RESEARCH STATEMENT - YOTAM SMILANSKY

Introduction. The field of aperiodic order is devoted to the study of mathematical models of quasicrystals and the ways in which certain aspects of order and disorder manifest themselves. Within this field, my own research is focused on the distribution and dynamics of point sets and tilings, and belongs to the interface between dynamical systems, number theory and metric geometry, with strong connections to combinatorics and mathematical physics. The experimental discovery of aperiodically ordered materials in the 1980's, most notably Shechtman's work for which he was awarded the 2011 Nobel Prize in Chemistry, has greatly contributed to the rapid growth of interest in aperiodic order over the last few decades. Several central mathematical models have emerged, including aperiodic infinite words, cut-and-project sets and substitution tilings such as the Penrose tiling of the plane. In parallel, new notions of order and disorder that go beyond crystallography and spectral theory have been developed.

My contributions to aperiodic order are centered around two core problems in the field. One problem concerns the different ways a point set can be lattice-like, random-like or anything in between, the so-called *hierarchy of order*. In my work this is demonstrated by bounded displacement and biLipschitz equivalence classes of point sets [FSS, SS2, SS3], and their gap distributions and pair correlations [Sm1, Sm3]. The second problem concerns the study of models of aperiodic order and their relations to other mathematical concepts. For example, incommensurable multiscale substitution schemes and the tilings they generate [SS1] extend the classical substitution tiling setup, joining other recent constructions in strengthening existing links to symbolic, hyperbolic and fractal dynamics, and to problems in uniform distribution and discrepancy [Sm2]. Another example is [RSW], in which we build on the foundations of [MS] and consider cut-and-project sets as elements of a homogeneous space of quasicrystals, a point of view that allows us to integrate advanced tools from homogeneous dynamics to formulate and approach problems that originate in the geometry of numbers and the theory of lattices. Further details about my research and future directions appear below.

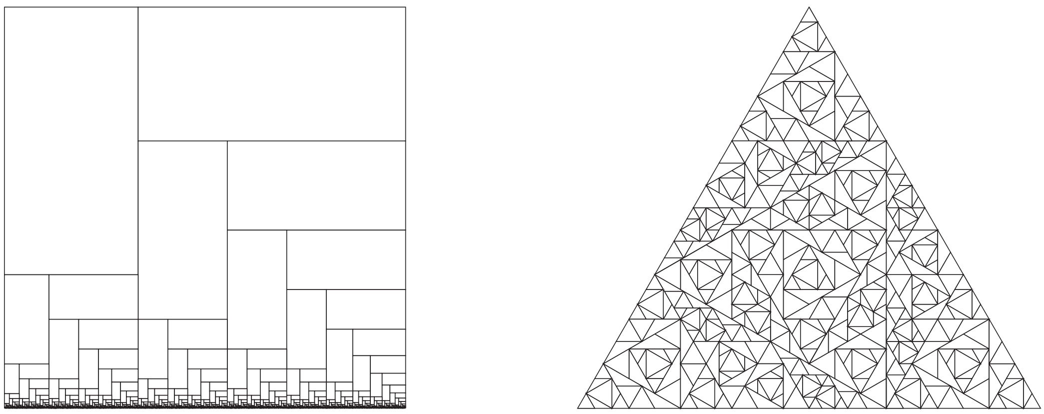

Figure. Patches of hyperbolic [SS4] (left) and multiscale substitution tilings [SS1].

Preliminaries. A set $\Lambda \subset \mathbb{R}^d$ is *Delone* if it is *uniformly discrete* and *relatively dense*, that is, if there exist $r, R > 0$ so that every ball of radius r contains at most one point of Λ and Λ intersects every ball of radius R. Examples include lattices, vertex sets of aperiodic tilings and cut-and-project sets, see Baake and Grimm's comprehensive introduction [BG]. Delone sets in $\mathbb{R}^d$ are often viewed as elements of the space $\mathscr{C}(\mathbb{R}^d)$ of closed subsets of $\mathbb{R}^d$ equipped with a natural compact topology, or as elements of a dynamical system with respect to a group action such as translations or volume preserving affine transformations.

