Classification and Statistics of Cut-and-Project Sets

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**Cut-and-Project Sets**

- Fix $\mathbb{R}^n = \mathbb{R}^d \oplus \mathbb{R}^m$ with projections $\pi_{\text{phys}}$ and $\pi_{\text{int}}$
  - physical space $V_{\text{phys}}$
  - internal space $V_{\text{int}}$
- Fix a lattice or a grid $L \subset \mathbb{R}^n$ and a window $W \subset V_{\text{int}}$

![Diagram](attachment://diagram.png)
Cut-and-Project Sets

- Fix \( \mathbb{R}^n = \mathbb{R}^d \oplus \mathbb{R}^m \) with projections \( \pi_{\text{phys}} \) and \( \pi_{\text{int}} \)
  
- Fix a lattice or a grid \( \mathcal{L} \subseteq \mathbb{R}^n \) and a window \( W \subseteq V_{\text{int}} \)

\[ \Lambda = \Lambda (\mathcal{L}, W) := \pi_{\text{phys}} (\mathcal{L} \cap \pi_{\text{int}}^{-1} (W)) \subseteq V_{\text{phys}} \]

The cut-and-project set associated with \( \mathcal{L} \) and \( W \)
**Assumptions and Basic Properties**

\( \Lambda(\mathcal{L},W) \) is irreducible if \( \pi_{\text{int}}(\mathcal{L}) = V_{\text{int}} \),

\( \pi_{\text{phys}} \) is 1-1 on \( \mathcal{L} \), and if \( W \) is regular:

- Bounded \( \Rightarrow \) \( \Lambda \) is uniformly discrete
- Non-empty interior \( \Rightarrow \) \( \Lambda \) is relatively dense \( \Rightarrow \) Delone
- Boundary of measure zero \( \Rightarrow \) \( \Lambda \) has asymptotic density \( D(\Lambda) \)
Assumptions and Basic Properties

\[ \Lambda(\mathcal{L}, \mathcal{W}) \text{ is irreducible if } \pi_{\text{int}}(\mathcal{L}) = V_{\text{int}}, \]

\[ \pi_{\text{phys}} \text{ is 1-1 on } \mathcal{L}, \text{ and if } \mathcal{W} \text{ is regular:} \]

- Bounded \( \Rightarrow \) \( \Lambda \) is uniformly discrete
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Boundary of measure zero \( \Rightarrow \) \( \Lambda \) has asymptotic density \( D(\Lambda) \)

Motivations and Relations

- **Geometric** A Delone set \( \Gamma \) is **Meyer** if \( \Gamma - \Gamma \) is also Delone

\[ \Rightarrow \text{ Every } \Lambda(\mathcal{L}, \mathcal{W}) \text{ is Meyer} \]

**Meyer:** Every Meyer set is contained in some \( \Lambda(\mathcal{L}, \mathcal{W}) \)
• **Arithmetic** Interesting sets have representations as $Λ(Λ,W)$

I An algebraic integer $>1$ is **Pisot** if all its conjugates lie inside the unit disk.

Let $K = \mathbb{Q}(\sqrt{2})$ with a ring of integers $\mathcal{O}_K$, and set $\mathcal{L} = \{(x, \overline{x}) | x \in \mathcal{O}_K\}$ the Minkowski embedding

$\Rightarrow$ Pisot numbers in $\mathcal{O}_K = \Lambda(\mathcal{L}, (-1,1)) \cap (1,\infty)$

II Relaxing assumption and allowing $\text{Vint} = \text{adeles} \Rightarrow$ primitive vectors

• **Dynamical** Delone sets are elements of $\mathcal{E}(\mathbb{R}^d) :=$ closed subsets of $\mathbb{R}^d$ which carries a natural topology. Set $X_\Lambda := \{\Lambda+t | t \in \mathbb{R}^d\}$ then Hof, Schlottmann: Dynamics of $(X_\Lambda, \mathbb{R}^d) \Rightarrow$ pure point diffraction
Example: The Ammann-Beenker Point Set

Let $K = \mathbb{Q}(\sqrt{2})$, and set

$$L = \{ (x_1, x_2, \bar{x}_1, \bar{x}_2) \mid x_1, x_2 \in \mathbb{Z}, \frac{1}{\sqrt{2}} (x_1 - x_2) \in \mathbb{Z} \}$$

$W = \{ \}$

$\Lambda(L, W)$ is then the vertex set of the Ammann-Beenker tiling, which can also be defined via a substitution rule with inflation constant $\lambda = 1 + \sqrt{2}$

