Order and Disorder in Multiscale Substitution Tilings

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Partially based on joint work with Yaar Solomon
Plan of Talk

- Introduction
- Bounded displacement and bilipschitz equivalence
- Multiscale substitution tilings
Delone Sets

A uniformly discrete and relatively dense set \( \Lambda \subseteq \mathbb{R}^d \) is called Delone.
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Examples  Lattices, sets induced by tilings and cut-and-project sets

A basic problem is to classify and measure how ordered or disordered a given Delone set is, compared to a lattice.
Lattice-like Properties

For $x \in \Lambda$, $r > 0$ the $r$-patch of $\Lambda$ at $x$ is $P_{\Lambda,r}(x) = (\Lambda - x) \cap B(0,r)$

- Finite local complexity (FLC)

$$\forall \, r > 0 \, \# \{ P_{\Lambda,r}(x) \mid x \in \Lambda \} < \infty$$

From Baake and Grimm's Aperiodic Order Vol 1
Lattice-like Properties

For \( x \in \Lambda \), \( r > 0 \) the \( r \)-patch of \( \Lambda \) at \( x \) is \( P_{\Lambda,r}(x) = (\Lambda - x) \cap B(0,r) \)

- **Finite local complexity (FLC)**
  \[ \forall \, r > 0 \quad \# \{ P_{\Lambda,r}(x) \mid x \in \Lambda \} < \infty \]

- **Repetitivity** \( \forall \, r > 0 \quad \exists \, R = R(r) \) so that every \( R \)-ball contains a copy of every \( r \)-patch. Linear repetitivity if \( R(r) \) is linear. Uniform patch frequency if patches appear in well-defined frequencies.
Lattice-like Properties

For \( x \in \Lambda, r > 0 \) the \( r \)-patch of \( \Lambda \) at \( x \) is \( P_{\Lambda,r}(x) = (\Lambda - x) \cap B(0,r) \)

- **Finite local complexity (FLC)**
  \[ \forall r > 0 \not\exists \#\{ P_{\Lambda,r}(x) \mid x \in \Lambda \} < \infty \]

- **Repetitivity** \( \forall r > 0 \exists R = R(r) \) so that every \( R \)-ball contains a copy of every \( r \)-patch. Linear repetitivity if \( R(r) \) is linear. Uniform patch frequency if patches appear in well-defined frequencies.

- **Self-similarity** there exists \( \alpha > 1 \) so that \( \alpha \Lambda \subset \Lambda \)
Spaces and Dynamical Systems of Delone Sets

Set $X_\Lambda = \{ \Lambda + t | t \in \mathbb{R}^d \}$, where the closure is with respect to a natural topology on Delone sets (induced by the Hausdorff metric restricted to centered balls).

- $\Lambda$ is (almost) repetitive $\Rightarrow$ The dynamical system $(X_\Lambda, \mathbb{R}^d)$ is minimal (every orbit is dense)
- (almost) linear repetitivity $\Rightarrow$ unique ergodicity (unique invariant measure)

(Radin '92, Solomyak '97, Damanik '01, Lagarias '03, Frettloeh '14)
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BD and BL Equivalence Relations

• Delone sets $\Lambda, \Gamma \in \mathbb{R}^d$ are bounded displacement (BD) equivalent if $\exists$ bijection $\varphi: \Lambda \rightarrow \Gamma$ that moves every point a bounded distance.