1. Multiscale substitution schemes. In [KSS, Sm2, SS1] and [Sm3] we introduced and studied multiscale substitution schemes, their associated graphs and the sequences of partitions and tilings they generate. Unlike the classical setup for substitution tilings, multiple scaling constants appear and substitutions are not applied simultaneously to all tiles but according to their volumes. This procedure defines new geometric objects with properties that distinguish them from the classical constructions.

A *multiscale substitution scheme* σ in $\mathbb{R}^d$ consists of a finite set τ_σ of n *prototiles* of unit volume, and *substitution rules* $\varrho(T_i)$, each a partition of $T_i \in \tau_\sigma$ into finitely many rescaled copies of elements of τ_σ. A simple example is the α-Kakutani scheme in $\mathbb{R}$, with the unit interval I its single prototile and a substitution rule consisting of αI and $(1-\alpha)I$. An associated directed weighted graph G_σ is accordingly defined, with vertices representing the prototiles and edges modeled by tiles in the $\varrho(T_i)$'s, with weights determined by the tile's scale. If G_σ is strongly connected then σ is *irreducible*, and if G_σ is *incommensurable*, that is, if it contains two closed orbits a, b with $a \notin b\mathbb{Q}$, then so is σ. For example, the graph associated with the α-Kakutani scheme has a single vertex and two loops of lengths $\log(1/\alpha)$ and $\log(1/1-\alpha)$, and is incommensurable for almost every $\alpha \in (0,1)$, e.g. for $\alpha = 1/3$.

A Kakutani sequence is defined by setting $\pi_0 := T_i \in \tau_\sigma$, and defining the partition π_{m+1} of T_i by substituting tiles of maximal volume in π_m according to σ. To generate a tiling, position a prototile T_i around the origin, and define the patch $F_t(T_i)$ by inflating T_i by e^t, while substituting every tile that appears in the process according to σ once its volume is greater than 1. The patches $\{F_t(T_i) : t \geqslant 0\}$ exhaust the space, and limits taken with respect to the topology on $\mathscr{C}(\mathbb{R}^d)$ define a tiling space $\mathbb{X}_\sigma$ of multiscale substitution tilings of $\mathbb{R}^d$. Earlier examples include Sadun's generalized pinwheel tilings [Sa] and the fusion tiling considered in [FS, §A.5] by Frank and Sadun.

Recent results. In [Sm2] I proved that irreducible Kakutani sequences are uniformly distributed, generalizing Kakutani's result for α-Kakutani sequences [Ka]. The observation that walks in G_σ correspond to tiles in element of $(\pi_m)_{m\geqslant 0}$ plays an important role in the proof. In my joint work with Solomon [SS1] we established structural, geometric, statistical and dynamical properties of incommensurable multiscale tilings. Tiles are similar to prototiles but appear in a dense set of volumes, and so tilings are of infinite local complexity. A suitable variant of uniform patch frequencies was established, in which patches are counted together with dilations. Extending ideas of Lee and Solomyak [LS], we showed that the tiling dynamical system $(\mathbb{X}_\sigma, \mathbb{R}^d)$ is minimal and uniquely ergodic, that is, every orbit is dense and there exists a unique translation invariant probability measure. Unlike the case of classical substitution tilings, we showed that tilings in $\mathbb{X}_\sigma$ are never uniformly spread in the sense of BD equivalence (see below), and by our main result in [SS2] the space $\mathbb{X}_\sigma$ contains representatives of continuously many BD equivalence classes. We established almost repetitivity but later showed in [SS3] that it is never linear, once again in contrast to the classical setup. Lastly, I provided explicit representations of tile counting formulas in [Sm3], given in terms of the combinatorial, volume and entropy information carried by the underlying scheme σ, which extend also to expected values in random partitions and have applications to gap distributions in the 1-dimensional case.

Our results on the distribution of paths on incommensurable graphs [KSS], jointly with Kiro and Smilansky, proved a valuable tool in this study, particularly the asymptotic formulas for the number of paths of length at most t between two fixed vertices and the number of walks of length exactly t from a vertex to some point on the graph. These are implied by the pole structure of the corresponding Laplace transforms via Wiener-Ikehara's Tauberian theorem.

