

Parallel Algorithms

Matching

\Leftrightarrow Bipartite G

n vertices on either side

① Does this graph have a perfect matching?

② Find one.

Determinant

Given $n \times n$ matrix A

Find $\det(A)$.

Then

There are $\text{polylog}(n)$ parallel time
algorithms for determinant
with $\text{poly}(n)$ processors.

④ Let $\lambda_1, \dots, \lambda_n$ be the n eigenvalues of A .
 = roots of $\det(\lambda I - A) = 0$.

$$P(\lambda) = \det(\lambda I - A) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

Roots of $P(\lambda) = \{\lambda_1, \dots, \lambda_n\}$

So $(-1)^n \prod_i \lambda_i = \text{const term of } P(\lambda)$.

$$\begin{aligned} &= P(0) \\ &= \det(-A) \end{aligned}$$

$$= (-1)^n \cdot \det(A).$$

① Try to compute $\sum \lambda_i, \sum \lambda_i^2, \sum \lambda_i^3, \dots$

$$\sum \lambda_i^n$$

② Express $\prod \lambda_i$ in terms of these.

$$\begin{array}{ll}
 \textcircled{1} & a+b \\
 & a^2+b^2 \\
 & 2ab = (a+b)^2 - (a^2+b^2) \\
 & x^2 - (a+b)x + ab
 \end{array}$$

$$\begin{array}{ll}
 \textcircled{2} & a+bt+c \\
 & a^2+b^2+c^2 \\
 & a^3+b^3+c^3 \\
 & 2(abt+bct+ca) = (a+b+c)^2 - (a^2+b^2+c^2) \\
 & (a^3+b^3+c^3 - 3abc) = (a+b+c) \\
 & (a^2+b^2+c^2 - ab - bc - ca) \\
 & x^3 - (a+b+c)x^2 + (abt+bct+ca)x \\
 & - abc = 0
 \end{array}$$

Newton Identities

$$\text{Let } s_j = \sum \lambda_i^j$$

Let $e_l = l^{\text{th}}$ elementary symmetric
fn of $\lambda_1, \dots, \lambda_n$

$$= \sum_{\substack{S \subseteq [n] \\ |S|=l}} \left(\prod_{i \in S} \lambda_i \right)$$

$$W = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ s_1 - 2 & 0 & \cdots & 0 \\ s_2 - s_1 + 3 & & & 0 \\ \vdots & & & \\ s_n - s_{n-1} - \cdots - s_1 + m \end{bmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}$$

~~10~~

$$e_l s_{l+1} - e_2 s_{l-2} + \dots + (-1)^{l-2} s_1 e_{l-1} + \underline{s_l e_l}$$

$$= \underline{s_l}$$

Gives

$$\begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = \begin{bmatrix} w^{-1} \end{bmatrix} \cdot \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}$$

Only depends on s_i 's.

Algorithm to compute e_i 's.

① Compute $\otimes s_i$'s.

$$s_f = \sum \lambda_i^f = \text{Tr}(A^f)$$

First compute $A, A^2, A^4, \dots, A^{2^k}, \dots$

$\underbrace{\hspace{10em}}$

$\log n$ rounds.

So total $\log^2 n$ time.
(parallel).

Then compute (in parallel forall)

$$A^j = \prod_{r \in S_j} A^{2^r}$$

$\overbrace{\hspace{10em}}$
 $O(\log n \log \log n)$ time

②

$$\text{Then } \vec{e} = \vec{w}^\dagger \cdot \vec{s}$$

where

$$w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Want to compute w^\dagger .

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}^{-1}$$

$$\boxed{T^{-1} = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & T_2^{-1} \end{bmatrix}}$$

$$= \begin{bmatrix} A^{-1} & 0 \\ D & C^{-1} \end{bmatrix}$$

$$A^{-1}B + DC = 0$$

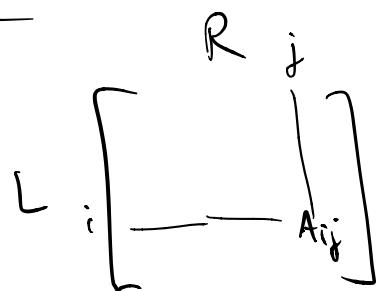
$$D = - A^{-1}B C^{-1}$$

Gives divide & conquer algorithm
to find ~~the~~ inverse of
lower triangular matrices
in parallel time $O(\log^2 n)$
with $\text{poly}(n)$ processes.

So we get $\det(A)$.

