SOME RESULTS ON THE REPRESENTATION THEORY OF VERTEX OPERATOR ALGEBRAS AND INTEGER PARTITION IDENTITIES

Shashank Kanade
April 9, 2015
INTRODUCTION
9 = 9
= 8 + 1
= 7 + 2
= 6 + 3
= 5 + 3 + 1

There are total 30 such ways.
RR 1
Partitions of $n$ such that each part $\equiv 1, 4 \pmod{5}$ are equinumerous with partitions whose adjacent parts differ by at least 2.

RR 2
Partitions of $n$ such that each part $\equiv 2, 3 \pmod{5}$ are equinumerous with partitions whose adjacent parts differ by at least 2 and whose smallest part is at least 2.
Rogers-Ramanujan 1

9 = 9
   = 8 + 1
   = 7 + 2
   = 6 + 3
   = 5 + 3 + 1

9 = 9
   = 6 + 1 + 1 + 1
   = 4 + 4 + 1
   = 4 + 1 + 1 + 1 + 1 + 1
   = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1

Rogers-Ramanujan 2

9 = 9
   = 7 + 2
   = 6 + 3

9 = 7 + 2
   = 3 + 3 + 3
   = 3 + 2 + 2 + 2
**d_i(n)**

Number of partitions of \( n \) with adjacent parts differing by at least 2, such that the smallest allowed part is \( i \).

**RR 1**

\[
\sum_{n \geq 0} d_1(n)q^n = \frac{1}{(1 - q)(1 - q^4)(1 - q^6)(1 - q^9) \cdots}
\]

**RR 2**

\[
\sum_{n \geq 0} d_2(n)q^n = \frac{1}{(1 - q^2)(1 - q^3)(1 - q^7)(1 - q^8) \cdots}
\]
GÖ-GO 2

Partitions of \( n \) with each part \( \equiv 1, 4, 7 \pmod{8} \) are equinumerous with partitions whose adjacent parts differ by at least 2 with difference at least 4 for adjacent \( \text{even} \) parts.

\[
20 = 9 + 6 + 4 + 1 \quad \times
\]

GÖ-GO 2

Partitions of \( n \) with each part \( \equiv 3, 4, 5 \pmod{8} \) are equinumerous with partitions whose adjacent parts differ by at least 2 with difference at least 4 for adjacent \( \text{even} \) parts, moreover, smallest allowed part is 3.
Lepowsky-Milne ‘78

Consider $A_1^{(1)}$, principal specialization.

$$\text{ch}(3\Lambda_0) = \text{ch}(3\Lambda_1)$$

$$= F(q) \cdot \frac{1}{(1 - q)(1 - q^4)(1 - q^6)(1 - q^9)\cdots}$$

$$\text{ch}(2\Lambda_0 + \Lambda_1) = \text{ch}(\Lambda_0 + 2\Lambda_1)$$

$$= F(q) \cdot \frac{1}{(1 - q^2)(1 - q^3)(1 - q^7)(1 - q^8)\cdots}$$

Lepowsky-Wilson ‘78-‘85

Explained the RR sum sides and proved the RR identities by inventing principal Heisenberg subalgebras and Z-algebras.
### Identities

<table>
<thead>
<tr>
<th>Identities</th>
<th>Algebra</th>
<th>Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rogers-Ramanujan</td>
<td>$A_1^{(1)}$</td>
<td>3</td>
</tr>
<tr>
<td>Gordon-Andrews</td>
<td>$A_1^{(1)}$</td>
<td>Odd</td>
</tr>
<tr>
<td>Andrews-Bressoud</td>
<td>$A_1^{(1)}$</td>
<td>Even</td>
</tr>
<tr>
<td>Capparelli</td>
<td>$A_2^{(2)}$</td>
<td>3</td>
</tr>
<tr>
<td>Nandi’s conjectures</td>
<td>$A_2^{(2)}$</td>
<td>4</td>
</tr>
<tr>
<td>Göllnitz-Gordon</td>
<td>$A_5^{(2)}$</td>
<td>2</td>
</tr>
<tr>
<td>K.-Russell Conjectures</td>
<td>$D_4^{(3)}$</td>
<td>3</td>
</tr>
</tbody>
</table>
Feigin-Stoyanovsky ‘93–‘94

A certain subspace $W_{\Lambda} \subset L(\Lambda)$ exhibits difference 2 conditions. RR sum sides for level 1, Gordon-Andrews sum sides at higher levels. Products came from geometric arguments.

