they give $\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)^2 = \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)(\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) + 2\Phi'_{a,\epsilon}(\boldsymbol{\xi}_n))$. This term must be positive to give the desired estimate. The second order term in the expansion naively appears to be of a high order since H_3 introduced two factors of L. However, rearrangements yield a cancellation of the highest order part of this term and the remainder is a lower order error term. For both the first and second commutators, there are many rearrangements of phase space localising functions. These are shown to be lower order in the standard way using lemma 8.12. The remainder term is shown to be lower order directly from the commutator order reduction lemma, lemma 8.10.

Proposition 8.15. *For $u \in \mathbb{S}$ and $0 \leq n \leq m \leq \frac{1}{2}$*

$$\langle u, [H_3, \Gamma_{n,m}]u\rangle \geq \|L^{\frac{2+n-m-\epsilon}{2}}(-V_l'\rho_*^{-1})^{\frac{1}{2}}X_\uparrow(\mathbf{x}_m)\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\|^2 - O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) \tag{8.130}$$

Proof. The difference between $H_3 = (L^2-1)V_l$ and L^2V_l is a local decay term. To simplify the calculation this is first removed.

$$\langle u, [H_3, \Gamma_{n,m}]u\rangle = \langle u, [(L^2-1)V_l, \Gamma_{n,m}]u\rangle \tag{8.131}$$

$$= \langle u, [L^2V_l, \Gamma_{n,m}]u\rangle - \langle u, [V_l, \Gamma_{n,m}]u\rangle \tag{8.132}$$

By the same argument as in lemma 8.14, $\langle u, [V_l, \Gamma_{n,m}]u\rangle = O(L^1)$. Now it is sufficient to consider the commutator $[L^2V_l, \Gamma_{n,m}]$.

Lemma 8.7, the Leibniz rule, can be applied with $G = H_3$. Since $[L^2V_l, \chi_\alpha] = 0$, there are only three terms in this expansion.

$$\langle u, [L^2V_l, \Gamma_{n,m}]u\rangle = \langle u, \chi_\alpha L^{n-\epsilon}([L^2V_l, \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)]\gamma_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.133}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)[L^2V_l, \gamma_{L^m}]\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.134}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\gamma_{L^m}[L^2V_l, \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)])\chi_\alpha u\rangle \tag{8.135}$$

To simplify the calculations, the commutator will be expanded and the expectation value will be taken on each piece separately. In this expansion, the commutators involving $\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)$ are expanded to second order, and γ_{L^m} is expanded using both expansions $\gamma_{L^m} = g_{L^m}\frac{\partial}{\partial r_*} + \frac{1}{2}g'_{L^m} = \frac{\partial}{\partial r_*}g_{L^m} - \frac{1}{2}g'_{L^m}$.

$$L^{n-\epsilon}([L^2V_l, \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)]\gamma_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.136}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)[L^2V_l, \gamma_{L^m}]\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.137}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\gamma_{L^m}[L^2V_l, \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)]) \tag{8.138}$$

$$= L^{2+n-\epsilon}((\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)'[V_l, \boldsymbol{\xi}_n] + \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)''[[V_l, \boldsymbol{\xi}_n], \boldsymbol{\xi}_n] + R_3)(\frac{\partial}{\partial r_*}g_{L^m} - \frac{1}{2}g'_{L^m})\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.139}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)g_{L^m}[V_l, \frac{\partial}{\partial r_*}]\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.140}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)(g_{L^m}\frac{\partial}{\partial r_*} + \frac{1}{2}g'_{L^m})(\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)'[V_l, \boldsymbol{\xi}_n] + \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)''[[V_l, \boldsymbol{\xi}_n], \boldsymbol{\xi}_n] + R_3)) \tag{8.141}$$

This complicated expression can be broken into six different terms, based on which term in the commutator is used and which term in the expansion of γ_{L^m} is used. These are indexed from highest order to lowest order. The leading term, I_1, will give the terms in the estimate, and the rest will all be shown to be lower order error terms, $O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1)$.

$$I_1 = L^{2+n-\epsilon}(\Phi'_{a,\epsilon}(\boldsymbol{\xi}_n)[V_l, \boldsymbol{\xi}_n]\frac{\partial}{\partial r_*}g_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.142}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)g_{L^m}[V_l, \frac{\partial}{\partial r_*}]\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.143}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)g_{L^m}\frac{\partial}{\partial r_*}\Phi'_{a,\epsilon}(\boldsymbol{\xi}_n)[V_l, \boldsymbol{\xi}_n]) \tag{8.144}$$

$$I_2 = L^{2+n-\epsilon}\frac{1}{2}(- \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)'[V_l, \boldsymbol{\xi}_n]g'_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) + \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)g'_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)'[V_l, \boldsymbol{\xi}_n]) \tag{8.145}$$

$$I_3 = L^{2+n-\epsilon}(\Phi''_{a,\epsilon}(\boldsymbol{\xi}_n)[[V_l, \boldsymbol{\xi}_n], \boldsymbol{\xi}_n]\frac{\partial}{\partial r_*}g_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.146}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)g_{L^m}\frac{\partial}{\partial r_*}\Phi''_{a,\epsilon}(\boldsymbol{\xi}_n)[[V_l, \boldsymbol{\xi}_n], \boldsymbol{\xi}_n]) \tag{8.147}$$

$$I_4 = L^{2+n-\epsilon}\frac{1}{2}(- \Phi''_{a,\epsilon}(\boldsymbol{\xi}_n)[[V_l, \boldsymbol{\xi}_n], \boldsymbol{\xi}_n]g'_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.148}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)g'_{L^m}\Phi''_{a,\epsilon}(\boldsymbol{\xi}_n)[[V_l, \boldsymbol{\xi}_n], \boldsymbol{\xi}_n]) \tag{8.149}$$

$$I_5 = L^{2+n-\epsilon}(R_3\frac{\partial}{\partial r_*}g_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) + \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)g_{L^m}\frac{\partial}{\partial r_*}R_3) \tag{8.150}$$

$$I_6 = L^{2+n-\epsilon}\frac{1}{2}(- R_3g'_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) + \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)g_{L^m}\frac{\partial}{\partial r_*}R_3) \tag{8.151}$$

The leading order term, I_1, is composed of two types of terms. One type comes from the commutator $[H_3, \gamma_{L^m}]$, and the other comes from combining the first order term in the expansion of $[H_3, \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)]$ with the derivative terms in γ_{L^m}. To get the desired estimate, it is necessary to rearrange the phase space localising functions. This involves commuting the phase space localisation, which introduces additional lower order error terms. In the calculations, leading order terms are written first here, followed by those which will be shown to be lower order or error terms.

The term I_1 is rearranged so that all the leading order subterms contain the ρ_* localisation factor, $(-V_l'g_{L^m})$, which first appears in the commutator $[V_l, \gamma_{L^m}]$, and is the localisation in the statement of the lemma. The commutators from rearranging localisations are lower order by lemma 8.12.

$$I_1 = L^{2+n-\epsilon}(\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)'(-V_l')(-iL^{n-1})\frac{\partial}{\partial r_*}g_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.152}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)g_{L^m}[V_l, \frac{\partial}{\partial r_*}]\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.153}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)g_{L^m}\frac{\partial}{\partial r_*}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)'(-V_l')(-iL^{n-1})) \tag{8.154}$$

$$= L^{2+n-\epsilon}(\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)'\boldsymbol{\xi}_n(-V_l'g_{L^m})\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.155}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)(-V_l'g_{L^m})\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.156}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)(-V_l'g_{L^m})\boldsymbol{\xi}_n\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)') \tag{8.157}$$

$$+L^{2+n-\epsilon}(\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)'[-V_l', \boldsymbol{\xi}_n]g_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)+ \tag{8.158}$$

$$\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)g_{L^m}[\boldsymbol{\xi}_n\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)', -V_l']) \tag{8.159}$$

$$= L^{2+n-\epsilon}(\boldsymbol{\xi}_n\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)'(-V_l'g_{L^m})\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.160}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)(-V_l'g_{L^m})\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.161}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)(-V_l'g_{L^m})\boldsymbol{\xi}_n\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)') \tag{8.162}$$

$$+L^{1+2n-\epsilon}B \tag{8.163}$$

Following this, the $\boldsymbol{\xi}_n$ localising functions are gathered to the right of the ρ_* space localisation. Once they are gathered this way, their sum is the operator $\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)^2$. One factor of $\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)$ can be moved back through the ρ_* localisation, so that $\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)$ appears on both sides of the ρ_* localisation. Rearranging in this way introduces new commutators, which are not immediately estimated since they involve $(-V_l'g_{L^m})$. This is a localising function in ρ_* and ρ_*L^m so that the previous lemmas which estimate commutators can not be applied.

$$I_1 = L^{2+n-\epsilon}(-V_l'g_{L^m})\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)(\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) + 2\boldsymbol{\xi}_n\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)') \tag{8.164}$$

$$+L^{2+n-\epsilon}([\boldsymbol{\xi}_n\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)', -V_l'g_{L^m}]\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.165}$$

$$+ [\Phi_{a,\epsilon}(\boldsymbol{\xi}_n), -V_l'g_{L^m}]\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.166}$$

$$+ [\Phi_{a,\epsilon}(\boldsymbol{\xi}_n), -V_l'g_{L^m}]\boldsymbol{\xi}_n\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)') \tag{8.167}$$

$$+L^{1+2n-\epsilon}B \tag{8.168}$$

$$= L^{2+n-\epsilon}(\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)(-V_l'g_{L^m})\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)) \tag{8.169}$$

$$L^{2+n-\epsilon}([(-V_l'g_{L^m}), \Phi_{c,\epsilon}(\boldsymbol{\xi}_n)]\Phi_{c,\epsilon}(\boldsymbol{\xi}_n) \tag{8.170}$$

$$+ [\boldsymbol{\xi}_n\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)', -V_l'g_{L^m}]\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.171}$$

$$+ [\Phi_{a,\epsilon}(\boldsymbol{\xi}_n), -V_l'g_{L^m}]\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.172}$$

$$+ [\Phi_{a,\epsilon}(\boldsymbol{\xi}_n), -V_l'g_{L^m}]\boldsymbol{\xi}_n\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)') \tag{8.173}$$

$$+L^{1+2n-\epsilon}B \tag{8.174}$$

Of the terms in the previous expression it is clear that the first is the term desired in the statement of the theorem and that the expectation value of the last is in $O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1)$. The other terms all involve a commutator of the form $[F(\boldsymbol{\xi}_n), (-V_l'g_{L^m})]$ multiplied by $L^{2+n-\epsilon}$ and bounded operators.