• $\Lambda, \Gamma$ are bilipschitz (BL) equivalent if $\exists L > 0$ and bijection $\varphi: \Lambda \rightarrow \Gamma$

  \[
  \forall x, y \in \Lambda \quad \frac{1}{L} \leq \frac{\|\varphi(x) - \varphi(y)\|}{\|x - y\|} \leq L
  \]

• $\Lambda$ is uniformly spread if it is BD to $\alpha \mathbb{Z}^d$ for some $\alpha > 0$, and rectifiable if it is BL to $\mathbb{Z}^d$.
BD and BL Equivalence Relations

- Not all Delone sets are rectifiable \((\text{Burago-Kleiner, McMullen '98, Cortez '16})\)
- Linear repetitivity \(\Rightarrow\) rectifiability \((\text{Aliste-Prieto, Coronel, Gambaudo '13})\)

\[(\text{use Burago-Kleiner '02 sufficient condition for rectifiability})\]

**Theorem (RS3'21)** True also for almost linear repetitivity.
BD and BL Equivalence Relations

- Not all Delone sets are rectifiable (Burago, Kleiner, McMullen '98, Cortez '16)
- Linear repetitivity \( \Rightarrow \) rectifiability (Aliste-Prieto, Coronel, Gambaudu '13)

(Use Burago-Kleiner '02 sufficient condition for rectifiability)

**Theorem** (SS3≥'21) True also for almost linear repetitivity.

- Sets associated with tilings with a single tile are uniformly spread 
  \( \Rightarrow \) lattices & periodic sets (Duneau, Oguey '90, Halls marriage theorem)
**BD and BL Equivalence Relations**

- Not all Delone sets are rectifiable \( (\text{Burago, McMullen '98, Cortez '10}) \)
- Linear repetitivity \( \Rightarrow \) rectifiability \( (\text{Aliste-Prieto, Coronel, Gambaudu '13}) \)

(Use Burago-Kleiner '02 sufficient condition for rectifiability)

**Theorem (SS3 ≥'21)** True also for almost linear repetitivity.

**Laczkovich '92** For a Delone set \( \Lambda \subseteq \mathbb{R}^d \) the following are equivalent:

- \( \Lambda \) is uniformly spread
- There exist \( \alpha, C > 0 \) so that \( \forall \ A \in Q_d = \{\text{finite unions of lattice cubes}\} \)
  \[ \text{discrepancy} \approx |\#(A \cap \Lambda) - \alpha \cdot \text{vol}(A)| \leq C \cdot \text{vol}_{d-1}(\partial A) \]
BD Equivalence Criterion and Dichotomy

**Theorem** (FSS ’21 and SS2 ’21) The following are equivalent:

- $\Lambda$ and $\Pi$ are not BD equivalent
- There exists a sequence of sets $A_m \in \mathcal{Q}_d$ so that
  \[
  \lim_{m \to \infty} \frac{|\#(A_m \cap \Lambda) - \#(A_m \cap \Pi)|}{\text{vol}_{d-1}(\partial A_m)} = 0
  \]

**Theorem** (SS2 ’21) Let $X$ be a minimal space of Delone sets.

- Either $\exists \Lambda \in X$ uniformly spread, and then every $\Lambda \in X$ is such.
- Or $X$ contains continuously many distinct BD class representatives.
**BD Equivalence Criterion and Dichotomy**

**Theorem** (FSS ’21 and SS2 ’21) The following are equivalent:

- $\Lambda$ and $\Pi$ are not BD equivalent
- There exists a sequence of sets $A_m \in Q_d$ so that

\[
\frac{|\#(A_m \cap \Lambda) - \#(A_m \cap \Pi)|}{\text{vol}_{d-1}(\partial A_m)} \xrightarrow{m \to \infty} \infty
\]

**Theorem** (SS2 ’21) Let $X$ be a minimal space of Delone sets.

- Either $\exists \Lambda \in X$ uniformly spread, and then every $\Lambda \in X$ is such.
- Or $X$ contains continuously many distinct BD class representatives.

$\Rightarrow$ Substitution tilings (Solomon ’14)

$\Rightarrow$ Cut-and-project sets (Haynes, Kelly, Weiss ’14, Frettlöh, Garber ’18)

$\Rightarrow$ Incommensurable multiscale tilings (SS1 ’21)
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Substitution Tilings

A tiling is a collection of tiles with disjoint interiors that covers $\mathbb{R}^d$. A substitution rule on a set of prototiles is a tessellation of each prototile by rescaled prototiles, with a fixed scale $c \in (0,1)$.