Capparelli-Lepowsky-Milas ‘01

Explained the recursion $F(x, q) = F(xq, q) + xqF(xq^2, q)$ using exact sequences among $W(\Lambda_0)$ and $W(\Lambda_1)$.

*Intertwining operators* gave the maps.

Calinescu-Lepowsky-Milas ‘08

Proved the assumed presentation result for $W_{\Lambda_1}$. 

· Finding new identities: K.-Russell
· Finding “motivated proofs:” K.-Lepowsky-Russell-Sills, Coulson-K.-Lepowsky-McRae-Qi-Russell-Sadowski
· Interpreting these proofs using the theory of vertex algebras: Thesis
· In vertex operator algebra theory — PRODUCTS TO SUMS: Thesis
· In vertex operator algebra theory — SUMS TO PRODUCTS: Thesis
SUMS TO PRODUCTS
\[ \mathcal{A} = \mathbb{C}[x_{-1}, x_{-2}, \ldots] \]

\[ l_{\Lambda_0} = \text{Ideal generated by} \left\{ r_n = \sum_{i=1}^{n-1} x_{-i}x_{-n+i}; n \geq 2 \right\}. \]

\[ r_{-2} = x_{-1}x_{-1} = x_{-1}^2 \]

\[ r_{-3} = x_{-1}x_{-2} + x_{-2}x_{-1} = 2x_{-1}x_{-2} \]

\[ r_{-4} = x_{-1}x_{-3} + x_{-2}x_{-3} + x_{-3}x_{-1} = 2x_{-1}x_{-3} + x_{-2}^2 \]

**Definition: Principal Subspace**

We call \( W_{\Lambda_0} = \mathcal{A}/l_{\Lambda_0} \) a *principal subspace.*
Recall:

\[ r_{-2} = x_{-1}x_{-1} = x^2_{-1} \]
\[ r_{-3} = x_{-1}x_{-2} + x_{-2}x_{-1} = 2x_{-1}x_{-2} \]
\[ r_{-4} = x_{-1}x_{-3} + x_{-2}x_{-3} + x_{-3}x_{-1} = 2x_{-1}x_{-3} + x^2_{-2} \]

and so on . . .

- Has a *basis* of monomials satisfying difference-2 conditions. For a proof using Gröbner bases, see Bruschek-Mourtada-Schepers ‘13. Slightly different space considered.
- In this paper, it comes up while calculating Hilbert-Poincaré series of arc space of a double point.
- Shows up in a lot of different problems — we’ll see that later.
Question

Where are the products?

Idea (J. Lepowsky)

First use the Jacobi triple product identity

\[
\frac{1}{(1 - q)(1 - q^4)(1 - q^6)(1 - q^9) \cdots} = \sum_{\lambda \geq 0} (-1)^\lambda \cdot \frac{q^{\lambda (5\lambda - 1)/2} (1 + q^\lambda)}{\prod_{n \geq 1} (1 - q^n)}
\]

Observe that this is an alternating sum:

Could be explained via Euler-Poincaré principle applied to a resolution.
\[ \cdots C_3 = \bigoplus_{i_1, i_2 \leq -2} A\xi_{i_1, i_2} \xrightarrow{\partial_3} C_2 = \bigoplus_{i_1 \leq -2} A\xi_{i_1} \xrightarrow{\partial_2} C_1 = A \xrightarrow{\partial_1} C_0 = \mathcal{W}_{\Lambda_0} \rightarrow 0 \]