By rewriting the commutator $[F(\boldsymbol{\xi}_n), (-V_l'g_{L^m})]$, the four remaining commutator terms can also be shown to be $O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1)$. It is necessary to separate $-V_l'g_{L^m}$ into a function of ρ_* and of ρ_*L^m to estimate the commutator. Naively applying the Leibniz rule for commutators, gives a contribution from the commutator with g_{L^m} which is $L^{1+m+2n-\epsilon}B$, which is not $O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1)$ in expectation value. The decomposition

$$-V_l'g_{L^m} = (-V_l'\rho_*^{-1})(\rho_*L^mg_{L^m})L^{-m} \tag{8.175}$$

ensures that the commutators are all $O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1)$ in expectation. This decomposition is possible since $-V_l'$ vanishes linearly at $\rho_* = 0$.

$$L^{2+n-\epsilon}[F(\boldsymbol{\xi}_n), (-V_l' g_{L^m})] = L^{2-m+n-\epsilon}[F(\boldsymbol{\xi}_n), (-V_l'\rho_*^{-1})L^m\rho_*] \tag{8.176}$$

The Leibniz rule can now be applied and the resulting commutators shown to be lower order operators by lemma 8.12. Note that the class of F considered here consists of $\Phi_{c,\epsilon}(x)$, $\Phi_{b,\epsilon}(x)$, and $\Phi_{a,\epsilon}(x)$, all of which are covered by lemma 8.12, and that $(V_l'\rho_*^{-1})$ and $L^m\rho_* g_{L^m} = \mathbf{x}_m \arctan(\mathbf{x}_m) = X_\uparrow(\mathbf{x}_m)^2$ both have bounded first and second derivative as required by lemma 8.12.

$$L^{2+n-\epsilon}[F(\boldsymbol{\xi}_n), (-V_l' g_{L^m})] = L^{2-m+n-\epsilon}[F(\boldsymbol{\xi}_n), (-V_l'\rho_*^{-1})]X_\uparrow(\mathbf{x}_m)^2 \tag{8.177}$$
$$+ L^{2-m+n-\epsilon}(-V_l'\rho_*^{-1})[F(\boldsymbol{\xi}_n), X_\uparrow(\mathbf{x}_m)^2] \tag{8.178}$$
$$= L^{2-m+n-\epsilon}L^{n-1}BL^m\rho_* + L^{2+n-\epsilon}L^{n+m-1}B \tag{8.179}$$
$$= L^{1+2n-\epsilon}B\rho_* + L^{1+2n-\epsilon}B \tag{8.180}$$

Since χ_α has compact support,

$$\|L^{\frac{1+2n-\epsilon}{2}}B\rho_*\chi_\alpha u\|^2 \leq C\|L^{\frac{1+2n-\epsilon}{2}}\chi_\alpha u\| \tag{8.181}$$
$$= O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) \tag{8.182}$$

This completes the estimate of the leading order term, I_1.

$$\langle u, \chi_\alpha I_1 \chi_\alpha u\rangle = \|L^{\frac{2+n-\epsilon}{2}}(-V_l'\rho_*)^{\frac{1}{2}}X_\uparrow(\mathbf{x}_m)\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\|^2 + O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) \tag{8.183}$$

It now remains to show that $I_2, \ldots, I_6$ are $O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1)$. Discussion of these continues from highest order to lowest. The first of these is I_2 which contains terms involving g_{L^m}' and $\boldsymbol{\xi}_n\Phi_{a,\epsilon}'(\boldsymbol{\xi}_n)$ terms. Note that because two different expansions of γ_{L^m} were used in equation 8.152, the signs on the two factors of g_{L^m}' are different. To simplify I_2, the factors of g_{L^m}' are expanded.

$$I_2 = L^{2+n-\epsilon}\frac{1}{2}\big(- \Phi_{a,\epsilon}'(\boldsymbol{\xi}_n)[V_l, \boldsymbol{\xi}_n]g_{L^m}'\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.184}$$
$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)g_{L^m}'\Phi_{a,\epsilon}'(\boldsymbol{\xi}_n)[V_l, \boldsymbol{\xi}_n]\big) \tag{8.185}$$
$$= L^{2+n-\epsilon}\frac{1}{2}\big(- \Phi_{a,\epsilon}'(\boldsymbol{\xi}_n)(-iL^{n-1})(-V_l')L^m X_\downarrow(\mathbf{x}_m)\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.186}$$
$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)X_\downarrow(\mathbf{x}_m)\Phi_{a,\epsilon}'(\boldsymbol{\xi}_n)(-iL^{n-1})(-V_l')\big) \tag{8.187}$$
$$= \frac{i}{2}L^{1+m+2n-\epsilon}\big(- \Phi_{a,\epsilon}'(\boldsymbol{\xi}_n)V_l'X_\downarrow(\mathbf{x}_m)\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.188}$$
$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)X_\downarrow(\mathbf{x}_m)\Phi_{a,\epsilon}'(\boldsymbol{\xi}_n)V_l'\big) \tag{8.189}$$

This leaves an expression for I_2 as the sum of two terms. The goal now is to rearrange these terms so that the highest order parts from each cancel. As usual, the rearrangement will introduce commutators. Naively attempting to commute $\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)$ or $\Phi_{a,\epsilon}'(\boldsymbol{\xi}_n)$ with g_{L^m} will leave a commutator of the form $L^{2m+3n-\epsilon}B$, which is not $O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1)$ in expectation value. Once again, an additional factor of $\rho_*\rho_*^{-1}$ is introduced and used to absorb a factor of L^m into a function of $\rho_* L^m$.

$$I_2 = \frac{i}{2}L^{1+2n-\epsilon}\big(- \Phi_{a,\epsilon}'(\boldsymbol{\xi}_n)\rho_* L^m X_\downarrow(\mathbf{x}_m)(V_l'\rho_*^{-1})\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.190}$$
$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)L^m X_\downarrow(\mathbf{x}_m)\Phi_{a,\epsilon}'(\boldsymbol{\xi}_n)\rho_*(V_l'\rho_*^{-1})\big) \tag{8.191}$$

Functions of ρ_* are now moved to obtain the same localising functions, $(V_l'\rho_*^{-1})$ and $\rho_* L^m X_\downarrow(\mathbf{x}_m) = \mathbf{x}_m X_\downarrow(\mathbf{x}_m)$, in both terms. This rearrangement introduces additional commutators which can be estimated by lemma 8.12.

$$I_2 = \frac{i}{2}L^{1+2n-\epsilon}\big(-\Phi'_{a,\epsilon}(\boldsymbol{\xi}_n)(\mathbf{x}_m X_\downarrow(\mathbf{x}_m))\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)(V_l'\rho_*^{-1}) \tag{8.192}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)(\mathbf{x}_m X_\downarrow(\mathbf{x}_m))\Phi'_{a,\epsilon}(\boldsymbol{\xi}_n)(V_l'\rho_*^{-1})\big) \tag{8.193}$$

$$+\frac{i}{2}L^{1+2n-\epsilon}\big(-\Phi'_{a,\epsilon}(\boldsymbol{\xi}_n)(\mathbf{x}_m X_\downarrow(\mathbf{x}_m))[(V_l'\rho_*^{-1}),\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)] \tag{8.194}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)X_\downarrow(\mathbf{x}_m)L^m[\Phi_{a,\epsilon}(\boldsymbol{\xi}_n),\rho_*](V_l'\rho_*^{-1})\big) \tag{8.195}$$

$$= \frac{i}{2}L^{1+2n-\epsilon}\big(-\Phi'_{a,\epsilon}(\boldsymbol{\xi}_n)(\mathbf{x}_m X_\downarrow(\mathbf{x}_m))\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)(V_l'\rho_*^{-1}) \tag{8.196}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)(\mathbf{x}_m X_\downarrow(\mathbf{x}_m))\Phi'_{a,\epsilon}(\boldsymbol{\xi}_n)(V_l'\rho_*^{-1})\big) \tag{8.197}$$

$$+L^{3n-\epsilon}B_1 + L^{m+3n-\epsilon}B_2 \tag{8.198}$$

The $\boldsymbol{\xi}_n$ localisation is now grouped to the left of all ρ_* and $\rho_* L^m$ localisation. The leading order terms are both $\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\Phi'_{a,\epsilon}(\boldsymbol{\xi}_n)(\mathbf{x}_m X_\downarrow(\mathbf{x}_m))(V_l'\rho_*^{-1})$, but with opposite signs, so they cancel. The remaining terms are commutators which are found to be $L^{m+3n-\epsilon}B$ in the standard way.

$$I_2 = \frac{i}{2}L^{1+2n-\epsilon}\big(-\Phi'_{a,\epsilon}(\boldsymbol{\xi}_n)[(\mathbf{x}_m X_\downarrow(\mathbf{x}_m)),\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)](V_l'\rho_*^{-1}) \tag{8.199}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)[(\mathbf{x}_m X_\downarrow(\mathbf{x}_m)),\Phi'_{a,\epsilon}(\boldsymbol{\xi}_n)](V_l'\rho_*^{-1})\big) \tag{8.200}$$

$$+L^{3n-\epsilon}B_1 + L^{m+3n-\epsilon}B_2 \tag{8.201}$$

$$= L^{m+3n-\epsilon}B_3 + L^{3n-\epsilon}B_1 \tag{8.202}$$

Because the powers of L involved are all less than $1+2n$, the desired estimate on I_2 holds.

$$\langle\chi_\alpha u, I_2\chi_\alpha u\rangle = O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) \tag{8.203}$$

Next I_3 is considered. This involves a second commutator $[[V_l,\boldsymbol{\xi}_n],\boldsymbol{\xi}_n]$ which can be explicitly computed as $(V_l'')(-L^{2n-2})$.

$$I_3 = L^{2+n-\epsilon}\big(\Phi''_{a,\epsilon}(\boldsymbol{\xi}_n)[[V_l,\boldsymbol{\xi}_n],\boldsymbol{\xi}_n]\frac{\partial}{\partial r_*}g_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.204}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)g_{L^m}\frac{\partial}{\partial r_*}\Phi''_{a,\epsilon}(\boldsymbol{\xi}_n)[[V_l,\boldsymbol{\xi}_n],\boldsymbol{\xi}_n]\big) \tag{8.205}$$

$$= L^{2+n-\epsilon}\big(\Phi''_{a,\epsilon}(\boldsymbol{\xi}_n)(V_l'')(-L^{2n-2})\frac{\partial}{\partial r_*}g_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.206}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)g_{L^m}\frac{\partial}{\partial r_*}\Phi''_{a,\epsilon}(\boldsymbol{\xi}_n)(V_l'')(-L^{2n-2})\big) \tag{8.207}$$