Bipartite G , no vertices on either side

Edmonds Matrix



$$A_{ij} = \begin{cases} 0 & \text{if } i, j \text{ not adjacent} \\ x_{ij} & \text{if adjacent.} \end{cases}$$

$$\det(A) = \sum_{\substack{\text{matchings} \\ M \subseteq [n] \times [n] \\ \text{of the graph.}}} (-1)^{\text{sign}(M)} \prod_{(i,j) \in M} x_{ij}$$

Algorithm to test if G has a perfect matching:
 Substitute Random values b_{ij}
 for each x_{ij} in $A(\vec{b})$
 Compute $\det(A(\vec{b}))$
 If $\neq 0$, then G has a perfect matching

If $= 0$, claim that
 G has no p.m.

For general graphs:

Tutte matrix. n vertex graph

$A \text{ } n \times n$

$$A_{ij} = \begin{cases} 0 & \text{if not an edge} \\ x_{ij} & \text{if } i < j \text{ is an edge} \\ -x_{ji} & \text{if } i > j \text{ is an edge} \end{cases}$$

5 7

$$\begin{bmatrix} & & & & \\ 5 & & & & x_{57} \\ & & & & \\ 7 & & -x_{57} & & \end{bmatrix}$$

If G is bipartite; Tutte matrix is



Fact G has a p.m. iff $\det(\text{Tutte mat}(G))$ is a nonzero polynomial.

Red-Blue matchings

Consider bipartite graph.

edges colored red / blue.

Want: p.m. with exactly k red edges.

R/B Edmonds matrix.

$$A = \begin{bmatrix} & & & 0 \\ & x_{ij}y & & x_{ij} \\ & & & x_{ij} \\ & & & \end{bmatrix}$$

red B

$$\det(A) = \sum_{k=0}^n \left(\sum_{\substack{\text{p.m.} \\ \text{with} \\ k \text{ red edges}}} y^k 2^{n-k} \right)$$

$\det(A)$ has nonzero coeff for $y^k 2^{n-k}$
iff \exists such a p.m.

$A(\textcircled{b}_{ij})(x_{ij}), Y, Z$

$$\det \left(A \left(\begin{pmatrix} \vec{b} \\ \downarrow \\ (b_{ij})_{\substack{i,j=1}}^n \end{pmatrix}, Y, Z \right) \right) = \sum_{k=0}^n c_k Y^k Z^{n-k}$$

Aly: For random \vec{b} , find
 \rightarrow coeff c_k of $\det(A(\vec{b}, Y, Z))$

\rightarrow If $c_k \neq 0$, b has a R.p.m.

If $c_k \neq 0$, claim that
G does not.

Finding a perfect matching

Isolation Lemma [MVV '85]

We have U . $|U| = m$

$$S_1, S_2, \dots, S_t \subseteq U$$

Pick for each $u \in U$, a weight $w(u) \in \{1, 2, \dots, \underline{2m}\}$ uniformly at random.

Then $P_n \left[\begin{array}{l} \text{there is a unique} \\ \text{set } S_j \text{ s.t. } \sum_{u \in S_j} w(u) \\ \text{is minimized} \end{array} \right] \geq \frac{1}{2}$.

For each $u \in U$

define bad event E_u :

{ Some S_j that contains u has minimum weight }

AND some S_ℓ that does not contain u has minimum weight },

$$\Pr_n[E_u] \leq \frac{1}{2^m}.$$

$$\Pr_n[\exists u \text{ s.t. } E_u \text{ happens}] \leq \frac{1}{2}.$$

~~so~~ $\Pr_n[\text{Isolation}] \geq 1 - \frac{1}{2} = \frac{1}{2}.$

Finding a p.m. in parallel.

1. For each edge $e \in G$

Pick $w(e) \in \{1, 2, \dots, 2^{n^2}\}$

uniformly at random.

2. Modified Edmonds Matrix.

$$A_{ij} = \begin{cases} 0 & \text{if } (i, j) \text{ not edge} \\ x_{ij} y^{w(ij)} & \text{_____ is edge} \end{cases}$$

$$\det(A) = \sum_{l=0}^{2^{n^3}} \binom{n^3}{l} y^l$$

Coeff of $y^l \neq 0 \iff \exists$

p.m. M s.t. $\sum_{e \in M} w(e) = l$.

Substitute distinct prime values

$x_{ij} = b_{ij}$ and
~~and~~ find ~~all~~ weffs of ~~all~~ x^l
for all l .

The smallest nonzero one
is likely the product of
primes corresponding to
the ~~any~~ chosen
matching.