Repeated applications of the substitution rule followed by a rescaling define larger and larger patches.
A multiscale substitution scheme $\sigma$ in $\mathbb{R}^d$ consists of a substitution rule on unit volume prototiles $T_1, \ldots, T_n$, where various different scales appear and satisfy a simple incommensurability condition.

A time-dependant substitution semiflow $F_t$ defines a family of patches: At time $t=0$, $F_t(T) = T$, and as $t$ increases the patch is inflated by $e^t$ and tiles of volume $>1$ are substituted.
Some Predecessors

- Rauzy's fractal '81
  multiple (but commensurable) scales

- Conway and Radin's pinwheel tiling '94
  $\theta = \arctan \frac{1}{2} \Rightarrow$ same triangle incommensurable directions

- Sadun's generalized pinwheel tilings '98

- $\alpha$-Kakutani sequences in $[0,1]$ '76 $\alpha$ $\bar{\alpha}$
  always split longest interval

- S'20: multiscale substitution Kakutani sequences of partitions

\[ \begin{array}{cccccccc}
\includegraphics[width=1cm]{image1} & \includegraphics[width=1cm]{image2} & \includegraphics[width=1cm]{image3} & \includegraphics[width=1cm]{image4} & \includegraphics[width=1cm]{image5} & \includegraphics[width=1cm]{image6} & \includegraphics[width=1cm]{image7} & \includegraphics[width=1cm]{image8} \\
\end{array} \]
Incommensurable Multiscale Substitution Tilings

**Theorem (SS1.2i)** Let $\mathbf{T}$ be an incommensurable tiling.

- Tiles and patches appear in a dense set of scales $\Rightarrow$ not FLC
- Periodic orbits of $F_{\mathbf{T}}$ $\Rightarrow$ self similar tilings
- $\mathbf{T}$ has uniform patch frequencies
- $(X_{\mathbf{T}}, \mathbb{R}^d)$ is minimal ($\mathbf{T}$ is almost repetitive) and uniquely ergodic

Proof follows Lee-Solomyak's 199"pixelization" approach

**Theorem (SS3.2i)** Almost repetitivity is not linear
The Associated Graph $G_6$

A directed weighted graph is defined according to $G_6$

- **Vertices** model the prototiles
- **Edges** model the tiles appearing in the substitution rule with
  - **Lengths** $= \log(1/\text{scale})$

$G_6$ is incommensurable if $G_6$ contains two closed paths of lengths $\frac{a}{6} \notin \mathbb{Q}$. Incommensurable multiscale substitution schemes generate a new distinct class of tilings of $\mathbb{R}^d$. 
Counting in Multiscale Substitution Tilings

Substitution \# tiles in patches = entries of powers of the substitution matrix $S$

$$\Rightarrow S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Multiscale \{Tiles in $F_t(T_i)$\} $\leftrightarrow$ \{Directed walks of length $t$ in $G_6$ originating at vertex $i$\}

Example the $\frac{1}{3}$-Kakutani scheme in $\mathbb{R}$:

$$\log_3 \rightarrow \log^{\frac{3}{2}}$$

the patches $F_0(I), F_{\log^3(I)}, F_{2\log^3(I)}$ and their respective walks
Counting in Multiscale Substitution Tilings

Theorem \((S \geq 21, \text{relying on Kiro, Smilansky } \times 2 '20)\)

\[
\#\{\text{tiles in } F_t(T)\} = \frac{v^T (S_\sigma - V_\sigma) 1}{v^T H_\sigma 1} \cdot \frac{e^{dt}}{\text{vol}(F_t(T))} + \text{ERROR TERM}, \quad t \to \infty
\]

Combinatorics matrix

\[
(S_\sigma)_{ij} = \sum_{T \text{ of type } j \text{ in } T_i} 1 \quad \# \text{reds in white}
\]

\[
S_\sigma = \begin{pmatrix} 8 & 3 \\ 1 & 3 \end{pmatrix}
\]