To further simplify this, it is rewritten in terms of $\boldsymbol{\xi}_n$ instead of $\frac{\partial}{\partial r_*}$.

$$I_3 = iL^{1+2n-\epsilon}\big(\Phi''_{a,\epsilon}(\boldsymbol{\xi}_n)(V_l'')\boldsymbol{\xi}_n g_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.208}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)g_{L^m}\boldsymbol{\xi}_n\Phi''_{a,\epsilon}(\boldsymbol{\xi}_n)(V_l'')\big) \tag{8.209}$$

This is rearranged to group $\boldsymbol{\xi}_n$ and $\Phi''_{a,\epsilon}(\boldsymbol{\xi}_n)$ together. Since $x\Phi''_{a,\epsilon}(x)$ is smooth and decays like $x^{\frac{-5+\epsilon}{4}}$ as $x\to\infty$, $x\Phi''_{a,\epsilon}(x)$ is a bounded function and $\boldsymbol{\xi}_n\Phi''_{a,\epsilon}(\boldsymbol{\xi}_n)$ is a bounded operator. The rearrangement introduces commutators which are estimated in the standard way by lemma 8.12.

$$I_3 = L^{1+2n-\epsilon}\big((-i\boldsymbol{\xi}_n\Phi''_{a,\epsilon}(\boldsymbol{\xi}_n))V_l''g_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) \tag{8.210}$$

$$+ \Phi_{a,\epsilon}(\boldsymbol{\xi}_n)g_{L^m}(-i\boldsymbol{\xi}_n\Phi''_{a,\epsilon}(\boldsymbol{\xi}_n))V_l''\big) \tag{8.211}$$

$$+L^{1+2n-\epsilon}\big(\Phi''_{a,\epsilon}(\boldsymbol{\xi}_n)[V_l'',-i\boldsymbol{\xi}_n]g_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\big) \tag{8.212}$$

$$= L^{1+2n-\epsilon}B_1 + L^{3n-\epsilon}B_2 \tag{8.213}$$

Since the powers of L involved are bounded by $1+2n$, the expectation value of I_3 with respect to $\chi_\alpha u$ is $O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1)$.

To estimate I_4, the commutator $[[V_l,\boldsymbol{\xi}_n],\boldsymbol{\xi}_n]$ and the derivative g'_{L^m} are explicitly expanded to show I_4 is a product of bounded operators and powers of L.

$$\begin{aligned}
&L^{2+n-\epsilon}\frac{1}{2}\big(-\Phi''_{a,\epsilon}(\boldsymbol{\xi}_n)[[V_l,\boldsymbol{\xi}_n],\boldsymbol{\xi}_n]g'_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) && (8.214)\\
&\qquad+\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)g'_{L^m}\Phi''_{a,\epsilon}(\boldsymbol{\xi}_n)[[V_l,\boldsymbol{\xi}_n],\boldsymbol{\xi}_n]\big) && (8.215)\\
&=\frac{1}{2}L^{2+n-\epsilon}\big(-\Phi''_{a,\epsilon}(\boldsymbol{\xi}_n)(V'_l)(-L^{2n-2})L^mX_\downarrow(\mathbf{x}_m)\Phi_{a,\epsilon}(\boldsymbol{\xi}_n) && (8.216)\\
&\qquad+\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)L^mX_\downarrow(\mathbf{x}_m)\Phi''_{a,\epsilon}(\boldsymbol{\xi}_n)(V'_l)(-L^{2n-2})\big) && (8.217)\\
&=L^{m+3n-\epsilon}B && (8.218)
\end{aligned}$$

Since $m+3n-\epsilon<1+2n$, the expectation value of I_4 with respect to $\chi_\alpha u$ is $O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1)$.

Finally the terms I_5 and I_6 involving $R_3=L^{3n-3}B$ are considered. These are each the sum of two inner products, but by the symmetry and anti symmetry properties of the operators involved, it is sufficient to consider just one of the inner products for each. The expectation value of I_5 is considered, and the operators χ_α, $\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)$ and $\frac{\partial}{\partial r_*}$ are moved to the left side of the inner product.

$$\langle u,\chi_\alpha\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)L^{2+n-\epsilon}\frac{\partial}{\partial r_*}g_{L^m}R_3\chi_\alpha u\rangle=\langle-\frac{\partial}{\partial r_*}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u, L^{2+n-\epsilon}g_{L^m}R_3\chi_\alpha u\rangle \quad (8.219)$$

The Cauchy-Schwartz inequality is applied and the resulting norms are found to be of a bounded operator acting on $\frac{\partial}{\partial r_*}\chi_\alpha u$ and on $L^{4n-1-\epsilon}\chi_\alpha u$. By corollary 7.9, $\|\frac{\partial}{\partial r_*}\chi_\alpha u\|^2$ is time integrable and hence $O(L^1)$ and $O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1)$. Since $4n-1-\epsilon<\frac{1+2n}{2}$, $\|L^{4n-1-\epsilon}\chi_\alpha u\|^2$ is $O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1)$.

$$\begin{aligned}
\langle u,\chi_\alpha\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)L^{2+n-\epsilon}\frac{\partial}{\partial r_*}g_{L^m}R_3\chi_\alpha u\rangle &\geq-\big(\|\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\frac{\partial}{\partial r_*}\chi_\alpha u\|\|L^{2+n-\epsilon}L^{3n-3}B\chi_\alpha u\|\big) && (8.220)\\
&\geq-\big(\|\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\frac{\partial}{\partial r_*}\chi_\alpha u\|^2+\|L^{2+n-\epsilon}L^{3n-3}B\chi_\alpha u\|^2\big) && (8.221)\\
&\geq-\big(\|\frac{\partial}{\partial r_*}\chi_\alpha u\|^2+\|L^{4n-1-\epsilon}\chi_\alpha u\|^2\big) && (8.222)\\
&\geq-O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) && (8.223)
\end{aligned}$$

The term I_6 is estimated in expectation value by expanding g'_{L^m} and finding the commutator is a bounded operator times $L^{4n+m-1-\epsilon}$. Since $4n+m-1-\epsilon<1+2n$, this expectation value in $O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1)$.

$$\begin{aligned}
\langle u,\chi_\alpha L^{2+n-\epsilon}g'_{L^m}R_3\chi_\alpha u\rangle&=\langle u,\chi_\alpha L^{2+n-\epsilon}X_\downarrow(\mathbf{x}_m)R_3\chi_\alpha u\rangle && (8.224)\\
&=\langle u,\chi_\alpha L^{2+m+n-\epsilon}L^{3n-3}B\chi_\alpha u\rangle && (8.225)\\
&=\langle u,\chi_\alpha L^{4n+m-1-\epsilon}B\chi_\alpha u\rangle && (8.226)\\
&=O(L^1) && (8.227)
\end{aligned}$$

This term is $O(L^1)$ because $4n+m-1-\epsilon\leq\frac{3}{2}$. Since $\langle u,\chi_\alpha R_3L^{2+n-\epsilon}\gamma_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\rangle$ is the complex conjugate of $\langle u,\chi_\alpha\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)L^{2+n-\epsilon}\gamma_{L^m}R_3\chi_\alpha u\rangle$ it is also L^1 in time.

The expectation value of all the terms computed is

$$\begin{aligned}
\langle u,[H_3,\Gamma_{n,m}]u\rangle&=\langle u,L^{2+n-m-\epsilon}\chi_\alpha\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)(-V'_l\rho_*^{-1})X_\uparrow(\mathbf{x}_m)^2\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\rangle-O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) && (8.228)\\
&=\langle u,L^{2+n-m-\epsilon}\chi_\alpha\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)X_\uparrow(\mathbf{x}_m)(-V'_l\rho_*^{-1})X_\uparrow(\mathbf{x}_m)\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\rangle-O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) && (8.229)\\
&=\|L^{\frac{2+n-m-\epsilon}{2}}(-V'_l\rho_*^{-1})^{\frac{1}{2}}X_\uparrow(\mathbf{x}_m)\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\|^2-O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1). && (8.230)
\end{aligned}$$

□

8.4 Derivative Bounds

In this subsection, we remove the localisation on the $\mathbf{x}_m$ phase space variable. The resulting estimates are localised in $\boldsymbol{\xi}_n$ only. Our estimates will be in the regions corresponding to $\Phi_{|\bullet|\le 1}$ and $\Psi_{L^{-\delta}\le|\bullet|\le 1}$. The first lemma shows that $\Gamma_{n,\frac{1}{2}}$ majorates $L^{\frac{\frac{3}{2}+m-\epsilon}{2}}$, and the second, that $\Gamma_{n,n}$ majorates $L^{1-\delta-\frac{\epsilon}{2}}$.

We eliminate the $\mathbf{x}_n$ dependence by combining lemmas 8.13 and 8.15, and noting $X_\downarrow(x)+X_\uparrow(x)>C$. Since all the Φ type localisations have the bulk of their support in the same region, we can replace one of the Φ -type localisations by another.

Lemma 8.16. *For $u\in\mathbb{S}$ and $n\in[0,\frac{1}{2}]$, there is a constant C such that*

$$\langle u,[H,\Gamma_{n,\frac{1}{2}}]u\rangle \ge C\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}\Phi_{|\boldsymbol{\xi}_n|\le 1}\chi_\alpha u\|^2 - O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) \tag{8.231}$$

Proof. Since commutators are linear in each term, $[H,\Gamma_{n,\frac{1}{2}}]$ is given by the sum of the three commutators $[H_i,\Gamma_{n,m}]$. These were computed in lemmas 8.13, 8.14, and 8.15. Ignoring $O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1)$ terms, the commutator with H_1 gave two terms, the commutator with H_2 gave no terms, and the commutator with H_3 gave one term.

$$\begin{aligned}
\langle u,[H,\Gamma_{n,m}]u\rangle =&\langle u,[H_1,\Gamma_{n,m}]u\rangle + \langle u,[H_2,\Gamma_{n,m}]u\rangle + \langle u,[H_3,\Gamma_{n,m}]u\rangle && (8.232)\\
=&C_1\|L^{\frac{m+n-\epsilon}{2}}X_\downarrow(\mathbf{x}_m)^{\frac{1}{2}}\frac{\partial}{\partial r_*}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\|^2 && (8.233)\\
&+C_2\|L^{\frac{3m+n-\epsilon}{2}}X_\downarrow(\mathbf{x}_m)\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\|^2 && (8.234)\\
&+\|L^{\frac{2+n-m-\epsilon}{2}}(-V_l'\rho_*^{-1})^{\frac{1}{2}}X_\uparrow(\mathbf{x}_m)\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\|^2 - O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) && (8.235)
\end{aligned}$$

The first two terms came from H_1 and are localised in the region $\mathbf{x}_m\le 1$. The last came from H_3 and is localised in the region $\mathbf{x}_m\ge 1$.

