Multiscale Substitution Tilings

Yotam Smilansky, Rutgers

Dynamical Systems e-Seminar

HUJI, 2020

Joint work with Yaar Solomon, BGU
Multiscale Substitution Schemes

A multiscale substitution scheme \( \sigma = (\Sigma_\sigma, \rho_\sigma) \) consists of

Prototiles \( \Sigma_\sigma = (T_1, \ldots, T_n) \)

of unit volume in \( \mathbb{R}^d \)

Substitution rule \( \rho_\sigma \) defining a
partition \( \rho_\sigma(T_i) \) of each \( T_i \in \Sigma_\sigma \)
into unions of rescaled prototiles

Examples:

\( S \)

\( \frac{1}{3}S \quad \frac{3}{5}S \)

\( U \)

\( \frac{2}{5}U \quad \frac{2}{5}D \)

\( D \)

\( \frac{3}{5}U \quad \frac{3}{5}D \)
Standard Substitution Tiling Construction

A fixed single scale $\varphi$  
(prototiles may vary in volume)

Every iteration defines a larger patch of tiles
Standard Substitution Tiling Construction

Taking appropriate limits defines substitution tilings of \( \mathbb{R}^d \)
- Induce Delone sets (rel. dense and unif. discrete)
- Have a finite # of tiles up to translations \( \text{sometimes FLC} \)
- Substitution matrix \( S \in M_n(\mathbb{Z}) \) (irreducible, sometimes primitive)
  \[ S_{ij} = \#\{\text{copies of } T_j \text{ in } \rho_0(T_i)\} \]
- Perron-Frobenius theory

The substitution matrix is
\[ S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \]

Perron-Frobenius eigenvalue \( \phi^2 \)
Substitution Flow

Position $T_i \in \mathcal{G}$ so that the origin is in the interior of $T_i$. Define the substitution flow $F_t(T_i)$ for $t \geq 0$ (time)

- At time $t=0$, $F_0(T_i) = T_i$ (a patch consisting of 1 tile)
- As $t$ increases, inflate the patch by $e^t$ and substitute tiles of volume $> 1$ via $p_o$ (substitution rule)
Multiscale substitution tilings are tilings of $\mathbb{R}^d$ that are limits of translations of patches $\{F_t(T_i) : t > 0, T_i \in \mathbb{Z}^d\}$, with respect to the Chabauty-Fell topology on the space of closed subsets of $\mathbb{R}^d$, induced by the metric

$$D(A_1, A_2) = \inf \left( \left\{ r > 0 : A_1 \cap B(0, r) \subset A_2^+ \right\} \cup \{1\} \right)$$

"Sets are close if restricting to a large centered ball, each is contained in a small neighborhood of the other."

The tiling space $X_0$ consists of all tilings generated by $S$. 
Stationary Tilings

Choose $s \in \mathbb{R}$ and an initial position of $T_i$ so that the patch $F_s(T_i)$ contains $T_i$ as a tile in the same position (under the assumptions we will introduce this is possible). Then for all $k \in \mathbb{N}$, $F_{ks}(T_i)$ contains $F_{(k-1)s}(T_i)$, so

$$S = \bigcup_{k=0}^{\infty} F_{ks}(T_i) \in X_0$$

is a stationary tiling, satisfying $F_s(S) = S$. 
Example With Square Scheme

Set $s = \log \frac{s}{3}$ and consider $F_{k,s}(S)$ for $k = 0, 1, \ldots, s$.
Graph Model For Substitution Schemes

A directed weighted graph $G_6$ is associated with $6$

Vertices model the prototiles in $T_6$.

Edges originating at a vertex model the tiles appearing in the substitution rule of the corresponding prototile.

Lengths determined by the scales of the tiles ($\log \frac{1}{\lambda}$)
Incommensurability and Irreducibility

A substitution scheme is incommensurable if $G_6$ contains two closed paths of lengths $a, b$ so that $\frac{a}{b} \notin \mathbb{Q}$. It is irreducible if $G_6$ is strongly connected.

Penrose–Robinson scheme

Rauzy scheme

The $\alpha$-Kakutani scheme
Does This Generate New Tilings?

Yes! If we assume incommensurability. In fact

- Commensurable multiscale substitution tilings can be generated as standard substitution tilings generated by some fixed scale scheme.

- Incommensurable tilings can not.