Preliminary results & future directions. A recent observation by Petite [Pet] is that multiscale substitution tilings of $\mathbb{R}^d$ may be viewed as intersections of appropriately constructed tilings in the hyperbolic upper half-space $\mathbb{H}^{d+1}$ with horizontal horospheres. In our work in preparation [SS4] we study this class of hyperbolic tilings, which is also related to examples considered by Böröczky [Bo] and Penrose [Pen], see also [Rad, DF], and essentially illustrated by M. C. Escher in his 1957 print *Regular Division of the Plane VI* [Es]. A substitution rule in $\mathbb{R}^d$ is transformed into a $d+1$-dimensional patch by giving tiles a *height* associated with their scales, and the inflation-substitution procedure is replaced by a gluing procedure that glues isometric copies of patches to tiles. For example, the 1/3-Kakutani scheme in $\mathbb{R}$ generates a tiling of the hyperbolic upper half-plain $\mathbb{H}^2$, as illustrated on page 1 above. These hyperbolic tilings contain only finitely many tiles up to hyperbolic isometries, do not have any horizontal periods, and the associated tiling space has a rich orbit structure under the isometry groups given by the geodesic $\mathbb{R}$-action (inflation flow corresponding to F_t) and the horospheric $\mathbb{R}^d$-action (horizontal translations). We show that the horospheric flow is minimal, while the geodesic flow has both dense orbits and periodic orbits, the latter satisfying a prime orbit theorem consistent with the hyperbolic dynamics of Parry and Pollicott [PP].

The mathematical diffraction measure associated with a Delone set is closely related to its pair correlations, and we are working to extend methods developed in [BGM] for primitive substitutions to incommensurable tilings of the real line, for which the gap distributions are described in [Sm3]. Numerical examination suggests that unlike the case of gap distributions, the pair correlations in such cases are reminiscent of those of a random Poisson process. A natural next step is to consider random and infinite substitution systems, and there is a surge of interest in such systems in the symbolic setting, with advances including [GRS, GS, MRST, MRW], see also [PvZ] for random partitions. The random incommensurable systems I considered in [Sm3] provide a new family of random tilings to explore, and the preliminary results on the associated graphs may prove useful. Uniform distribution of Kakutani sequences generated by an infinite substitution rule on a single prototile was shown in [PS] to follow from results in renewal theory. Generalizing our counting results to infinite graphs should imply generalizations to multiple prototiles with consequences to renewal theory.

2. Geometry of numbers in the space of quasicrystals. Decompose $\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^m$ into *physical* and *internal* spaces with projections π and π_{int}, respectively. Let $\mathcal{L} = g\mathbb{Z}^n$ be a grid in $\mathbb{R}^n$ such that $\pi_{\text{int}}(\mathcal{L})$ is dense in $\mathbb{R}^m$, where $g \in G_n := \mathrm{ASL}_n(\mathbb{R})$ the affine special linear group, and fix a bounded *window* $\mathcal{W} \subset \mathbb{R}^m$. The associated *cut-and-project* set is

$$\mathcal{P}(\mathcal{W}, \mathcal{L}) = \{\pi(x) \mid x \in \mathcal{L},\ \pi_{\text{int}}(x) \in \mathcal{W}\} \subset \mathbb{R}^d.$$

The group G_d acts on the physical space, and by Ratner's celebrated rigidity results [Rat] there exists a group H such that the closure of the orbit $G_d g\mathbb{Z}^n$ is $Hg\mathbb{Z}^n$. Every point in the orbit closure corresponds to a cut-and-project set, defining Marklof and Strömbergsson's *space of quasicrystals* [MS], in which questions similar to those that arise regarding typical behavior of lattices are naturally formulated. One distinction from lattices is the local pattern structure: while a lattice consists of a single pattern of a given fixed diameter, multiple distinct patterns appear in a cut-and-project set, and finer pattern statistics may be studied.