Here the specialisation $m=\frac{1}{2}$ is used, and only the second term from H_1, which includes $L^{\frac{\frac{3}{2}+n-\epsilon}{2}}$, and the term from H_3 are used. These two terms are rearranged so that they include the same derivative localisation and can be combined.

The term from H_3 is

$$\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}(-V_l'\rho_*^{-1})^{\frac{1}{2}}X_\downarrow(\mathbf{x}_m)\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.236}$$

The second term from H_1 is rearranged with the goal of introducing localisation in $\boldsymbol{\xi}_n$ and ρ_* which is the same as the localisation appearing in the term coming from H_3. The $\mathbf{x}_m$ localisation will be complementary. First $\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)$ is brought to the left of the phase space localisations.

$$\begin{aligned}
\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}X_\downarrow(\mathbf{x}_m)\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \ge&\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)X_\downarrow(\mathbf{x}_m)\chi_\alpha u\| && (8.237)\\
&-\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}[X_\downarrow(\mathbf{x}_m),\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)]\chi_\alpha u\| && (8.238)
\end{aligned}$$

Since $\Phi_{a,\epsilon}(x)\ge C\Phi_{c,\epsilon}(x)$, $\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)$ can be replaced by $\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)$ while preserving the inequality, and $\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)$ can be commuted back through the $\mathbf{x}_m$ localisation.

$$\begin{aligned}
\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}X_\downarrow(\mathbf{x}_m)\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \ge&C\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)X_\downarrow(\mathbf{x}_m)\chi_\alpha u\| && (8.239)\\
&-\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}[X_\downarrow(\mathbf{x}_m),\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)]\chi_\alpha u\| && (8.240)\\
\ge&C\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}X_\downarrow(\mathbf{x}_m)\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| && (8.241)\\
&-\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}[\Phi_{c,\epsilon}(\boldsymbol{\xi}_n),X_\downarrow(\mathbf{x}_m)]\chi_\alpha u\| && (8.242)\\
&-\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}[X_\downarrow(\mathbf{x}_m),\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)]\chi_\alpha u\| && (8.243)
\end{aligned}$$

The two commutators are shown to be lower order by lemma 8.12, and since the resulting error terms include power of L^{1+2n} in expectation value, the error terms are $O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1)$. Since $V_l\rho_*^{-1}$ is a bounded function, an additional factor of this localisation can be introduced freely.

$$\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}X_\downarrow(\mathbf{x}_m)\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.244}$$

$$\geq C\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}X_\downarrow(\mathbf{x}_m)\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| - \|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}L^{n-\frac{1}{2}}B\chi_\alpha u\| - \|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}L^{n-\frac{1}{2}}B\chi_\alpha u\| \tag{8.245}$$

$$\geq C\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}(-V_l'\rho_*^{-1})^{\frac{1}{2}}X_\downarrow(\mathbf{x}_m)\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| - O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) \tag{8.246}$$

Since $X_\downarrow(\mathbf{x}_m)$ and $(-V_l'\rho_*^{-1})$ commute, equations (8.246) and (8.236) can be combined.

$$\langle u, [H, \Gamma_{n,\frac{1}{2}}]u\rangle \geq C(\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}X_\downarrow(\mathbf{x}_m)(-V_l'\rho_*^{-1})^{\frac{1}{2}}\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\|^2 \tag{8.247}$$

$$+ \|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}X_\uparrow(\mathbf{x}_m)(-V_l'\rho_*^{-1})^{\frac{1}{2}}\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\|^2) \tag{8.248}$$

$$- O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) \tag{8.249}$$

Both of these terms involve localisation in $\mathbf{x}_{\frac{1}{2}}$ acting on $(-V_l'\rho_*^{-1})^{\frac{1}{2}}\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u$. The two localisations satisfy the relationship

$$X_\downarrow(x)^2 + X_\uparrow(x)^2 = (\frac{1}{1+(\frac{x}{2M})^2})^2 + (x\sqrt{\frac{\arctan(x)}{x}})^2 \geq C, \tag{8.250}$$

so that they can be combined to give a better estimate.

$$\langle u, [H, \Gamma_{n,\frac{1}{2}}]u\rangle \geq C\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}(-V_l'\rho_*^{-1})^{\frac{1}{2}}\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\|^2 + O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) \tag{8.251}$$

To eliminate the factor of $(-V_l'\rho_*^{-1})^{\frac{1}{2}}$, a new function is introduced. This function, denoted f, is a smooth, compactly supported function and equal to the inverse of $(-V_l'\rho_*^{-1})^{\frac{1}{2}}$ on $\text{supp}\chi_\alpha$. Since it is a bounded function, f can be introduced into the norm on the right hand side and then commuted through $\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)$ with $(-V_l'\rho_*^{-1})^{\frac{1}{2}}$ at the cost of an additional commutator.

$$\langle u, [H, \Gamma_{n,\frac{1}{2}}]u\rangle \geq C\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}f(-V_l'\rho_*^{-1})^{\frac{1}{2}}\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\|^2 - O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) \tag{8.252}$$

$$\geq C\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)f(-V_l'\rho_*^{-1})^{\frac{1}{2}}\chi_\alpha u\|^2 - \|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}[f(-V_l'\rho_*^{-1})^{\frac{1}{2}}, \Phi_{c,\epsilon}(\boldsymbol{\xi}_n)]\chi_\alpha u\|^2 \tag{8.253}$$

$$- O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) \tag{8.254}$$

Since f is equal to the inverse of $(-V_l'\rho_*^{-1})^{\frac{1}{2}}$ on the support of χ_α, $f(-V_l'\rho_*^{-1})^{\frac{1}{2}}\chi_\alpha = \chi_\alpha$. The commutators can be shown to be lower order by lemma 8.12, and since $-\frac{1}{2}+3n-\frac{\epsilon}{2} < 1+2n$ the error terms are $O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1)$.

$$\langle u, [H, \Gamma_{n,\frac{1}{2}}]u\rangle \geq C\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)f(-V_l'\rho_*^{-1})^{\frac{1}{2}}\chi_\alpha u\|^2 - \|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}L^{n-1}B\chi_\alpha u\|^2 \tag{8.255}$$

$$- O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) \tag{8.256}$$

$$\geq C\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\|^2 - O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) \tag{8.257}$$

Since $\Phi_{c,\epsilon}(x) \geq \Phi_{|x|\leq 1}$, $\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)$ can be replaced by $\Phi_{|\boldsymbol{\xi}_n|\leq 1}$ to complete the proof.

$$\langle u, [H, \Gamma_{n,\frac{1}{2}}]u\rangle \geq C\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}\Phi_{|\boldsymbol{\xi}_n|\leq 1}\chi_\alpha u\|^2 - O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) \tag{8.258}$$

□

Our next goal is a result with localisation $\Psi_{L^{-\delta}\leq|(|\leq 1}\boldsymbol{\xi}_n)$. We begin by introducing new Ψ type localisations, which are used as smooth approximations to $\Psi_{L^{-\delta}\leq|\bullet|\leq 1}$.

Definition 8.17. *The function $\Psi_1 : [0,\infty) \to [0,1]$ is defined to be a smooth C^∞ function which has support on $[\frac{1}{2},\infty)$ and is identically one on $[1,\infty)$. The function $\Psi_2 : [0,\infty) \to [0,1]$ is defined to be a Schwartz class function which has support on $[0,2]$ and is identically 1 on $[0,1]$. These are extended as even functions. The extensions are also Schwartz class, since the original functions are constant in a neighbourhood of zero. The operator $\Psi(L^{-\delta},\boldsymbol{\xi}_n)$ is defined by*

$$\Psi(L^{-\delta},\boldsymbol{\xi}_n) = \Psi_1(L^{\delta}\boldsymbol{\xi}_n)\Psi_2(\boldsymbol{\xi}_n) \tag{8.259}$$

The following lemma allows us to replace one Φ type localisation with Ψ type localisation and to move this replacement through $\mathbf{x}_m$ localisation.

Lemma 8.18. *Given δ and ϵ positive, there is a constant C_1 such that for all $v \in \mathbb{S}$ and for F_2 either $\Phi_{a,\epsilon}$ or $\Phi_{c,\epsilon}$,*

$$\|\Psi_{L^{-\delta}\leq|\boldsymbol{\xi}_n|\leq 1}v\| \leq \|\Psi(L^{-\delta},\boldsymbol{\xi}_n)v\| \tag{8.260}$$

$$C_1\|L^{-\delta}(\Psi(L^{-\delta},\bullet)F_2^{-1})(\boldsymbol{\xi}_n)v\| \leq \|\boldsymbol{\xi}_n v\| \tag{8.261}$$

There is a constant C_2, such that for any differentiable function F_1 and for F_2 either $\Phi_{a,\epsilon}$ or $\Phi_{c,\epsilon}$,

$$\|L^{1-2n-\delta}[F_1(\mathbf{x}_n),(\Psi(L^{-\delta},\bullet)F_2^{-1})(\boldsymbol{\xi}_n)]\| \leq C_2\|F'\|_\infty \tag{8.262}$$

Proof. Since L, $\frac{\partial}{\partial r_*}$, and $\boldsymbol{\xi}_n$ are all commuting operators, any functions of these operators, defined by the spectral theorem, also commute. The spherical harmonic decomposition is into orthogonal subspaces preserved by these operators. Therefore, it is sufficient to prove the first two results for functions with a single spherical harmonic component and to prove the third result by considering the operator on the right as an operator on a single spherical harmonic.