From now on all schemes are irreducible and incommensurable.
Tile Shape and Denseness of Scales

- For every tiling in $X_0$, all tiles are similar to rescaled copies of the prototiles in $\mathcal{Z}_0$.
- Tiles appear in a dense set of scales within certain intervals of possible scales bounded away from 0. It follows that tilings induce Debrue sets.
- The same holds for any legal patch - a subpatch of $F_t(T_i)$ for some $t \geq 0$ and $T_i \in \mathcal{Z}_0$. 
Scale Complexity Theorem

For stationary $S = \bigcup_{k \in \mathbb{K}_o} F_k S(T)$ the complexity function $c_S(k)$ counts the number of distinct scales in which tiles appear in $F_k S(T)$. If $c_S(l) = c_S(l+1)$ for some $l \in \mathbb{N}$ then $c_S(k) = c_S(l)$ for all $k \geq l$, and such $l$ exists if and only if $\sigma$ is commensurable.

A ‘Sturmian’ tiling, in which $c_S(k+1) = c_S(k) + 1$ for all $k \geq 0$.
Explicit Counting Formulas

Consider the following matrices in $M_n(\mathbb{R})$ associated with $\sigma$. Here $\Sigma^*$ denotes summation over all tiles $T$ of type $j$ in $p_0(T_i)$:

- Substitution matrix $(S_\sigma)_{ij} = \# \{ \text{rescaled copies of } T_j \text{ in } p_0(T_i) \} = \Sigma^* 1$
- Weighted substitution matrix $(W_\sigma)_{ij} = \Sigma^* \text{vol}(C_T)$
- Entropy matrix $(H_\sigma)_{ij} = \Sigma^* \text{vol}(C_T) \log(\text{vol}(C_T))$

$$S_\sigma = \begin{pmatrix} 8 & 5 \\ 1 & 3 \end{pmatrix}$$

$$W_\sigma = \begin{pmatrix} \frac{15}{25} & \frac{8}{25} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$H_\sigma = \begin{pmatrix} -\frac{12}{25} \log\frac{1}{25} - \frac{5}{25} \log\frac{1}{25} & -\frac{4}{25} \log\frac{1}{25} - \frac{4}{25} \log\frac{1}{25} \\ -\frac{1}{4} \log\frac{1}{4} & -\frac{3}{4} \log\frac{1}{4} \end{pmatrix}$$


- $\log^2 y$

always true

PF eigenvalue $= 1$

Right PF eigenvector $= 1 = (1, 1)^T$

Left PF eigenvector $= w = \left(\frac{1}{4}, \frac{8}{25}\right)^T$
Explicit Counting Formulas

Consider the following matrices in $M_n(\mathbb{R})$ associated with $\sigma$.

Here $\Sigma^*$ denotes summation over all tiles $T$ of type $j$ in $\rho_\sigma(T_i)$.

- Substitution matrix $(S_\sigma)_{ij} = \# \{ \text{rescaled copies of } T_j \text{ in } \rho_\sigma(T_i) \} = \Sigma^*_1$
- Weighted substitution matrix $(W_\sigma)_{ij} = \Sigma^* \text{vol}(T)$
- Entropy matrix $(H_\sigma)_{ij} = \Sigma^* \text{vol}(T) \log(\text{vol}(T))$

\[
\# \{ \text{tiles of type } r \text{ in } F_t(T) \} = \frac{[w^T(S_\sigma - W_\sigma)]_r \ \text{vol}(F_t(T)) + o(\text{vol}(F_t(T)))}{w^T H_\sigma 1}
\]

\[
\text{vol} \left( \text{tiles of type } r \text{ in } F_t(T) \right) = \frac{[w^T H_\sigma]_r \ \text{vol}(F_t(T)) + o(\text{vol}(F_t(T)))}{w^T H_\sigma 1}
\]

where $w \in \mathbb{R}^n$ is a left PF eigenvector of $W_\sigma$. 
Distribution of Tile Scales

If in addition we define

- Density matrix \((D_\delta(x))_{ij} = \sum^* g_{i\alpha}(x), g_{i\alpha}(x) = \begin{cases} \frac{\alpha}{x^{d+1}}, & 0 < x < 1 \\ 0 & \end{cases}\)

\#\{tiles in \(F_T\) of type \(r\) and scale in \([a,b]\)\}

\[
\frac{d}{\omega^TH''_\alpha} \int_a^b (w^TD_\delta(x))_r dx \cdot \text{vol}(F_T) + o(\text{vol}(F_T))
\]