- \( \xi_{\ldots, i, \ldots, j, \ldots} = -\xi_{\ldots, j, \ldots, i, \ldots} \)
- \( \partial_1 \) is the projection map \( A \rightarrow A/\mathcal{I}_{\Lambda_0} \).
- \( \partial_{k+1}(\xi_{-i_1, -i_2, \ldots, -i_k}) = \sum_{n=1}^{k} (-1)^{n-1} \cdot r_{-i_n} \cdot \xi_{-i_1, -i_2, \ldots, -\widehat{i_n}, \ldots, -i_k} \)
  \[ \partial(r_{-i}) = 0 \]
  \[ \partial(\xi_{-i}) = r_{-i} \]
  \[ \partial(\xi_{-i, -j}) = r_{-i} \xi_{-j} - r_{-j} \xi_{-i} \]
- \( H_n = \text{Ker}(\partial_n)/\text{Im}(\partial_{n+1}) \)
  \( H_0 = H_1 = 0. \)
\[ \cdots C_3 = \bigoplus_{i_1, i_2 \leq -2} A\xi_{i_1, i_2} \xrightarrow{\partial_3} C_2 = \bigoplus_{i_1 \leq -2} A\xi_{i_1} \xrightarrow{\partial_2} C_1 = A \xrightarrow{\partial_1} C_0 = W_{\Lambda_0} \to 0 \]

**Interpretation**

\[ \partial_2 : \bigoplus_{i_1 \leq -2} A\xi_{i_1} \to A \]

\( \text{Ker}(\partial_2) \) is precisely the *the space of relations* amongst the \( r_n \)s.

\[ \partial_3 : \bigoplus_{i_1, i_2 \leq -2} A\xi_{i_1, i_2} \to \bigoplus_{i_1 \leq -2} A\xi_{i_1} \]

\( \text{Im}(\partial_3) \) is precisely the the space of *trivial* relations amongst the \( r_n \)s: 
\[ r_n \cdot r_m - r_m \cdot r_n = 0. \]

\[ H_2 = \text{Ker}(\partial_2)/\text{Im}(\partial_3) \] measures the space of “non-trivial” relations.
\[ \cdots C_3 = \bigoplus_{i_1, i_2 \leq -2} A \xi_{i_1, i_2} \xrightarrow{\partial_3} C_2 = \bigoplus_{i_1 \leq -2} A \xi_{i_1} \xrightarrow{\partial_2} C_1 = A \xrightarrow{\partial_1} C_0 = W_{\Lambda_0} \rightarrow 0 \]

Euler-Poincaré principle

With \( \chi \) being the “dimension” (actually, the \((x, q)\)-character)

\[
\chi(W_{\Lambda_0}; x, q) = \sum_{n \geq 1} (-1)^{n+1} \left( \chi(C_n; x, q) - \chi(H_n; x, q) \right).
\]

The Problem

Find the precise structure of \( H_n \)s and calculate \( \chi(H_n; x, q) \).
There is a derivation $L_{-1}$ of $\mathcal{A}$, that can be extended to the $C_j$’s:

\[
L_{-1} \cdot x_{-j} = jx_{-j_{-1}}
\]

\[
L_{-1} \cdot r_{-j} = (j - 1)r_{-j_{-1}}
\]

\[
L_{-1} \cdot (\xi_{-i_1, \ldots, -i_k}) = (i_1 - 1)\xi_{-i_1_{-1}, -i_2, \ldots, -i_k} + (i_2 - 1)\xi_{-i_1, -i_2_{-1}, \ldots, -i_k}
+ \cdots + (i_k - 1)\xi_{-i_1, -i_2, \ldots, -i_{k-1}}
\]

\[
L_{-1}(a \cdot c) = L_{-1}(a)c + aL_{-1}(c)
\]

\[
[\partial, L_{-1}] = 0
\]