For a fixed value of l,

$$\Psi(l^{-\delta},x) = \Psi_1(l^{\delta}|x|)\Psi_2(|x|) \tag{8.263}$$

$$\geq \chi([1,\infty),l^{\delta}|x|)\chi([0,1],x) \tag{8.264}$$

$$\geq \chi([l^{-\delta},\infty),|x|)\chi([0,1],x) \tag{8.265}$$

$$\geq \chi([l^{-\delta},1],|x|) \tag{8.266}$$

$$\geq \Psi_{l^{-\delta}\leq|x|\leq 1} \tag{8.267}$$

Therefore, by the spectral theorem, for any function $v \in \mathbb{S}$, on each spherical harmonic

$$\|\Psi_{L^{-\delta}\leq|\boldsymbol{\xi}_n|\leq 1}v\| \leq \|\Psi(L^{-\delta},\boldsymbol{\xi}_n)v\| \tag{8.268}$$

Now the case when F_2 is either $\Phi_{a,\epsilon}$ or $\Phi_{c,\epsilon}$ is considered. Both $\Phi_{a,\epsilon}(x)$ and $\Phi_{c,\epsilon}(x)$ are smooth, strictly positive functions, so in either case $F_2(x)$ has a bounded inverse for $|x| \in \mathrm{supp}(Z_2) \subset [0,2]$, and $(\Psi_2 F_2^{-1})(\boldsymbol{\xi}_n)$ is a bounded operator. Since all the operators involved commute and $\Psi_1(l^{\delta}|x|)$ is bounded, $(\Psi(L^{-\delta},\bullet)F_2^{-1})(\boldsymbol{\xi}_n)$ is a well defined, bounded operator. Again, for a fixed l,

$$|x| \geq \frac{1}{2}l^{-\delta}\chi([\frac{1}{2}l^{-\delta},\infty),|x|) \tag{8.269}$$

$$\geq \frac{1}{2}l^{-\delta}\chi([\frac{1}{2},\infty),l^{\delta}|x|) \tag{8.270}$$

$$\geq \frac{1}{2}l^{-\delta}\Psi_1(l^{\delta}|x|) \tag{8.271}$$

$$\geq Cl^{-\delta}\Psi_1(l^{\delta}|x|)(\Psi_2 F_2^{-1})(x) \tag{8.272}$$

$$\geq Cl^{-\delta}(\Psi(L^{-\delta},\bullet)F_2^{-1})(x) \tag{8.273}$$

On each spherical harmonic, by the spectral theorem,

$$\|\boldsymbol{\xi}_n v\| = \|(|\boldsymbol{\xi}_n|)v\| \geq C\|L^{-\delta}(\Psi(L^{-\delta},\bullet)F_2^{-1})(\boldsymbol{\xi}_n)v\| \tag{8.274}$$

Finally the commutator is calculated. Again this is proven on a single spherical harmonic shell.

$$\begin{aligned}
(\Psi(L^{-\delta},\bullet)F_2^{-1})(\boldsymbol{\xi}_n) =&\Psi_1(L^{\delta}\boldsymbol{\xi}_n)(\Psi_2F_2^{-1})(\boldsymbol{\xi}_n) &(8.275)\\
L^{1-2n-\delta}[F_1(\mathbf{x}_n),(\Psi(L^{-\delta},\bullet)F_2^{-1})(\boldsymbol{\xi}_n)] =&L^{1-2n-\delta}[F_1(\mathbf{x}_n),\Psi_1(L^{\delta}\boldsymbol{\xi}_n)\Psi_2(\boldsymbol{\xi}_n)] &(8.276)\\
=&L^{1-2n-\delta}[F_1(\mathbf{x}_n),\Psi_1(L^{\delta}\boldsymbol{\xi}_n)]\Psi_2(\boldsymbol{\xi}_n) &(8.277)\\
&+L^{1-2n-\delta}\Psi_1(L^{\delta}\mathbf{x}_n)[F_1(\boldsymbol{\xi}_n),(\Psi_2F_2^{-1})(\boldsymbol{\xi}_n)] &(8.278)
\end{aligned}$$

Each of these terms is now estimated. Since only one spherical harmonic is being considered, the operator L can be replaced with the constant l. This simplifies the discussion of the commutator involving $\Psi_1(L^{\delta}\boldsymbol{\xi}_n)$.

Since the Ψ_1 is compactly supported and C^{∞}, it follows that Ψ_2' is Schwartz class and that $\|\mathcal{F}[\Psi_2']\|_1 \leq \infty$.

$$\begin{aligned}
\|L^{1-2n-\delta}[F_1(\mathbf{x}_n),\Psi_1(l^{\delta}\boldsymbol{\xi}_n)](\Psi_2F_2^{-1})(\boldsymbol{\xi}_n)\| \leq&Cl^{-\delta}\|l^{1-2n}[F_1(\mathbf{x}_n),\Psi_1(l^{\delta}\boldsymbol{\xi}_n)]\| &(8.279)\\
\leq&Cl^{-\delta}\|F_1'\|_{\infty}\|\mathcal{F}[(\Psi_1(l^{\delta}|\bullet|))']\|_1 &(8.280)
\end{aligned}$$

At this point, the derivative of the function in a scaled variable, $(\Psi_1(l^{\delta}\bullet))'$, is evaluated to be a scaled version of the derivative evaluated at the scaled variable, $l^{\delta}\Psi_1'(l^{\delta}\bullet)$. To evaluate the Fourier transform of this, it is noted that $\|\mathcal{F}[f(\lambda\bullet)]\|_1 = \|\mathcal{F}[f(\bullet)]\|_1$.

$$\begin{aligned}
\|L^{1-2n-\delta}[F_1(\mathbf{x}_n),\Psi_1(L^{\delta}\boldsymbol{\xi}_n)](\Psi_2F_2^{-1})(\boldsymbol{\xi}_n)\| \leq&Cl^{-\delta}\|F_1'\|_{\infty}\|l^{\delta}\mathcal{F}[\Psi_1'(l^{\delta}|\bullet|)]\|_1 &(8.281)\\
\leq&C\|F_1'\|_{\infty}\|\mathcal{F}[\Psi_1']\|_1 &(8.282)\\
\leq&C\|F_1'\|_{\infty} &(8.283)
\end{aligned}$$

This completes the estimate on the first of the commutator terms.

Since Ψ_2 is Schwartz class, and F_2 is smooth and has bounded inverse on $\mathrm{supp}(\Psi_2)$, $(\Psi_2F_2^{-1})$ is Schwartz class, and $\|\mathcal{F}[(\Psi_2F_2^{-1})']\|_1 \leq \infty$.

$$\begin{aligned}
\|L^{1-2n-\delta}\Psi_1(L^{\delta}\boldsymbol{\xi}_n)[F_1(\mathbf{x}_n),(\Psi_2F_2^{-1})(\boldsymbol{\xi}_n))\| \leq&C\|L^{1-2n-\delta}[F_1(\mathbf{x}_n),(\Psi_2F_2^{-1})(\boldsymbol{\xi}_n)]\| &(8.284)\\
\leq&C\|L^{1-2n}[F_1(\mathbf{x}_n),(\Psi_2F_2^{-1})(\boldsymbol{\xi}_n)\| &(8.285)\\
\leq&C\|F_1'\|_{\infty}\|\mathcal{F}[(\Psi_2F_2^{-1})']\|_1 &(8.286)\\
\leq&C\|F_1'\|_{\infty} &(8.287)
\end{aligned}$$

This completes the estimate on the second of the commutator terms. □

We now prove a $\Psi_{L^{-\delta}\leq|\bullet|\leq 1}$ localised estimate. This is analogous to the $\Phi_{|\bullet|\leq 1}$ localised estimate in lemma 8.16; although, the proof is more complicated because the Ψ type localisations are more complicated to work with. Because $\boldsymbol{\xi}_n$ is bounded below on the support of $\Psi_{L^{-\delta}\leq|\boldsymbol{\xi}_n|\leq 1}$, we can dominate more powers of L with this localisation.

Lemma 8.19. *For $u \in \mathbb{S}$ and $n \in [0, \frac{1}{2}]$,*

$$\langle u, [H, \Gamma_{n,n}]u\rangle \geq \|L^{1-\frac{\epsilon}{2}-\delta}\Psi_{L^{-\delta}\leq|\boldsymbol{\xi}_n|\leq 1}\chi_\alpha u\|^2 - O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) \tag{8.288}$$

Proof. Once again, lemmas 8.13, 8.14, and 8.15 are used to estimate $\langle u, [H, \Gamma_{n,m}]u\rangle$. In lemma 8.16, the combination of these three lemmas was given, and this is repeated here.

$$\begin{aligned}
\langle u, [H, \Gamma_{n,m}]u\rangle \geq&C_1\|L^{\frac{m+n-\epsilon}{2}}X_{\downarrow}(\mathbf{x}_m)^{\frac{1}{2}}\frac{\partial}{\partial r_*}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\|^2 &(8.289)\\
&+C_2\|L^{\frac{3m+n-\epsilon}{2}}X_{\downarrow}(\mathbf{x}_m)\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\|^2 &(8.290)\\
&+\|L^{\frac{2+n-m-\epsilon}{2}}(-V_l\rho_*^{-1})^{\frac{1}{2}}X_{\uparrow}(\mathbf{x}_m)\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\|^2 + O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) &(8.291)
\end{aligned}$$

For this lemma, $m = n$ will be used. The terms that will be used are the first term from H_1 which involves $\frac{\partial}{\partial r_*}$ and its adjoint, and the term from H_3.