From Baake and Grimm's Aperiodic Order Vol 1
Action of $\text{ASL}_d(\mathbb{R})$ and Main Goals

$\text{ASL}_d(\mathbb{R}) = \text{SL}_d(\mathbb{R}) \times \mathbb{R}^d = \{ \text{volume and orientation preserving affine maps } \mathbb{R}^d \to \mathbb{R}^d \}$

- Describe probability measures on cut-and-project sets in $\mathbb{R}^d$, $d \geq 2$, that are affine group invariant and ergodic.
  - We obtain a complete classification

- Describe counting statistics for typical cut-and-project sets with respect to such measures
  - We obtain counting results for both points and patches
Ratner-Marklof-Strömbergsson Measures [MS14]

- Fix $d+m=n$, $\mathbb{R}^n = V_{\text{phys}} \oplus V_{\text{int}}$, $W_c V_{\text{int}}$. Define an embedding
  \[ \text{ASL}_d(\mathbb{R}) \hookrightarrow \text{ASL}_n(\mathbb{R}) \quad (g, v) \mapsto (\tilde{g}, \tilde{v}) = \left( \begin{pmatrix} g & 0_{d,m} \\ 0_{m,d} & I_m \end{pmatrix}, (0_m) \right) \]

- Let \( \mathcal{L} \in \mathcal{Y}_n = \text{ASL}_n(\mathbb{R}) / \text{ASL}_n(\mathbb{Z}) = \text{space of grids} \), then the orbit of a cut-and-project set \((g, v)\), \( \wedge(\mathcal{L}, W) = \wedge(\tilde{(g,v)}, \mathcal{L}, W) \) in the space of grids \( \mathcal{Y}_n \)
Ratner-Marklof-Strömbergsson Measures [MS14]

- Fix $d+m=n$, $\mathbb{R}^n = V_{\text{phys}} \oplus V_{\text{int}}$, $W_c V_{\text{int}}$. Define an embedding

$$ASL_d(\mathbb{R}) \subseteq ASL_n(\mathbb{R}) \quad (g, v) \mapsto \left( g \begin{pmatrix} \mathbb{I} & 0_{d,m} \\ 0_{m,d} & \mathbb{I}_{m} \end{pmatrix}, \left( v \right) \right)$$

- Let $\mathcal{L} \in Y_n = ASL_n(\mathbb{R}) / ASL_n(\mathbb{Z}) = \text{space of grids}$, then

$$\Lambda(\mathcal{L}, W) = \Lambda\left( \left( g, v \right), \mathcal{L}, W \right)$$

**Ratner:** Orbits closures $ASL_d(\mathbb{R}) \mathcal{L} \subseteq Y_n$ support $ASL_d(\mathbb{R})$-invariant probability measures described using Haar measures on algebraic groups

$$ASL_d(\mathbb{R}) < H < ASL_n(\mathbb{R}) \quad ASL_d(\mathbb{R}) \mathcal{L} = H \mathcal{L}$$

- Let $\bar{\mu}$ be an $ASL_d(\mathbb{R})$-invariant ergodic measure on $Y_n$, and

$$\psi(\mathcal{L}) = \Lambda(\mathcal{L}, W)$$

Then $\mu := \psi_* \bar{\mu} \text{ RMS measure on } \{\Lambda(h\mathcal{L}, W) | h \in H\}$
Classification of Measures

**Theorem** Any $\text{ASL}_d(\mathbb{R})$-invariant ergodic measure assigning full measure to irreducible cut-and-project sets is an RMS measure.
Classification of Measures

**Theorem** Any $\text{ASCDCIR}(\mathbb{R})$-invariant ergodic measure assigning full measure to irreducible cut-and-project sets is an RMS measure.

**Theorem** For every RMS measure, there exist $d\leq k \leq n$ and a real number field $K$ so that $H = H' \times \mathbb{R}^n$, $H' \subset \text{SL}_n(\mathbb{R})$. $H'$ arises via restriction of scalars from the field $K$ and one of the following groups:

- $\text{SL}_k$, then $n = k \cdot \deg(K/\mathbb{Q})$. Over $\mathbb{R}$ (and up to conjugation)

$$H' = \left\{ \left( \begin{array}{cccc}
A_1 & & & \\
& \ddots & & \\
& & A_{\deg(K/\mathbb{Q})} & \\
& & & A_j \in \text{SL}_k(\mathbb{R})
\end{array} \right) \right\}$$