$L_{-1}\text{Ker}(\partial) \subset \text{Ker}(\partial)$

$L_{-1}\text{Im}(\partial) \subset \text{Im}(\partial)$
There is an automorphism $\sigma$ of $A$, that can be extended to the $C_j$s:

$$
\sigma(x_{-1}) = 0,
\sigma(x_{-i}) = x_{-i+1}
$$

$$
\sigma(\xi_{-i_1, \ldots, -i_k}) = \xi_{-i_1+2, \ldots, -i_k+2}
$$

$$
\sigma(a \cdot c) = \sigma(a)\sigma(c)
$$

$$
[\partial, \sigma] = 0
$$

$$
\sigma\text{Ker}(\partial) \subset \text{Ker}(\partial)
$$

$$
\sigma\text{Im}(\partial) \subset \text{Im}(\partial)
$$
Some obvious elements in the $\text{Ker}(\partial_2)$:

$$\mu_{-4} = 2x_{-2} \xi_{-2} - x_{-1} \xi_{-3}$$

$$\partial(\mu_{-4}) = 2x_{-2} \cdot x^2_{-1} - x_{-1} \cdot 2x_{-1}x_{-2} = 0$$

$$\partial(L_{-1}^S \cdot \mu_{-4}) = 0 \text{ for } s \in \mathbb{N}.$$
Theorem (K.)

The second homology $H_2$ is generated by the elements $L_{-1}^s \cdot \mu_{-4}$ for $s \in \mathbb{N}$. 
Remark
The proof is very similar to the proof of presentation of $W_{\Lambda_0}$ in Calinescu-Lepowsky-Milas ‘08.
Use the “minimal counter-example” technique.

Remark
The derivation $L_{-1}$ does not enter the proof. Only the automorphism $\sigma$ is needed.

Recall:
\[
\sigma : \quad x_{-1} \mapsto 0, \quad x_j \mapsto x_{j+1}, \quad \xi_{-2} \mapsto 0, \quad \xi_{-3} \mapsto 0, \quad \xi_{-i} \mapsto \xi_{-i+2}.
\]
Reason:
$\mu_{-4}, \mu_{-5}, \mu_{-6}$ determine the elements of $\text{Ker}(\partial_2)$ of charge 3.
\[ \mathfrak{g} = \mathfrak{sl}_2 = \mathbb{C}\{x_\alpha, \alpha, x_{-\alpha}\} \]

\[ \langle a, b \rangle = \text{Tr}(ab) \]

\[ \mathfrak{n} = \mathbb{C}x_\alpha \]

\[ \widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \]

\[ [x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{m+n} + \langle x, y \rangle n\delta_{m+n,0}c \]

\[ [c, \widehat{\mathfrak{g}}] = 0 \]

\[ \widehat{\mathfrak{g}} \cong A^{(1)}_1 \]

\[ \bar{\mathfrak{n}} = \mathfrak{n} \otimes \mathbb{C}[t, t^{-1}] \subset \widehat{\mathfrak{g}} \]

\[ \bar{\mathfrak{n}}_- = \mathfrak{n} \otimes t^{-1}\mathbb{C}[t^{-1}] \subset \widehat{\mathfrak{g}} \]

\[ [\bar{\mathfrak{n}}_-, \bar{\mathfrak{n}}_-] = 0. \]
Relations to Principal Subspaces

\( L(\Lambda) \): Irreducible, integrable \( \hat{\mathfrak{g}} \) – module
generated by highest wt. vector \( v_\Lambda \)

\[ W_\Lambda := \mathcal{U}(\mathfrak{n}) \cdot v_\Lambda \cong \mathcal{U}(\mathfrak{n}_-) \cdot v_\Lambda \]

\[ \mathcal{U}(\mathfrak{n}_-) \cong \mathbb{C}[x_{-1}, x_{-2}, \ldots] = \mathcal{A} \]

\[ f_\Lambda : \mathcal{U}(\mathfrak{n}_-) \longrightarrow W_\Lambda \]

\[ 1 \mapsto v_\Lambda \]

Theorem (Calinescu-Lepowsky-Milas ‘08)