Corollaries

- For any \(\alpha\): \#\{copies of \(\alpha T_r\) in \(F_T\)\} \(= o(\text{vol}(F_T))\).
- Gap distribution for point sets defined as tile boundaries of 1 dimensional tilings

\[
\frac{\#\{\text{neighboring points in } [-N,N] \text{ of distance in } [a,b]\}}{\#\{\text{points in } [-N,N]\}} \rightarrow \frac{1}{w^T(S_\delta - W_\delta)1} \int_a^b w^TD_\delta(x)1 dx
\]
Counting: The Substitution Flow and the Graph

- Every tile in $F_t(T_i)$ corresponds to a unique metric path of length $t$ in $G_0$ which originates at vertex $i$.
- If the tile is a copy of $\alpha T_j$, then the path terminates on an edge terminating at vertex $j$, at distance $\log \frac{1}{x}$ from $j$.

\[ \frac{1}{3} \text{ Kakulani:} \]

- Counting is translated to problems on the distribution of paths on incommensurable graphs \cite{Kiro, Smilansky x2, 2020}
Bounded Displacement Equivalence

- Delone sets $\Lambda, \Gamma$ are BD equivalent if $\exists$ bijection $\varphi : \Lambda \rightarrow \Gamma$ with $\sup_{x \in \Lambda} \| x - \varphi(x) \| < \infty$ (BD map).
  
  A tiling is uniformly spread if $\exists \Lambda$, obtained by picking a point from each tile, which is BD equivalent to a lattice.

- Laczkovich criterion: being uniformly spread is equivalent to sufficiently bad tile counting error terms.

- Tilings in $X_6$ are never uniformly spread.

- Moreover, the set of BD equivalence classes represented in $X_6$ has cardinality of the continuum [S', Solomon].
Uniform Patch Frequencies Theorem

Tilings $T \times X_\delta$ have uniform patch frequencies:

For any legal patch $P$ in $T$ and a bounded interval $I$ that contains a left neighborhood of 1:

$$\lim_{q \to \infty} \frac{\# \text{ appearances of } \alpha P \text{ in } T \text{ inside } A_q + h}{\text{vol} (A_q)} =: \text{freq}(P, I, T)$$

exists uniformly in $h \in \mathbb{R}^d$, is positive and independent of $T \times X_\delta$ and of a choice of a van Hove sequence $(A_q)$ in $\mathbb{R}^d$.

(A sequence of bounded measurable sets $\lim_{q \to 0} \frac{\text{vol} \left( (\partial A_q)^r \right)}{\text{vol}(A_q)} = 0$)

(A sequence of bounded measurable sets $\lim_{q \to 0} \frac{\text{vol} \left( (\partial A_q)^r \right)}{\text{vol}(A_q)} = 0$)
Dynamics Theorems

- $X_0$ is invariant under translations in $\mathbb{R}^d$ and under $F_t$. $F_t(T-x) = F_t(T) - e^t x$ for $t>0$, $x \in \mathbb{R}^d$ (horospherical and geodesic).

- Periodic orbits of $F_t$ correspond to stationary tilings.

- The dynamical system $(X_0, \mathbb{R}^d)$ is minimal.

  - Tilings $T \in X_0$ are almost repetitive (relative denseness of return times to $\epsilon$-neighborhoods of patches, for all $\epsilon > 0$).

  - Every $T_1, T_2 \in X_0$ are almost locally indistinguishable (same $\epsilon$-neighborhoods up to translations, for all $\epsilon > 0$).

- $(X_0, \mathbb{R}^d)$ is uniquely ergodic.
Steps of Proof of Unique Ergodicity

- Cylinder sets are defined according to “pixelizations.” A cylinder set consists of all tilings with a pixelization compatible with a fixed coloring of a large cube.

- The ergodic average \( \frac{1}{\text{vol}(A_g)} \int_{A_g} \chi_c(S-x-h)dx \) converges to a countable sum of counting functions of appearances of patches in \( A_{g+h} \).

- Denseness of scales in which patches appear allow the interpretation of the sum as a Riemann-Stieltjes integral, allowing an evaluation using the uniform patch frequencies.
Short Summary of New Tools

- substitution matrix → substitution graph
- discrete-time substitution and inflation → substitution flow
- primitivity → incommensurability
- Perron-Frobenius → distribution of paths on weighted graphs
- uniform patch frequency → UPF with scale intervals
- summation on patches → Riemann-Stieltjes integration
Thanks!