The terms from H_1 are estimated first. The first step is weakening the estimate to eliminate the square root.

$$\|L^{\frac{2n-\epsilon}{2}}X_\downarrow(\mathbf{x}_n)^{\frac{1}{2}}\frac{\partial}{\partial r_*}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \geq \|L^{n-\frac{\epsilon}{2}}X_\downarrow(\mathbf{x}_n)\frac{\partial}{\partial r_*}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.292}$$

Now $\frac{\partial}{\partial r_*}$ is treated as if it were a localisation in $\frac{\partial}{\partial r_*}$ and is commuted to the right of the chain of operators.

$$\begin{aligned}\|L^{\frac{2n-\epsilon}{2}}X_\downarrow(\mathbf{x}_n)\frac{\partial}{\partial r_*}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \geq& \|L^{n-\frac{\epsilon}{2}}\frac{\partial}{\partial r_*}X_\downarrow(\mathbf{x}_n)\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| &(8.293)\\ &- \|L^{n-\frac{\epsilon}{2}}[X_\downarrow(\mathbf{x}_n),\frac{\partial}{\partial r_*}]\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| &(8.294)\end{aligned}$$

The commutator from moving $\frac{\partial}{\partial r_*}$ through $X_\downarrow(\mathbf{x}_n)$ can be computed explicitly. It involves the function $X_\downarrow'(x) = -2xX_\downarrow(x)^2$, which is bounded.

$$\begin{aligned}[X_\downarrow(\mathbf{x}_n),\frac{\partial}{\partial r_*}] =& \frac{\partial}{\partial r_*}\left(1+\left(\frac{L^n\rho_*}{2M}\right)^2\right)^{-1} &(8.295)\\ =& -2L^n\mathbf{x}_nX_\downarrow(\mathbf{x}_n)^2 &(8.296)\\ =& L^nB. &(8.297)\end{aligned}$$

The "localisation" $L^n\frac{\partial}{\partial r_*}$ can be replaced by $L^{1-\delta}(\Psi(L^{-\delta},\bullet)\Phi_{a,\epsilon}^{-1})(\boldsymbol{\xi}_n)$ using lemma 8.18.

$$\begin{aligned}\|L^{\frac{2n-\epsilon}{2}}X_\downarrow(\mathbf{x}_n)\frac{\partial}{\partial r_*}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \geq& C\|L^{1-\frac{\epsilon}{2}-\delta}(\Psi(L^{-\delta},\bullet)\Phi_{a,\epsilon}^{-1})(\boldsymbol{\xi}_n)X_\downarrow(\mathbf{x}_n)\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| &(8.298)\\ &- \|L^{2n-\frac{\epsilon}{2}}B\chi_\alpha u\| &(8.299)\end{aligned}$$

The $(\Psi(L^{-\delta},\bullet)\Phi_{a,\epsilon}^{-1})(\boldsymbol{\xi}_n)$ is commuted back through the $\mathbf{x}_n$ localisation. The commutator involving $(\Psi(L^{-\delta},\bullet)\Phi_{a,\epsilon}^{-1})(\boldsymbol{\xi}_n)$ is shown to be lower order by lemma 8.18. Lemma 8.12 can not be directly applied to commutators involving $(\Psi(L^{-\delta},\bullet)\Phi_{a,\epsilon}^{-1})(\boldsymbol{\xi}_n)$, because $\Psi(L^{-\delta},\boldsymbol{\xi}_n) = \Psi_1(L^\delta\boldsymbol{\xi}_n)\Psi_2(\boldsymbol{\xi}_n)$ is a product of two operators with different localisations.

$$\begin{aligned}&\|L^{\frac{2n-\epsilon}{2}}X_\downarrow(\mathbf{x}_n)\frac{\partial}{\partial r_*}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| &(8.300)\\ \geq& C\|L^{1-\frac{\epsilon}{2}-\delta}X_\downarrow(\mathbf{x}_n)\Psi(L^{-\delta},\boldsymbol{\xi}_n)\chi_\alpha u\| &(8.301)\\ &- \|L^{1-\frac{\epsilon}{2}-\delta}[(\Psi(L^{-\delta},\bullet)\Phi_{a,\epsilon}^{-1})(\boldsymbol{\xi}_n),X_\downarrow(\mathbf{x}_n)]\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| - \|L^{2n-\frac{\epsilon}{2}}B\chi_\alpha u\| &(8.302)\\ \geq& C\|L^{1-\frac{\epsilon}{2}-\delta}X_\downarrow(\mathbf{x}_n)\Psi(L^{-\delta},\boldsymbol{\xi}_n)\chi_\alpha u\| - C\|L^{1-\frac{\epsilon}{2}-\delta}L^{2n-1+\delta}B\chi_\alpha u\| - \|L^{2n-\frac{\epsilon}{2}}B\chi_\alpha u\| &(8.303)\end{aligned}$$

Since $2n-\frac{\epsilon}{2}$ is the power of L appearing in both error terms, and this exponent is less than $\frac{1+2n}{2}$, the error terms are $O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1)$. As in lemma 8.16, an additional localisation by $(-V_l'\rho_*^{-1})$ is introduced.

$$\begin{aligned}&\|L^{\frac{2n-\epsilon}{2}}X_\downarrow(\mathbf{x}_n)\frac{\partial}{\partial r_*}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| &(8.304)\\ \geq& C\|L^{1-\frac{\epsilon}{2}-\delta}X_\downarrow(\mathbf{x}_n)(-V_l\rho_*^{-1})^{\frac{1}{2}}\Psi(L^{-\delta},\boldsymbol{\xi}_n)\chi_\alpha u\| - O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) &(8.305)\end{aligned}$$

Now the H_3 terms are estimated. The goal is to replace the localisation $\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)$ by $\Psi(L^{-\delta},\boldsymbol{\xi}_n)$. This requires commuting $(\Psi(L^{-\delta},\bullet)\Phi_{c,\epsilon}^{-1})(\boldsymbol{\xi}_n)$ through the ρ_* localisation.

$$\|L^{\frac{2+n-\epsilon}{2}}(-V_l'\rho_*^{-1})^{\frac{1}{2}}X_\uparrow(\mathbf{x}_n)\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| = \|L^{1-\frac{\epsilon}{2}}X_\uparrow(\mathbf{x}_n)(-V_l'\rho_*^{-1})^{\frac{1}{2}}\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.306}$$

The bounded localisation $(\Psi(L^{-\delta},\bullet)\Phi_{c,\epsilon}^{-1})()$ is introduced, and then commuted through the localisation in $\mathbf{x}_n$. A factor of L^{δ} is dropped to control commutator terms involving $\Psi(L^{-\delta},\boldsymbol{\xi}_n)$ at a later stage.

$$\|L^{\frac{2+n-m-\epsilon}{2}}(-V_l'\rho_*^{-1})^{\frac{1}{2}}X_\uparrow(\mathbf{x}_n)\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.307}$$
$$\geq C\|(\Psi(L^{-\delta},\bullet)\Phi_{c,\epsilon}^{-1})(\boldsymbol{\xi}_n)L^{1-\frac{\epsilon}{2}-\delta}X_\uparrow(\mathbf{x}_n)(-V_l'\rho_*^{-1})^{\frac{1}{2}}\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.308}$$
$$\geq C_1\|L^{1-\frac{\epsilon}{2}-\delta}X_\downarrow(\mathbf{x}_n)(\Psi(L^{-\delta},\bullet)\Phi_{c,\epsilon}^{-1})(\boldsymbol{\xi}_n)(-V_l'\rho_*^{-1})^{\frac{1}{2}}\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.309}$$
$$-C_2\|L^{1-\frac{\epsilon}{2}-\delta}[(\Psi(L^{-\delta},\bullet)\Phi_{c,\epsilon}^{-1})(\boldsymbol{\xi}_n),X_\uparrow(\mathbf{x}_n)](-V_l'\rho_*^{-1})^{\frac{1}{2}}\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.310}$$

Now the operator $(\Psi(L^{-\delta},\bullet)\Phi_{c,\epsilon}^{-1})(\boldsymbol{\xi}_n)$ is commuted through the localisation $(-V_l'\rho_*^{-1})$. This eliminates the $\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)$ localisation.

$$\|L^{\frac{2-\epsilon}{2}}(-V_l'\rho_*^{-1})^{\frac{1}{2}}X_\uparrow(\mathbf{x}_n)\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.311}$$
$$\geq C\|L^{1-\frac{\epsilon}{2}-\delta}X_\downarrow(\mathbf{x}_n)(-V_l'\rho_*^{-1})^{\frac{1}{2}}(\Psi(L^{-\delta},\bullet)\Phi_{c,\epsilon}^{-1})(\boldsymbol{\xi}_n)\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.312}$$
$$-C\|L^{1-\frac{\epsilon}{2}-\delta}X_\uparrow(\mathbf{x}_n)[(\Psi(L^{-\delta},\bullet)\Phi_{c,\epsilon}^{-1})(\boldsymbol{\xi}_n),(-V_l'\rho_*^{-1})^{\frac{1}{2}}]\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.313}$$
$$-\|L^{1-\frac{\epsilon}{2}-\delta}[(\Psi(L^{-\delta},\bullet)\Phi_{c,\epsilon}^{-1})(\boldsymbol{\xi}_n),X_\uparrow(\mathbf{x}_n)](-V_l'\rho_*^{-1})^{\frac{1}{2}}\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.314}$$
$$\geq C\|L^{1-\frac{\epsilon}{2}-\delta}X_\uparrow(\mathbf{x}_n)(-V_l'\rho_*^{-1})^{\frac{1}{2}}\Psi(L^{-\delta},\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.315}$$
$$-C\|L^{1-\frac{\epsilon}{2}-\delta}X_\uparrow(\mathbf{x}_n)[(\Psi(L^{-\delta},\bullet)\Phi_{c,\epsilon}^{-1})(,)(-V_l'\rho_*^{-1})^{\frac{1}{2}}]\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.316}$$
$$-\|L^{1-\frac{\epsilon}{2}-\delta}[(\Psi(L^{-\delta},\bullet)\Phi_{c,\epsilon}^{-1})(\boldsymbol{\xi}_n),X_\uparrow(\mathbf{x}_n)](-V_l'\rho_*^{-1})^{\frac{1}{2}}\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.317}$$

The commutator terms can now be estimated using lemma 8.18 and are found to be lower order error terms.

$$\|L^{\frac{2-\epsilon}{2}}(-V_l'\rho_*^{-1})^{\frac{1}{2}}X_\uparrow(\mathbf{x}_n)\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.318}$$
$$\geq C\|L^{1-\frac{\epsilon}{2}-\delta}X_\uparrow(\mathbf{x}_n)(-V_l'\rho_*^{-1})^{\frac{1}{2}}\Psi(L^{-\delta},\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.319}$$
$$-C\|L^{1-\frac{\epsilon}{2}-\delta}X_\uparrow(\mathbf{x}_n)[(\Psi(L^{-\delta},\bullet)\Phi_{c,\epsilon}^{-1})(\boldsymbol{\xi}_n),(-V_l'\rho_*^{-1})^{\frac{1}{2}}]\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.320}$$
$$-\|L^{1-\frac{\epsilon}{2}-\delta}[(\Psi(L^{-\delta},\bullet)\Phi_{c,\epsilon}^{-1})(\boldsymbol{\xi}_n),X_\uparrow(\mathbf{x}_n)](-V_l'\rho_*^{-1})^{\frac{1}{2}}\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.321}$$
$$\geq C\|L^{1-\frac{\epsilon}{2}-\delta}X_\uparrow(\mathbf{x}_n)(-V_l'\rho_*^{-1})^{\frac{1}{2}}\Psi(L^{-\delta},\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.322}$$
$$-C\|L^{1-\frac{\epsilon}{2}-\delta}X_\uparrow(\mathbf{x}_n)L^{2n+\delta-1}B_1\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.323}$$
$$-\|L^{1-\frac{\epsilon}{2}-\delta}L^{2n+\delta-1}B_2(-V_l'\rho_*^{-1})^{\frac{1}{2}}\Phi_{c,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.324}$$
$$\geq C\|L^{1-\frac{\epsilon}{2}-\delta}X_\uparrow(\mathbf{x}_n)(-V_l'\rho_*^{-1})^{\frac{1}{2}}\Psi(L^{-\delta},\boldsymbol{\xi}_n)\chi_\alpha u\| \tag{8.325}$$
$$+O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2,L^1) \tag{8.326}$$