Recent results. In my joint work with Rühr and Weiss [RSW] we studied the class of Ratner-Marklof-Strömbergsson (RMS) measures, which are ergodic G_d-invariant probability measures supported on cut-and-project sets, and described their classification in terms of the data of the group H, which is shown to arise via restriction of scalars from a real number field. We used the classification to prove analogues of results of Siegel, Weil and Rogers about a Siegel summation formula and identities and bounds involving higher moments. These in turn imply results about asymptotics, with error estimates, of point and pattern-counting statistics for typical cut-and-project sets, extending Schmidt's results for typical lattices [Sc].

Preliminary results & future directions. In our ongoing work [GSW] we give necessary and sufficient conditions for projections and intersections of cut-and-project sets along and with a fixed subspace of physical space, to be cut-and-project sets themselves. For planar cut-and-project sets associated with lattices that are Minkowski embeddings of quadratic algebraic integers, such as Penrose-like sets, we consider the distribution of *shapes* of cut-and-project sets that arise by projecting along, or intersecting with, lines through (algebraic) integer points. The special lattices involved allow a parameterization of shapes by closed orbits in the space of lattices, related to the distribution of $||(x, y)||/\gcd(x, y)$ for pairs of quadratic integers.

A classical problem in the geometry of numbers is to find the most economical way to cover $\mathbb{R}^d$ by overlapping Euclidean balls centered at points of a lattice, or to pack the space with disjoint balls centered at lattice points, see [CS]. It would be interesting to apply our aforementioned results to extend results concerning typical lattices [ORW] to typical cut-and-project sets. Do these provide economical point sets for such problems? What is the dependence on the internal dimension? The question of classifying all G_d-invariant probability measure supported on point sets was raised by Marklof [Mar] in connection with problems in mathematical physics. In [RSW] we proved that such measures supported on cut-and-project sets are RMS. Is this true for measures supported on the larger space of all Delone sets?

3. Bounded displacement and biLipschitz equivalence. Delone sets Λ, Γ are *biLipschitz (BL) equivalent* if there is a biLipschitz bijection between them, and *bounded displacement (BD) equivalent* if there exists a bijection $\phi : \Lambda \to \Gamma$ that moves every point a bounded distance. All lattices in $\mathbb{R}^d$ are BL equivalent, and those with the same covolume are BD equivalent. A Delone set is *rectifiable* if it is BL equivalent to a lattice, and *uniformly spread* if it is BD equivalent to one. A sufficient condition for rectifiability and a sufficient and necessary condition for a set to be uniformly spread were given by Burago and Kleiner [BK2] and by Laczkovich [La], respectively, both stated in terms of discrepancy bounds.

Recent results. A necessary and sufficient condition for BD non-equivalence for non-uniformly spread sets was given in [FSS, SS2], building on Laczkovich's original arguments. We applied it in [FSS], jointly with Frettlöh and Solomon, to construct arbitrarily many distinct substitution rules with a common substitution matrix that give rise to non-equivalent sets, and a mixed substitution system that generates representatives of continuously many equivalence classes. In [SS2] we established a dichotomy for BD equivalence in minimal spaces of Delone sets, with respect to translations: either the space $\mathbb{X}$ contains a uniformly spread set, in which case all sets in $\mathbb{X}$ are such, or otherwise $\mathbb{X}$ contains representatives of continuously many BD equivalence classes. This dichotomy applies to orbit closures of well-studied constructions in aperiodic order, including cut-and-project sets and sets associated with primitive and incommensurable substitution tilings. We later showed in [SS3] that almost linearly repetitive sets are rectifiable, generalizing [ACG] to the infinite local complexity case.

Preliminary results & future directions. Constructing non-rectifiable sets is a difficult problem. Existence of such sets was established by McMullen [McM] and Burago and Kleiner [BK1], but the existing remarkable examples of non-rectifiable sets [Mag, CN] are rather involved and require infinitely many steps and constants. Sets associated with incommensurable multiscale substitution tilings seem natural candidates for simply defined non-rectifiable constructions as they are never uniformly spread or almost linearly repetitive. We are currently investigating concrete examples, and in particular the square tiling introduced in [SS1], for which the aforementioned condition of [BK2] is not satisfied. Is there a BL analogue to the BD equivalence dichotomy established in [SS2]?