- $\text{Sp}_{2k}$, then $n = 2k \cdot \deg(K/\mathbb{Q})$ (arises only when $d=2$)
Special Cases and Examples

- \( \dim V_{\text{phys}} > \dim V_{\text{int}} \) or \( n \) prime \( \Rightarrow H = \text{ASL}_n(\mathbb{R}) \) (generic case)

- \( \dim V_{\text{phys}} = \dim V_{\text{int}} = 2 \) \( \Rightarrow \) Three options
  - The generic case \( (H = \text{ASL}_n(\mathbb{R})) \)
  - \( H = \text{Sp}_4(\mathbb{R}) \times \mathbb{R}^4 \) (can only arise if \( d = 2 \))
  - \( H = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A, B \in \text{SL}_2(\mathbb{R}) \right\} \times \mathbb{R}^4 \), corresponding to restriction of scalars for \( \text{SL}_2 \) and \( K = \mathbb{Q}(\sqrt{d}) \).

\( \Rightarrow \) Example \( K = \mathbb{Q}(\sqrt{2}) \) for Ammann-Beenker

\[ K \ni a + b\sqrt{d} \mapsto \begin{pmatrix} a & db \\ b & a \end{pmatrix} \in \mathcal{M}_{\deg(k/\mathbb{Q})}(\mathbb{Q}) = \mathcal{M}_2(\mathbb{Q}) \]

extended entry-wise to define \( \text{Res}_{K/\mathbb{Q}}(\text{SL}_k) \cong (\text{SL}_k)^{\deg(k/\mathbb{Q})} \) over \( \mathbb{R} \)
Effective Point Counting Following Schmidt

An unbounded ordered family is a collection of Borel subsets $\{\Omega_T \mid T \in \mathbb{R}_+\}$ of $\mathbb{R}^d$ so that

- $0 < T_1 < T_2 \Rightarrow \Omega_{T_1} \subseteq \Omega_{T_2}$
- For all $T$, $\text{vol}(\Omega_T) < \infty$
- $\text{vol}(\Omega_T) \to \alpha$
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- \( \text{vol}(\Omega_T) \to \alpha \)

**Theorem** Let \( \mu \) be an RMS measure. For every \( \epsilon > 0 \), every unbounded ordered family and for \( \mu \text{-a.e. cut-and-project set } \Lambda \)

\[
\#(\Lambda \cap \Omega_T) = D(\Lambda) \text{vol}(\Omega_T) + O(\text{vol}(\Omega_T)^{\frac{1}{2} + \epsilon})
\]

matches best known result even for lattices and \( \Omega_T = B(0,T) \)
Effective Patch Counting Following Schmidt

For a point $x \in \Lambda$ and $R > 0$, the $R$-patch of $\Lambda$ at $x$ is

$$P_{\Lambda,R}(x) := (\Lambda - x) \cap B(0, R)$$

Theorem (Let $\mu$ be an RMS measure and assume the window $W$ has $\dim_B \partial W < m = \dim V_{\text{int}}$. There is $\Theta > 0$ so that for any unbounded ordered family, for $\mu$-a.e $\Lambda$ and any patch $P$ in $\Lambda$:

$$\# \{ x \in \Lambda \cap \Omega_T : P_{\Lambda,R}(x) = P \} = D(\Lambda, P) \text{vol}(\Omega_T) + O(\text{vol}(\Omega_T)^{1-\Theta})$$

For $\dim_B \partial W = m - 1$ any $\Theta < \frac{1}{m+2}$ is good.
A Siegel Summation Formula and a Rogers Second Moment Bound

Let $f \in C_c(\mathbb{R}^d)$ and $\mu$ an RMS measure. Define a Siegel-Veech transform $\hat{f}(\Lambda) := \sum_{x \in \Lambda} f(x)$.

[MS14] There exists $c>0$ so that

$$\int \hat{f}(\Lambda) \, d\mu(\Lambda) = c \int_{\mathbb{R}^d} f(x) \, d\text{vol}(x)$$

Theorem: If in addition $f : \mathbb{R}^d \to [0,1]$ and $\hat{f} \in \mathcal{L}^2(\mu)$, then there exists $C>0$ so that

$$\int |\hat{f}(\Lambda) - \int \hat{f}(\Lambda) \, d\mu(\Lambda)|^2 \, d\mu(\Lambda) \leq C \int_{\mathbb{R}^d} f(x) \, d\text{vol}(x)$$

$\leftarrow$ Rogers second moment bound
Thank You!