Equations (8.305) and (8.326) can be combined with the initial estimate on the expectation value of $[H,\Gamma_{n,m}]$ to produce an intermediate result.

$$\langle u,[H,\Gamma_{n,n}]u\rangle \geq C(\|L^{1-\frac{\epsilon}{2}-\delta}X_\uparrow(\mathbf{x}_n)(-V_l'\rho_*^{-1})^{\frac{1}{2}}\Psi(L^{-\delta},\boldsymbol{\xi}_n)\chi_\alpha u\|^2 \tag{8.327}$$
$$+\|L^{1-\frac{\epsilon}{2}-\delta}X_\downarrow(\mathbf{x}_n)(-V_l'\rho_*^{-1})^{\frac{1}{2}}\Psi(L^{-\delta},\boldsymbol{\xi}_n)\chi_\alpha u\|^2) \tag{8.328}$$
$$-O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2,L^1) \tag{8.329}$$

Both these terms involve localisation in $\mathbf{x}_n$ acting on $(-V_l'\rho_*^{-1})^{\frac{1}{2}}\Psi(L^{-\delta},\boldsymbol{\xi}_n)\chi_\alpha u$. The sum of the localisations is bounded below by a constant.

$$X_\downarrow(x)^2+X_\uparrow(x)^2=(\frac{1}{1+(\frac{x}{2M})^2})^2+(x\sqrt{\frac{\arctan(x)}{x}})^2\geq C \tag{8.330}$$

Therefore, the two terms can be combined to provide a better estimate.

$$\langle u,[H,\Gamma_{n,n}]u\rangle \geq C\|L^{1-\frac{\epsilon}{2}-\delta}(-V_l'\rho_*^{-1})^{\frac{1}{2}}\Psi(L^{-\delta},\boldsymbol{\xi}_n)\chi_\alpha u\|-O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2,L^1) \tag{8.331}$$

To eliminate the factor of $(-V_l'\rho_*^{-1})^{\frac{1}{2}}$, a new function is introduced. This function, f, is a smooth, compactly supported function and equal to the inverse of $(-V_l'\rho_*^{-1})^{\frac{1}{2}}$ on $\mathrm{supp}\chi_\alpha$. Since it is bounded, it can be freely introduced into the norm. The ρ_* localisation can then be commuted through $\Psi(L^{-\delta}, \boldsymbol{\xi}_n)$.

$$\langle u, [H, \Gamma_{n,n}]u\rangle \geq C\|L^{1-\frac{\epsilon}{2}-\delta} f(-V_l'\rho_*^{-1})^{\frac{1}{2}}\Psi(L^{-\delta}, \boldsymbol{\xi}_n)\chi_\alpha u\| - O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) \tag{8.332}$$

$$\geq C\|L^{1-\frac{\epsilon}{2}-\delta}\Psi(L^{-\delta}, \boldsymbol{\xi}_n) f(-V_l'\rho_*^{-1})^{\frac{1}{2}}\chi_\alpha u\| \tag{8.333}$$

$$- \|L^{1-\frac{\epsilon}{2}-\delta}[f(-V_l'\rho_*^{-1})^{\frac{1}{2}}, \Psi(L^{-\delta}, \boldsymbol{\xi}_n)]\chi_\alpha u\| \tag{8.334}$$

$$- O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) \tag{8.335}$$

From the definition of f, the product of all the ρ_* localisation reduces to χ_α. The new commutator terms can be estimated by lemma 8.18 and are found to be lower order error terms.

$$\langle u, [H, \Gamma_{n,n}]u\rangle \geq C\|L^{1-\frac{\epsilon}{2}-\delta}\Psi(L^{-\delta}, \boldsymbol{\xi}_n)\chi_\alpha u\| - \|L^{1-\frac{\epsilon}{2}-\delta}L^{2n+\delta-1}B\chi_\alpha u\| \tag{8.336}$$

$$- O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) \tag{8.337}$$

$$\geq C\|L^{1-\frac{\epsilon}{2}-\delta}\Psi(L^{-\delta}, \boldsymbol{\xi}_n)\chi_\alpha u\| - O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) \tag{8.338}$$

Lemma 8.18 can now be used to replace the localisation in $\Psi(L^{-\delta}, \boldsymbol{\xi}_n)$ by localisation in $\Psi_{L^{-\delta}\leq|\boldsymbol{\xi}_n|\leq 1}$.

$$\langle u, [H, \Gamma_{n,n}]u\rangle \geq C\|L^{1-\frac{\epsilon}{2}-\delta}\Psi_{L^{-\delta}\leq|\boldsymbol{\xi}_n|\leq 1}\chi_\alpha u\| - O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) \tag{8.339}$$

□

8.5 Phase Space Induction

The previous section shows that $\Gamma_{n,\frac{1}{2}}$ and $\Gamma_{n,n}$ majorate $L^{\frac{\frac{3}{2}+n-\epsilon}{2}}$ and $L^{1-\delta-\frac{\epsilon}{2}}$ respectively. We would like to integrate the Heisenberg identity to conclude that the time integral of the expectation value of these powers of L are bounded. However, our definition of majoration allows the domination to occur only in a region of phase space and the lower order terms to be unlocalised.

The lower order terms are $O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1)$. To control these terms, we use a finite induction on n, to eventually control $L^{1-\varepsilon}$ without phase space localisation.

Following this, we use the control of $L^{1-\varepsilon}$ in the conformal estimate to prove point wise in time, weighted L^6 decay, with bounds involving an additional L^ε factor.

Theorem 8.20 (Phase Space Induction). *If $\varepsilon > 0$, then for $u \in \mathbb{S}$ a solution to the wave equation, $\ddot{u} + Hu = 0$,*

$$\|L^{1-\frac{\varepsilon}{2}}\chi_\alpha u\|^2 = O(L^1) \tag{8.340}$$

Proof. This is proven for $\varepsilon = \delta + \epsilon$.

We induct on n to prove, simultaneously, the two statements

$$\|L^{\frac{\frac{3}{2}+n-\delta-\epsilon}{2}}\chi_\alpha u\|^2 = O(L^1), \tag{8.341}$$

$$\|L^{1-\delta}\chi([L^{1-n}, \infty), \frac{\partial}{\partial r_*})\chi_\alpha u\|^2 = O(L^1). \tag{8.342}$$

The induction will run from $n = 0$, in steps of size δ, as long as

$$n \leq \frac{1}{2} - 2\delta - \epsilon. \tag{8.343}$$

Each step in the induction will be proven by Morawetz type arguments using $\Gamma_{n,n}$ and $\Gamma_{n,\frac{1}{2}}$.

The base case of (8.341) follows from the angular modulation result, theorem 7.8, which says, in the notation of this section $\|L^{\frac{3}{2}}\chi_\alpha u\|^2 = O(L^1)$. The base case of (8.342) follows from the corollary to the local decay estimate, corollary 7.35. For $n = 0$, since

$$l^{1-\delta}\chi([l, \infty), x) \leq l^{1\delta}\chi([l^{1-\delta}, \infty), x) \leq x, \tag{8.344}$$

by the spectral theorem and corollary 7.9,

$$\|L^{1-\delta}\chi([L,\infty),\frac{\partial}{\partial r_*})\chi_\alpha u\|^2 \leq \|\frac{\partial}{\partial r_*}\chi_\alpha u\|^2 = O(L^1). \tag{8.345}$$

The inductive step is now considered.

Morawetz type estimates with $\Gamma_{n,n}$ and $\Gamma_{n,\frac{1}{2}}$ will be used. Before these estimates are proven, $\|\Gamma_{n,m}u\|^2$ is shown to be bounded by the energy and a local decay term, under the condition $0 \leq n \leq m \leq \frac{1}{2}$.

To begin estimating the norm of $\Gamma_{n,m}u$, $\Gamma_{n,m}$ is expanded as a product of operators, and the factor of γ_{L^m} is expanded as a sum of two terms. These terms can be further simplified by eliminating bounded functions.

$$\begin{aligned}
\|\Gamma_{n,m}u\| =&\|\chi_\alpha\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\gamma_{L^m}L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| && (8.346)\\
\leq&\|\chi_\alpha\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\frac{\partial}{\partial r_*}g_{L^m}L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| && (8.347)\\
&+\|\chi_\alpha\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\frac{1}{2}\frac{L^m}{1+(\frac{\rho_* L^m}{2M})^2}L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| && (8.348)\\
\leq&\|\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\frac{\partial}{\partial r_*}g_{L^m}L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| + \|L^{m+n-\epsilon}\chi_\alpha u\| && (8.349)
\end{aligned}$$

Interpolation can be used to simplify $\|L^{m+n-\epsilon}\chi_\alpha u\|$.

$$\|\Gamma_{n,m}u\| \leq \|\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\frac{\partial}{\partial r_*}g_{L^m}L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| + C(E[u] + \|\left(1+\left(\frac{\rho_*}{2M}\right)^2\right)^{-1}u\|^2) \tag{8.350}$$

To control $\|\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\frac{\partial}{\partial r_*}g_{L^m}L^{n-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\|$, it is rewritten in terms of $\boldsymbol{\xi}_n$ and L, and its sub factors are rearranged.

$$\begin{aligned}
\|\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\frac{\partial}{\partial r_*}L^{n-\epsilon}g_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| =&\|L^{1-\epsilon}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\boldsymbol{\xi}_n g_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| && (8.351)\\
\leq&\|L^{1-\epsilon}g_{L^m}\boldsymbol{\xi}_n\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)^2\chi_\alpha u\| && (8.352)\\
&+\|L^{1-\epsilon}[\boldsymbol{\xi}_n\Phi_{a,\epsilon}(\boldsymbol{\xi}_n),g_{L^m}]\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha\| && (8.353)
\end{aligned}$$

The first of these two norms can be estimated by lemma 8.6. The second can be estimated using lemma 8.12 which states that the commutator is $L^{n+m-1}B$.

$$\begin{aligned}
\|\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\frac{\partial}{\partial r_*}L^{n-\epsilon}g_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \leq&\|L^{\frac{1+n\epsilon-2\epsilon}{1-\epsilon}}\chi_\alpha u\| + \|\frac{\partial}{\partial r_*}\chi_\alpha u\| && (8.354)\\
&+\|L^{n+m-\epsilon}B\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| && (8.355)
\end{aligned}$$

Since $\frac{1+n\epsilon-2\epsilon}{1-\epsilon} \leq 1$ and $n+m-\epsilon \leq 1$, each of the terms involving powers of L in this expression can be estimated by interpolation.

$$\|\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\frac{\partial}{\partial r_*}L^{n-\epsilon}g_{L^m}\Phi_{a,\epsilon}(\boldsymbol{\xi}_n)\chi_\alpha u\| \leq C(E[u] + \|\left(1+\left(\frac{\rho_*}{2M}\right)^2\right)^{-1}u\|^2) \tag{8.356}$$

This completes the estimate on $\|\Gamma_{n,m}u\|$.