Previous research in number theory. Previously I worked in number theory, studying sums of two squares. In [Sm1] I focused on the pair correlations of integer sums of two squares and their distribution in short intervals, and in [BSW], together with Bary-Soroker and Wolf, we defined a function field analogue of sums of two squares and established analogues of Fermat's classificationtand Landau's asymptotic density theorems both in the large degree limit and in the large field limit. The experience and tools I acquired in number theory have proved valuable in my subsequent research, and will hopefully continue and contribute to my future work.

References

[ACG] J. Aliste-Prieto, D. Coronel and J. M. Gambaudo. Linearly repetitive Delone sets are rectifiable, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **30**(2) (2013), 275–290.

[BG] M. Baake and U. Grimm. *Aperiodic order. Volume 1: A mathematical invitation*, Cambridge University Press, Cambridge (2013).

[BGM] M. Baake, F. Gähler and N. Mañibo. Renormalization of pair correlation meaures for primitive inflation rules and absence of absolutely continuous diffraction, *Comm. Math. Phys* **370**(2) (2019), 591–635.

[BSW] L. Bary-Soroker, Y. Smilansky and A. Wolf. On the function field analogue of Landau's theorem on sums of squares, *Finite Fields Appl.* **39** (2016), 195–215.

[Bo] K. Böröczky. Gömbkitöltések állandó görbületü terekben I, II, *Mat. Lapok*, **25** (1974), 265–306.

[BK1] D. Burago, B. Kleiner,Separated nets in Euclidean space and Jacobians of biLipschitz maps, *Geom. Func. Anal.* **8**(2) (1998), 273–282.

[BK2] D. Burago, B. Kleiner, Rectifying separated nets, *Geom. Func. Anal.* **12** (2002), 80–92.

[CS] J.H. Conway and N.J.A. Sloane. *Sphere packing, lattices and groups*, volume 290 of Grundlehren de mathemtische wissenschaften. Springer (1988).

[CN] M. I. Cortez and A. Navas. Some examples of repetitive, non-rectifiable Delone sets, *Geom. Top.* **20**(4) (2016), 1909–1939.

[DF] N. Dolbilin and D. Frettlöh. Properties of Böröczky tilings in high-dimensional hyperbolic spaces, *European J. Combin.* **31**(4) (2010), 1181–1195.

[Es] M. C. Escher. *The Regular Division of the Plane.* In *Escher on Escher: exploring the infinite*, Henry N. Abrams, Inc., (1989), 90–127. Translated by Karin Ford.

[FS] N. P. Frank and L. Sadun. Fusion tilings with infinite local complexity, *Topology Proc.* **43**, (2014), 235–276.

[FSS] D. Frettlöh, Y. Smilansky and Y. Solomon. Bounded displacement non-equivalence in substitution tilings, *J. Combin. Theory Ser. A* **177**, (2021), 105326.

[GRS] P. Gohlke, D. Rust and T. Spindeler. Shifts of finite type and random substitutions, *Discrete Contin. Dyn. Syst.* **39**(9) (2019), 5085–5103.

[GS] P. Gohlke and T. Spindeler. Ergodic frequency measures for random substitutions, *Studia Math.* **255**(3) (2020), 265–301.

[GSW] I. Gringlaz, Y. Smilansky and B. Weiss. Projections and intersections of cut-and-project sets, *in preparation.*

[Ka] S. Kakutani. A problem of equidistribution on the unit interval $[0,1]$. In *Measure theory*, Springer, Berlin, Heidelberg, (1976), 369–375.

[KSS] A. Kiro, Y. Smilansky and U. Smilansky. The distribution of path lengths on directed weighted graphs. In *Analysis as a tool in mathematical physics*, Birkhäuser, Basel, (2020), 351–372.

[La] M. Laczkovich. Uniformly spread discrete sets in $\mathbb{R}^d$, *J. Lond. Math. Soc.* **46**(2) (1992), 39–57.

[LS] J. Y. Lee and B. Solomyak. On substitution tilings and Delone sets without finite local complexity, *Disc. Cont. Dynam. Sys.* **39**(6) (2019), 3149–3177.