Volume matrix

\[
(V_\sigma)_{ij} = \sum_{T \text{ of type } j \text{ in } T_i} \text{vol}(T) \quad \text{total red area in white}
\]

\[
V_\sigma = \begin{pmatrix} 18 & 8 \frac{25}{3} \\ \frac{1}{4} & \frac{3}{4} \frac{25}{3} \end{pmatrix}
\]

Entropy matrix

\[
(H_\sigma)_{ij} = \sum_{T \text{ of type } j \text{ in } T_i} -\text{vol}(T) \cdot \log \text{vol}(T) \quad \text{contribution of reds to entropy of white}
\]

\[
H_\sigma = \begin{pmatrix} -\frac{1}{2} \log \frac{4}{25} & -\frac{1}{2} \log \frac{4}{25} & -\frac{1}{2} \log \frac{4}{25} & -\frac{1}{2} \log \frac{4}{25} \\ -\frac{1}{4} \log \frac{1}{4} & -\frac{1}{4} \log \frac{1}{4} & -\frac{1}{4} \log \frac{1}{4} & -\frac{1}{4} \log \frac{1}{4} \end{pmatrix}
\]

and \(v^T = \text{left Perron-Frobenius eigenvector of } V_\sigma\)

Theorem \((SS1 '21) \exists k \in \mathbb{N} \forall t_0 > 0 \exists t > t_0: \text{ERROR TERM} \geq C \frac{e^{dt}}{t^k}\)

\(\Rightarrow\) Incommensurable tilings are never uniformly spread.
Counting in Multiscale Substitution Tilings

**Theorem** (S ≥ 21, relying on Kiro, Smilansky x2 '20)

Similar asymptotic formulas for:

- \# \{\text{tiles of type } r \text{ and } \text{vol} \in [a, b] \text{ in } F_r(T)\}
- \text{volume}(\bigcup \{\text{tiles of type } r \text{ and } \text{vol} \in [a, b] \text{ in } F_r(T)\})
- Expected values for random partitions
Counting in Multiscale Substitution Tilings

**Theorem** (S≥21, relying on Kiro, Smilansky x2 '20) similar formulas for

- **Gap distribution** \( \Lambda = \) Delone set of tile boundaries in a 1-dim tiling

\[
\frac{\# \{ \text{Neighbors in } \Lambda \cap [-N,N] \text{ of distance } e [a,b]\}}{\# \{ \Lambda \cap [-N,N]\}} \to \int_a^b \frac{\nu^T C_\sigma(x) 1}{\nu^T H_\sigma 1} \, dx
\]

where \( (C_\sigma(x))_{ij} = \sum_T \frac{\text{vol } T}{x^2} \), \( \text{vol } T < x < 1 \)

- **Numerics** for pair correlations are consistent with **Poisson process**

```math
\begin{align*}
\text{list} &= \{0, 3^{\text{10}}\}; i = 1; \\
\text{Do[While[list[[i + 1]] - list[[i]] > 1,} & \\
\text{list = Insert[list, list[[i]] + (list[[i + 1]] - list[[i]])/3, i + 1]], (i, 91005)]; \\
\text{gaps} &= \text{Flatten[Table[N[Differences[list, i, j]], (j, i, 100)];} \\
\text{Histogram[gaps, (0, 100, 0.5), "PDF"]}
\end{align*}
```

```math
\begin{align*}
\text{averagegap} &= 1 / (-1/3 * \text{Log}[1/3] - (2/3) * \text{Log}[2/3]); \\
\text{list} &= \text{Accumulate[RandomVariate[ExponentialDistribution[averagegap], 90 000]]}; \\
\text{gaps} &= \text{Flatten[Table[N[Differences[list, 1, j]], (j, 1, 100)];} \\
\text{Histogram[gaps, (0, 100, 0.5), "PDF"]}
\end{align*}
```
Thank You!