We can now integrate the Heisenberg like identity to find

$$\begin{aligned}
\int_1^T\langle u,[H,\Gamma_{n,m}]u\rangle dt =&\int\frac{d}{dt}(\langle u,\Gamma_{n,m}\dot{u}\rangle - \langle\dot{u},\Gamma_{n,m}u\rangle)dt && (8.357)\\
=&-2\langle\dot{u},\Gamma_{n,m}u\rangle|_1^T && (8.358)\\
\leq&C(E[u] + \|\left(1+\left(\frac{\rho_*}{2M}\right)^2\right)^{-1}u(T)\|^2 + \|\left(1+\left(\frac{\rho_*}{2M}\right)^2\right)^{-1}u(1)\|^2) && (8.359)
\end{aligned}$$

The second term on the right can be dropped since, by the local decay result, the norm $\| \left(1+\left(\frac{\rho_*}{2M}\right)^2\right)^{-1} u(T)\|^2 \to 0$ on a sequence of times.

$$\int_1^T \langle u, [H,\Gamma_{n,m}]u\rangle dt \leq C(E[u] + \| \left(1+\left(\frac{\rho_*}{2M}\right)^2\right)^{-1} u(1)\|^2) \tag{8.360}$$

The derivative localisation results, lemmas 8.19 and 8.16 can now be applied. From the inductive hypothesis (8.341) and condition (8.343), the lower order terms are integrable:

$$\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2 = O(L^1), \tag{8.361}$$
$$O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) = O(L^1). \tag{8.362}$$

From inductive hypothesis (8.342) and the derivative localising result with $n = m$, lemma 8.19,

$$\begin{aligned}
\|L^{1-\delta}\chi([L^{1-n-\delta},\infty),\frac{\partial}{\partial r_*})\chi_\alpha u\|^2 =& \|L^{1-\delta}\chi([L^{1-n-\delta},L^{1-n}),\frac{\partial}{\partial r_*})\chi_\alpha u\|^2 && (8.363)\\
&+ \|L^{1-\delta}\chi([L^{1-n},\infty),\frac{\partial}{\partial r_*})\chi_\alpha u\|^2 && (8.364)\\
=& \|L^{1-\delta}\Psi_{L^{-\delta}\leq|\boldsymbol{\xi}_n|\leq 1}\chi_\alpha u\|^2 && (8.365)\\
&+ \|L^{1-\delta}\chi([L^{1-n},\infty),\frac{\partial}{\partial r_*})\chi_\alpha u\|^2 && (8.366)\\
\leq \langle u, [H,\Gamma_{n,n}]u\rangle + O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) & && (8.367)\\
&+ O(L^1). && (8.368)
\end{aligned}$$

Integration in time extends inductive hypothesis (8.342) to $n+\delta$

$$\int_1^\infty \|L^{1-\delta}\chi([L^{1-n-\delta},\infty),\frac{\partial}{\partial r_*})\chi_\alpha u\|^2 dt \leq C(E[u] + \| \left(1+\left(\frac{\rho_*}{2M}\right)^2\right)^{-1} u(1)\|^2). \tag{8.369}$$

From condition (8.343), it follows that $\frac{\frac{3}{2}+n-\epsilon}{2} < 1-\delta$. From the derivative localising result with $n = \frac{1}{2}$, lemma 8.16,

$$\begin{aligned}
\|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}\chi_\alpha u\|^2 =& \|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}\chi([0,L^{1-n}],\frac{\partial}{\partial r_*})\chi_\alpha u\|^2 && (8.370)\\
&+ \|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}\chi([L^{1-n},\infty),\frac{\partial}{\partial r_*})\chi_\alpha u\|^2 && (8.371)\\
\leq& \|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}\Phi_{|\boldsymbol{\xi}_n|\leq 1}\chi_\alpha u\|^2 + \|L^{1-\delta}\chi([L^{1-n},\infty),\frac{\partial}{\partial r_*})\chi_\alpha u\|^2 && (8.372)\\
\leq& \langle u, [H,\Gamma_{n,\frac{1}{2}}]u\rangle + O(\|L^{\frac{1+2n}{2}}\chi_\alpha u\|^2, L^1) + O(L^1). && (8.373)
\end{aligned}$$

Integration in time extends inductive hypothesis (8.341) up to $n+\delta$.

$$\int_1^\infty \|L^{\frac{\frac{3}{2}+n-\epsilon}{2}}\chi_\alpha u\|^2 dt \leq C(E[u] + \| \left(1+\left(\frac{\rho_*}{2M}\right)^2\right)^{-1} u(1)\|^2) \tag{8.374}$$

Since the induction continues as long as condition (8.343) holds, after the last application of equation (8.374),

$$\int_1^\infty \|L^{\frac{2-2\delta-2\epsilon}{2}}\chi_\alpha u\|^2 dt = O(L^1) \tag{8.375}$$

This proves the desired result with $\varepsilon = \delta - \epsilon$. Since δ and ϵ can be taken to be arbitrarily small, so can ε. □

Except for the loss of L^ϵ, this provides the control on the angular energy near the photon sphere required by corollary 6.9 to control the weighted L^6 norm. Since the wave equation is linear and L commutes with H, $L^\epsilon u$ is also a solution, and the energy of this function can be used to recover the additional factor of $L^\epsilon u$.

Theorem 8.21. *If $\tilde{u}$ is a solution to the wave equation* (2.21) *, and $u = r\tilde{u}$, then*

$$\|F^{\frac{1}{2}}\tilde{u}\|_{L^6(\tilde{\mathfrak{M}})} \leq t^{-\frac{1}{3}} C(\|u_0\|_2^2 + E[u_0, u_1] + E_{\mathcal{C}}[u_0, u_1] + \|L^\epsilon u\|^2 + E[L^\epsilon u_0, L^\epsilon u_1])^{\frac{1}{2}} \tag{8.376}$$

$$\|(\rho_*^2 + 1)^{\frac{-1}{2}}\tilde{u}\|_{L^2(\tilde{\mathfrak{M}})} \leq t^{-\frac{1}{2}} C(\|u_0\|_2^2 + E[u_0, u_1] + E_{\mathcal{C}}[u_0, u_1] + \|L^\epsilon u\|^2 + E[L^\epsilon u_0, L^\epsilon u_1])^{\frac{1}{2}} \tag{8.377}$$

Proof. If $\tilde{u}$ is a solution to equation (2.21), then u is a solution to equation (2.24). The weighted L^6 norms are related.

$$\|F^{\frac{1}{2}}\tilde{u}\|^6_{L^6(\tilde{\mathfrak{M}})} = \int F^3 |\tilde{u}|^6 d^3\mu \tag{8.378}$$

$$= \int F^3 r^{-4} |u| d^3\mu. \tag{8.379}$$

Since the wave equation equation (2.24) is linear and H commutes with L, if u is a solution, then so is $L^\epsilon u$. The phase space induction theorem, theorem 8.20, can be applied to $L^\epsilon u$.

$$\int \|L^{1-\epsilon}\chi_\alpha L^\epsilon u\|^2 dt = O(L^1), \tag{8.380}$$

$$\int \|L\chi_\alpha u\|^2 dt \leq C(\|L^\epsilon u\|^2 + E[L^\epsilon u_0, L^\epsilon u_1]) \tag{8.381}$$

This is the term which is needed to control the conformal charge in the reduction to angular energy on the photon sphere, corollary 6.9.

$$E_{\mathcal{C}}[u(t), \dot{u}(t)] \leq E_{\mathcal{C}}[u_0, u_1] + CtE[u_0, u_1]^{\frac{1}{2}}(E[u_0, u_1]^{\frac{1}{2}} + \|u_0\|) \tag{8.382}$$

$$+ Ct \int^t \int \chi_\alpha |\nabla_{S^2} u|^2 d^3\mu d\tau \tag{8.383}$$

$$\leq E_{\mathcal{C}}[u_0, u_1] + CtE[u_0, u_1]^{\frac{1}{2}}(E[u_0, u_1]^{\frac{1}{2}} + \|u_0\|) \tag{8.384}$$

$$+ Ct(\|L^\epsilon u_0\|^2 + E[L^\epsilon u_0, L^\epsilon u_1]) \tag{8.385}$$

Since the conformal charge controls the weighted L^6 norm, we have

$$\|F^{\frac{1}{2}}\tilde{u}\|_{L^6(\tilde{\mathfrak{M}})} \leq (\int F^3 r^{-4} |u| d^3\mu)^{\frac{1}{6}} \tag{8.386}$$

$$\leq Ct^{\frac{-1}{3}}(\|u_0\|^2 + E[u_0, u_1] + E_{\mathcal{C}}[u_0, u_1] + \|L^\epsilon u_0\|^2 + E[L^\epsilon u_0, L^\epsilon u_1]). \tag{8.387}$$

Similarly, since lemma 3.6 controls the weighted L^2 norm by the conformal charge,

$$\langle \tilde{u}, \frac{1}{\rho_*^2 + 1}\tilde{u}\rangle_{L^2(\tilde{\mathfrak{M}})} = \langle u, \frac{1}{\rho_*^2 + 1} u\rangle \tag{8.388}$$

$$\leq Ct^{-2} E_{\mathcal{C}}[u(t), \dot{u}(t)] \tag{8.389}$$

$$\leq Ct^{-2} E_{\mathcal{C}}[u_0, u_1] + Ct^{-1}(E[u_0, u_1] + \|u_0\|^2 + \|L^\epsilon u_0\|^2 + E[L^\epsilon u_0, L^\epsilon u_1]) \tag{8.390}$$

□

Acknowledgements

These results originally appeared as part of the Ph.D. Dissertation of P. Blue, "Decay estimates and phase space analysis for wave equations on some black hole metrics"[3], submitted in October 2004.

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