\( \text{Ker}(f_{\Lambda_0}) = l_{\Lambda_0} \) and \( \text{Ker}(f_{\Lambda_1}) = l_{\Lambda_0} + \mathcal{A}x_{-1} \).
\[ \cdots \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow L_{\Lambda_0} \]

\[ V_{r_0 r_1 \cdot \Lambda_0} \quad V_{r_0 \cdot \Lambda_0} \quad V_{\Lambda_0} \quad V_{\Lambda_0} \]

\[ V_{r_0 r_1 \cdot \Lambda_0} \longmapsto (2x_{-2} - x_{-1}L(-1))V_{r_0 \cdot \Lambda_0} \]

\[ V_{r_0 \Lambda_0} \longmapsto x_{-1}^2 V_{\Lambda_0} \]

\[ w \cdot \Lambda = w(\Lambda + \rho) - \rho \]
Stable unreduced Khovanov homology.

Apparently quite hard to compute for Torus knots. See katlas.org

\[ \text{Kh}(T(n, \infty)) = \lim_{m \to \infty} q^{-(n-1)(m-1)+1} \text{Kh}(T(n, m)). \]

This limit exists (Stošić).

**Conjecture (Gorsky-Oblomkov-Rasmussen ‘12)**

\[ \text{Kh}(T(n, \infty)) \text{ is dual to the homology of the Koszul complex} \]
\[ \text{determined by the elements } r_{-2}, \ldots, r_{-n-1}. \]
Every Koszul complex is a dg-algebra:

\[ \xi_{-i_1,\ldots,-i_j} = \xi_{-i_1} \wedge \cdots \wedge \xi_{-i_j}. \]

\[ Z = \bigoplus_{n \geq 0} \text{Ker}(\partial_n) \] is a sub-algebra with

\[ B = \bigoplus_{n \geq 0} \text{Im}(\partial_n), \] a two-sided ideal.

\[ H = \bigoplus_{n \geq 0} H_n \] is a graded algebra.
Conjecture (Gorsky-Oblomkov-Rasmussen ‘12)

For $T(n, \infty)$, $H$ is generated as by the elements $\mu_{-4}, \ldots, \mu_{-n-2}$, with
the defining relations being

\[
x(z)^2 = 0 \]
\[
x(z)\mu(z) = 0 \]
\[
 x''(z)\mu(z) - x'(z)\mu'(z) = 0 \]
\[
 \mu(z)\mu'(z) = 0, \]

recall that $\mu(z) = 2x'(z)\xi(z) - x(z)\xi'(z)$.

$\sigma$: unreduced $\rightarrow$ reduced.
RELATIONS TO MEURMAN-PRIMC’S WORK

\[ E_0 \rightarrow L(\Lambda_0) \]
\[ R = U(\mathfrak{g})x^2_{-1} \subset E_0 \]
\[ \bar{R} = \text{Coeffs of } r(x) = Y(r, x), r \in R \]
\[ \bar{R}_1 \otimes E_0 \rightarrow E_0 \]
\[ u \otimes v \leftrightarrow u_{-1}v \]
\[ Y(u, x) \otimes Y(v, x) \leftrightarrow Y(u, x)Y(v, x)_{\cdot}, u \in \bar{R}_1 \]

Meurman-Primc ‘99, Primc ‘02

The kernel of the map above is generated, in some sense, by:

\[ \frac{d}{dx} \left( x_\theta(x)^2 \right) \otimes x_\theta(x) - 2(x_\theta(z)^2) \otimes \frac{d}{dx} x_\theta(z) \]
\[ L(-1)x_{-1}^2 1 \otimes x_{-1} 1 - 2x_{-1}^2 1 \otimes L(-1)x_{-1} 1 \]
PRODUCTS TO SUMS
GÖLLNITZ-GORDON IDENTITIES AND $A_5^{(2)}$

**Theorem (K.)**

Spanning works!