[Mag] A. N. Magazinov. The family of bi-Lipschitz classes of Delone sets in Euclidean space has the cardinality of the continuum, *Proc. of the Steklov. Inst. of Math.* **275**, (2011), 87–98.

[MRW] N. Mañibo, D. Rust and J. Walton. Substitutions on compact alphabets, *preprint*, `arXiv:2204.07516`, (2022).

[Mar] J. Marklof. Kinetic limits of dynamical systems. In *Hyperbolic dynamics, fluctuations and large deviations*, Proc. Symp. Pure Math., American Mathematical Soc. (2015), 195–223.

[MS] J. Marklof and A. Strömbergsson. Free path lengths in quasicrystals, *Comm. Math. Phys.* **330**(2) (2014), 723–755.

[McM] C.T. McMullen. Lipschitz maps and nets in Euclidean space, *Geom. Func. Anal.* **8**(2) (1998), 304–314.

[MRST] E. Miro, D. Rust, L. Sadun and G.S. Tadeo. Topological mixing of random substitutions, *Israel J. Math.*, to appear, `arXiv:2103.02361`, (2021).

[ORW] O. Ordentlich, O. Regev and B. Weiss. New bounds on the density of lattice coverings, *J. Amer. Math. Soc* **35**(1) (2021), 295–308.

[PP] W. Parry and M. Pollicott. *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Astérisque 187-188, (1990).

[Pen] R. Penrose. Pentaplexity. A class of non-periodic tilings of the plane, *Math. Intell.* **2** (1979), 32–37.

[Pet] S. Petite. Personal communication.

[PS] M. Pollicott and B. Sewell. An infinite interval version of the α-Kakutani equidistribution problem, *Israel J. Math*, to appear, `arXiv:2103.10235`, (2021).

[PvZ] R. Pyke and W. R. van Zwet. Weak convergence results for the Kakutani interval splitting procedure, *Ann. Probab.* **32**(1) (2004), 380–423.

[Rad] C. Radin. Orbits of orbs: sphere packing meets Penrose tilings, *Amer. Math. Monthly* **111**(2) (2004), 137–149.

[Rat] M. Ratner. On Raghunathan's measure conjecture, *Ann. of Math.* **134** (1991), 545-–607.

[RSW] R. Rühr, Y. Smilansky and B. Weiss. Classification and statistics of cut and project sets, *preprint*, `arXiv:2012.13299`, (2020).

[Sa] L. Sadun. Some generalizations of the pinwheel tiling, *Disc. Comput. Geom.* **20** (1998), 79–110.

[Sc] W. M. Schmidt. A metrical theorem in geometry of numbers, *Trans. Amer. Math. Soc.* **95**(3) (1960), 516–529.

[Sm1] Y. Smilansky. Sums of two squares – pair correlation and distribution in short intervals, *Int. J. Number Theory* **9**(7) (2013), 1687–1711.

[Sm2] Y. Smilansky. Uniform distribution of Kakutani partitions generated by substitution schemes, *Israel J. Math.* **240** (2020), 667–710.

[Sm3] Y. Smilansky. Statistics and gap distributions in random Kakutani partitions and multiscale substitution tilings, *J. Math. Anal. Appl.* **516**(2) (2022), 126535.

[SS1] Y. Smilansky and Y. Solomon. Multiscale substitution tilings, *Proc. Lond. Math. Soc.* **123**(6) (2021), 517–564.

[SS2] Y. Smilansky and Y. Solomon. A dichotomy for bounded displacement equivalence of Delone sets, *Ergo. Theo. Dynam. Sys.* **42**(8) (2022), 2693-–2710.

[SS3] Y. Smilansky and Y. Solomon. Discrepancy and rectifiability of almost linearly repetitive Delone sets, *Nonlinearity*, to appear, `arXiv:2109.14564`, (2022).

[SS4] Y. Smilansky and Y. Solomon. A new class of hyperbolic tilings and their orbits, *in preparation.*

[So] Y. Solomon. Substitution tilings and separated nets with similarities to the integer lattice, *Israel J. Math.* **181** (2011), 445–460.

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