Figure 1: $A_5^{(2)}$

\[
\begin{align*}
\text{ch}(\Omega(L(\Lambda_0 + \Lambda_1))) & = \frac{1}{(1 - q)(1 - q^4)(1 - q^7) \cdots} \\
\text{ch}(\Omega(L(\Lambda_3))) & = \frac{1}{(1 - q^3)(1 - q^4)(1 - q^5) \cdots}
\end{align*}
\]
· Higher rank motivated proofs?
· Multi-colour generalizations of the Göllnitz-Gordon identities?
· Look at level 2 standard modules contained in the tensor product of two inequivalent level 1 standard modules for higher rank algebras in the same “series.”

\[ A_{5}^{(2)} \text{ Göllnitz-Gordon} \]

\[ A_{7}^{(2)} \text{ Rogers-Ramanujan [Bos-Misra ‘94]} \]

\[ A_{9}^{(2)} \text{ ?????} \]

\[ A_{11}^{(2)} \text{ Nandi’s identities} \]
PRODUCTS TO SUMS, AGAIN
RR 1

Partitions of $n$ such that each part $\equiv 1, 4 \pmod{5}$ are equinumerous with partitions whose adjacent parts differ by at least 2.

RR 2

Partitions of $n$ such that each part $\equiv 2, 3 \pmod{5}$ are equinumerous with partitions whose adjacent parts differ by at least 2 and whose smallest part is at least 2.

Question: Ehrenpreis

RR1 $\geq$ RR2: Reason — sum sides.

Can one see this using only products?
Motivated by Ehrenpreis’s question:

(RR1 ≥ RR2, without looking at sum-sides)

\[
G_1 = \prod_{\substack{m > 0, \\ m \not= 0, \pm 2(5) }} \frac{1}{1 - q^m} = 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + \cdots
\]

\[
G_2 = \prod_{\substack{m > 0, \\ m \not= 0, \pm 1(5) }} \frac{1}{1 - q^m} = 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + 2q^7 + \cdots
\]

Observe: \( G_1 \geq G_2 \).

\[
G_1 - G_2 = q + q^4 + q^5 + q^6 + q^7 + 2q^8 + 2q^9 + \cdots
\]

Let \( G_3 = (G_1 - G_2)/q = 1 + q^3 + q^4 + q^5 + q^6 + 2q^7 + 2q^8 + \cdots \).

Let \( G_4 = (G_2 - G_3)/q^2 = 1 + q^4 + q^5 + q^6 + 2q^7 + 2q^8 + \cdots \).

Continue: Let \( G_i = (G_{i-2} - G_{i-1})/q^{i-2} \).
Observe that each

\[ G_i = 1 + q^i + \cdots \]  

**Empirical Hypothesis**

Proof of the Empirical Hypothesis not only answers Ehrenpreis’s question but also immediately leads to a proof of RR.

**Idea (J. Lepowsky, A. Milas)**

Explain the sequence of \( G_i \)’s using exact sequences similarly to the principal subspace story:

\[ 0 \rightarrow \Omega(L_{G_2}) \rightarrow \Omega(L_{G_1}) \rightarrow ? \rightarrow 0. \]

Use *twisted relativized intertwining operators*. 
Theorem (K.)

There exist *explicitly constructed* twisted intertwining operators among certain mixed triples of untwisted and $\theta$-twisted modules for the vertex operator algebra $V = L_{\mathfrak{sl}_2}^-(\ell, 0)$, for a positive integer $\ell$.

Theorem (K.)

$V = V_{\mathbb{Z}\alpha}$: Rank 1 lattice vertex algebra, $\langle \alpha, \alpha \rangle = 2$.

$\theta$: Lift to $V$ of the “$-1$” automorphism of $\mathbb{Z}\alpha$.

$G = \mathbb{Z}/8$.

There exists a natural abelian intertwining algebra structure on the space $V \oplus V_{(\mathbb{Z}+1/2)\alpha} \oplus V^{T_1} \oplus V^{T_2}$ with grading group $G$.

The $Y$ map for the abelian intertwining algebra is comprised of twisted intertwining operators.
Questions?